Finite Element Method and Computational Structural Dynamics
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Lecture - 27
Finite elements of C^0 Continuity in 2-D and 3-D-IX

Hello friends! We have seen the Finite Element Modeling, using Triangular elements such as 3 noded triangular element that has the constant strain state i.e. constant derivative condition; and then we also discussed 6 noded triangle, that will have linear derivative field.

The 3 noded triangle element is also popularly known as linear strain triangle or LST, and then if we need to have cubic variation for the primary variable within element boundary, then we can have 10 noded triangle with 9 nodes on the edge of the triangle, and 1 node in the interior of the triangle.

And, we also discussed how the interpolation functions for these triangular elements can be very easily derived by using the concept of local coordinates, which are related to the partition of the area with coordinate x y in the interior of the domain. And, how the 1 point chord defines or a divides the entire area of the triangle into 3 parts. And, those ratios of the area with respect to the total area of the triangle, defines the value of the local coordinate system. This is taken to as a standard definition. Local coordinate varies from 0 to 1 as parallel grid lines being parallel to the element edges. And, then each point within the within the domain of the triangle, can be defined by an ordered pair of these 3, local coordinates L1, L2, L3. And, using this local coordinate system it is very easy to derive the interpolation function for higher order elements such as quadratic elements or cubic elements.

Though triangular elements are simplest two dimensional element for covering area, it is possible to cover a larger amount of area in 1 element by using rectangular geometry. So, that is the next element which we will discuss in 2 dimensional elements. In 3 dimensional case, the analog of triangular elements is a pyramid called tetrahedron, and the analog of a rectangular element in 3 dimensional domains is a cuboid. And, also in between we have prism or triangular prism. All of these enclose area and they can be used for deriving the finite element models.

So, we now discuss the development of a rectangular element and see how the interpolation functions are defined for rectangular elements. Because all that changes in the entire process of finite element analysis is the discretization using the appropriate element. And, then defining the interpolation functions, which defines how the primary variables are approximated within individual element.

Rest of the steps of entire finite element analysis are identical and they follow naturally after this.

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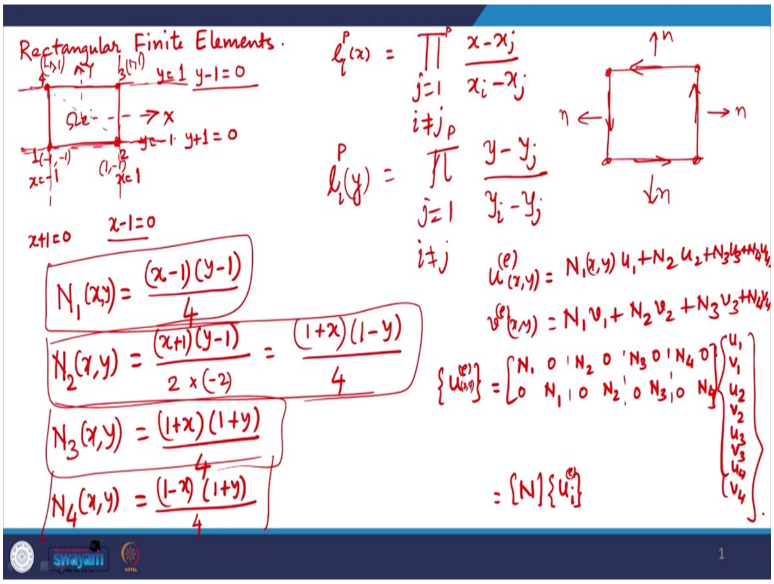

So, we consider rectangular element and there are two elements actually in the family. The first element is of course, 4 noded rectangle.

So, if I have coordinate system somewhere let us say x and y and I can have lines as $x = -1$, $x = +1$, $y = -1$, and $y = +1$ which represents gridlines.

So, you can see that the location of the nodes is at the intersection of the grid lines. So, this is the grid line for $x = -1$, this is a grid line for $x = +1$, this is the grid line for $y = -1$, this is the grid line for $y = +1$. And, intersection of these grid lines is what defines the location of nodes of define vertices of this rectangle. Rectangular element and this is the element domain Ω_e.

