Finite Element Method and Computational Structural Dynamics
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Lecture - 26
Finite Elements of C^0 Continuity in 2-D and 3-D-VIII

Hello friends. So, we saw in our previous lecture how the domain integrals that arise in the approximate solution of equations of motion in 2 dimensional continuum arise and how they are solved and how those integrals involving various powers of x in triangular geometry are evaluated. Now, after taking care of the domain integral there is still one boundary integral term that is left to be evaluated.

How do we evaluate the boundary integral in the triangular geometry? The triangular geometry boundary will of course be straight line.

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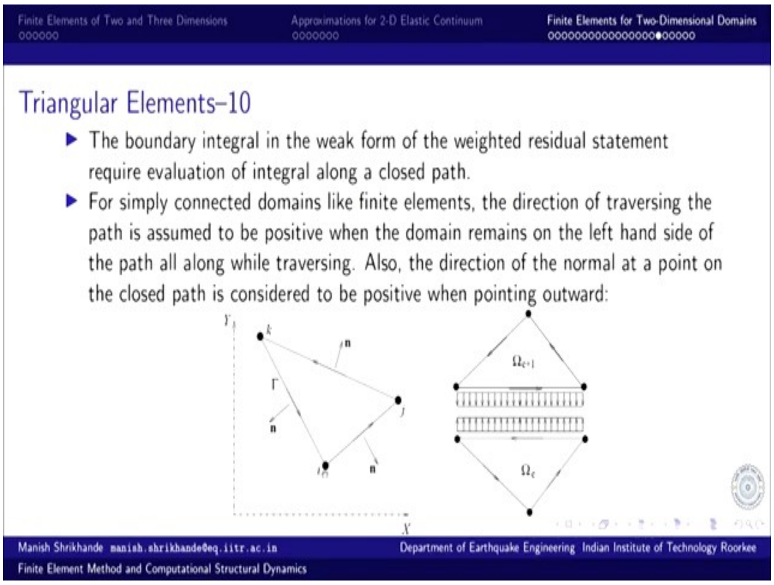

And it is to be evaluated along the closed path. So, this is the boundary integral that is to be evaluated in the closed path. So, I go from i and j. So, that is and from j to k and from k to i and that is the normal outward normal positive direction and the outward flux is of course, going positive outward normal.

So, there are two elements element e and element e plus 1. Boundary integral in the weak form of the weighted residual statement require integral along the closed path and for simply connected domains like finite elements which do not have any gaps or any hole within the interior of the element. So, it only refers to going along the edges of the element in the counter clockwise order.

The positive direction is considered to be the outward normal on the edge. That is the important thing to note. So, how do we evaluate these coefficients these boundary integrals? The boundary integrals that you look at like this.

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