Now, the equation of lines: this particular line is defined by $x+1=0$, this particular line is defined by equation $x-1=0$, this particular line is defined by $y+1=0$, and this particular line is defined by $y-1=0$.

And, in this we have the clue to derive the interpolation functions. The requirement for the interpolation function is that, interpolation function for i^{th} node should evaluate to unity at the corresponding node, and it should vanish at all other nodes. And, if the nodes are located at the intersection of the grid lines the simplest way to achieve vanishing condition of the interpolation function is to just multiply by appropriate grid lines.

So, for example, if I am trying to derive the interpolation function corresponding to node 1, all that I need to do is take the equation of line 2-3 multiply it with equation of line 3-4. And, it takes care of the condition that N1 should vanish at 2, 3 and 4. And, then I evaluate that product at the coordinates of node 1 and normalize with respect to that.

So, that is the normalizing condition and that makes the interpolation function for node 1 evaluate to unity. And, that is essentially what we call as Lagrange's interpolation, but in this case we are essentially talking about Lagrange's interpolation in one dimension.

So, essentially Lagrange's interpolation for node i is given by

$$l_i^p(x)=\prod_{j=1;\, j\neq i}^{p}\frac{x-x_j}{x_i-x_j}$$

So, essentially equation of line 0s of individual grid lines so, $x=x_j$ and the equation of that grid line would be $x-x_j=0$. So, I multiply with that and normalize it with that particular expression evaluated at the node i $x=x_i$ and then take the product.

Now, this is p^{th} degree polynomial. So, whatever number of terms we will have? p number of products. So, it will be p^{th} degree polynomial and together this is a variation with respect to x. So, in case of 2 dimensional elements all that I need to do. I will have similar kind of expression for y.

$$l_i^p(y)=\prod_{j=1;\, j\neq i}^{p}\frac{y-y_j}{y_i-y_j}$$

So, interpolation function for node 1 $N_1(x)$ would be given by equation of line 2-3 $x-1=0$, multiplied by equation of this line 3-4 $y-1=0$. And, I evaluate this at node 1, which has coordinates (-1, -1). So, the interpolation function for node 1 is:

$$N_1=\frac{(x-1)(y-1)}{4}$$

Similarly, interpolation function for node 2 as a function of x and y is simply product of equation of line 1-4 and equation of line 3-4. Normalized to have unit value at node 2.

$$N_2=\frac{(1+x)(1-y)}{4}$$

Similarly N_3 and N_4 can be written as:

$$N_3=\frac{(1+x)(1+y)}{4}$$

$$N_4=\frac{(1-x)(1+y)}{4}$$

So, these are the 4 interpolation functions, which define the variation u within the element.

$$u(x,y)=N_1u_1+N_2u_2+N_3u_3+N_4u_4$$

Similarly the displacement in y direction v is given by N 1 v 1 plus N 2 v 2 plus N 3 v 3 plus N 4 v 4.

$$v(x,y)=N_1v_1+N_2v_2+N_3v_3+N_4v_4$$

So, essentially we can write this as Nu_i , where N is a matrix of shape functions.

So, N_1 is a function of x and y within the element domain Ω_e for node 1. Similarly, node 2, node 3, node 4, and are the nodal values of the primary variables u_1 and v_1 displacement along x and y directions. And, together they give us the desired approximation for variation of primary variable within the element; u as a function of x and y.

And, once we have this variation, we can go back to the weak form and work out the domain integral evaluation. And, from that point onward it is essentially exactly the same process. It is an integral in 2 dimension and the expression is available as a variable in 2 dimensions.

Integrals can be evaluated easily. And, the domain integral would go in the counter clockwise direction. For domain integral this is the unit normal for each side.

And, again as discussed in the case of triangular elements, these domain boundary integrals, in the boundary term of the weak form needs to be evaluated only when the element boundary coincides with the boundary of the problem. Because, in this interior if any of these element boundaries in the interior of the problem domain, then the outward positive normal the direction of the fluxes secondary variable, they would cancel out by virtue of equilibrium.

And, these boundary terms need not be evaluated. And, when we are evaluating these at the boundary term this 2 dimensional can be converted into 1 dimensional integral by similar kind of mapping process, x y coordinate system can be converted into a local st coordinate system. Such that, one of the coordinate directions aligns with the edge of the boundary, that is being considered and origin being located at one of the nodes.

So, that the other coordinate direction takes the value of 0. And, this problem the problem becomes simple 1 dimensional integral and those can be evaluated analytically. So, that is the basic definition for triangular elements.

So, the 4 noded triangular element is the simplest 4 noded rectangular element, is the simplest rectangular element that we have. And as I said there are two families of rectangular elements, I did not take the names of those families. Because first element 4 noded rectangular element is common to both the families, but the two families of rectangular elements are Lagrange family and second one is known as Serendipity family.

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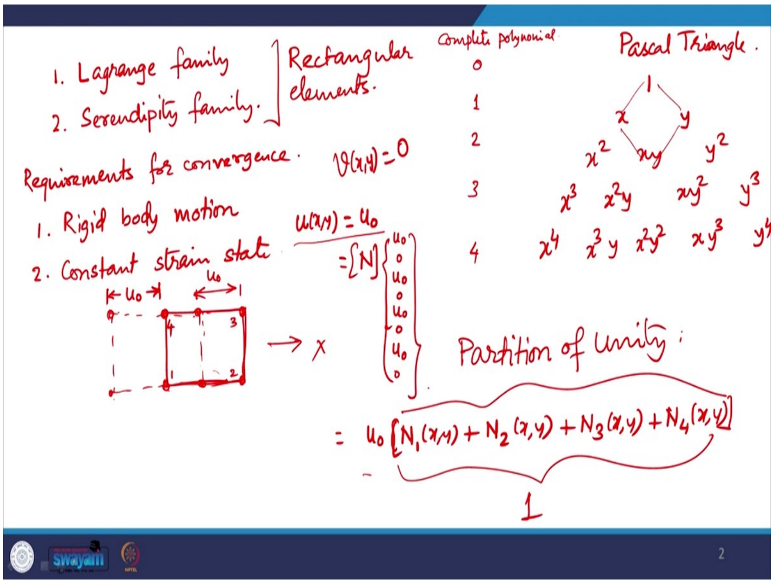

Now, the difference between the; between them geometry is of course, rectangle right. So, whether it is from Lagrange family or from serendipity family, we are looking at rectangular elements. So, both of them are rectangular elements. So, what is the difference?

Difference is in the construction of interpolation functions. In Lagrange family, we consider the placement of nodes at the intersection of the grid lines. So, that the interpolation functions are very easily derived by using product of Lagrangian interpolation functions in respective directions

So, I take the Lagrangian interpolation along x, I take multiply it with the Lagrangian interpolation of desired degree along y direction. And, then normalize it with respect to whatever node we are looking at, and we have the desired interpolation function very straight forward. And, that way, we can generate any degree of polynomial any higher degree elements does not really pose much of a challenge.

It is only finding out, we have standard way of defining Lagrange interpolation formula and we can just knowing the position of the nodes, in the knowing the coordinates of the nodes. We can simply find out what is the equation, what are the Lagrange interpolation in x direction, and multiply it with the Lagrange interpolation in y direction, and that is the interpolation function for that node.

And, of course, normalized that will be normalized to have unit value at that node. Serendipity family is little different, in the sense that it comes the name serendipity comes from a place called serendip, where the prince of serendip was known for accidental discoveries.

So, he never went out to discover or intentionally try to find something, but he had that knack it is said that he had that knack that he would stumble upon something useful. So, serendipity family regular rectangular elements were developed somewhat like that the interpolation function, they were often a matter of trial and error. You would keep on trying and trying and trying and suddenly you hit a golden pot and you find the set of functions which satisfy the desired requirement.

So, that is how serendipity family originally was derived and the name got stuck, but now we of course, have a very systematic way of developing interpolation functions even for serendipity family. So, let us take I mean the first element 4 noded rectangle is common to both of them. So, we now go to the next degree of accuracy that is second degree quadratic variation. So, the polynomial variation within the element has a quadratic variation.

So, one thing that we need to look at, as we discussed about this triangular elements there was always a 1 to 1 mapping. So, in Pascal triangle right, so, in Pascal triangle for 4 nodes we need these terms, for triangle we only need 1, 1 x and y, so, linear term complete.

So, complete polynomial. So, this is 0 here it is 1, this is 2, this is third degree, this is fourth degree right. So, for triangular elements 3 noded triangular elements, we only needed 3 terms. So, it was a polynomial approximation was complete in first degree there was no extra term.

But, while using the rectangular elements, although a rectangular element we could define a larger geometry, which would otherwise have taken 2 elements, 2 triangular elements to cover the same area. So, there is some kind of economy in the number of elements to be processed, because the; it takes a substantial amount of computational time to derive the element level equilibrium equations, to compute the element level equilibrium equations.

So, if I can do the work of two elements by using 1 element, that is a considerable saving right. So, that way rectangular elements are more efficient, but it comes at a cost of an additional term from the second degree polynomial x y. So, the approximations are all based on 4 terms; 4 noded rectangle they will have 4 terms in approximation.

And, those 4 terms will involve one constant term, one linear term in x 1 linear term in y and then a product term x y that comes from the second degree. But, the convergence rate would still be governed by the first degree that is the complete polynomial degree of complete polynomial in the approximation.

So, this is a little bit of problem, but not a very serious one and before going into the details of how do we develop the general principles of developing the interpolation models for Lagrange's family and serendipity family. Let us we also look at what are the requirements, I mean the how do they satisfy the requirements of the approximation, how does it satisfy the requirements for the convergence of finite element solution.

For the convergence of finite element solution we mentioned that requirements for convergence. So, that is rigid body modes and second one is constant strain state. So, constant strain state we have been talking about if we have this constant terms, I mean linear terms included in the approximation, they will have that value of derivative constant, derivative condition.

So, if the problem is such that the first derivative is constant, then the if as long as we retain these first degree polynomial terms in the approximation, that capacity is built into the solution, built into the approximate solution and that constant derivative condition will be represented right. But, how do we represent the rigid body motion?

So, for rigid body motion, what is required I mean if we look at it this particular thing, let us say element is required to undergo rigid body motion and it is now translated. So, this is let us say u 0 right. So; that means, the entire element has translated by constant amount u 0 and without straining.

So, every node so, u 1 is moving by u 0 along x direction and y direction the displacement is 0 for simplicity. So, u 1 the displacement node 1, node 2, node 3, node 4. So, node 1 moved by u 1, node 4 also moved by u 1, u 0 and node 2 moved by u 0 and node 3 also moved by u 0. So, entire block moves as a rigid body there is no relative deformation. So, there is no straining in the element.

So, if that happens then this motion should be captured. So, when I say u_0 , the nodal displacement of each of these 4 nodes is same u_0 . So, $u(x,y)$ is equal to u_0 and this should be equal to N multiplied by $\{u_0, 0, u_0, 0, u_0, 0, u_0, 0\}^T$, that is the vertical component the displacement along y direction is 0. So, v at x y is 0 so, that is what we happen we have this v_1 is equal to 0, v_2 is equal to 0, v_3 is equal to 0, v_4 is equal to 0, but all the 4 nodes move in x direction by the same amount u_0 .

So, essentially this looks like $u_0(N_1(x,y)+N_2(x,y)+N_3(x,y)+N_4(x,y))$. So on the left hand side we have u_0 , on the right hand side we have u_0 multiplied by this sum of interpolation functions of all nodes. Now, this rigid body mode can be captured only if this is equal to 1.

And, this is what we known what we call as partition of unity. So, this is very important test or very important requirement for the interpolations to satisfy. We will explore this more and the rules of Lagrange's family of rectangular elements and serendipity family of rectangular elements in our next lecture.

Thank you.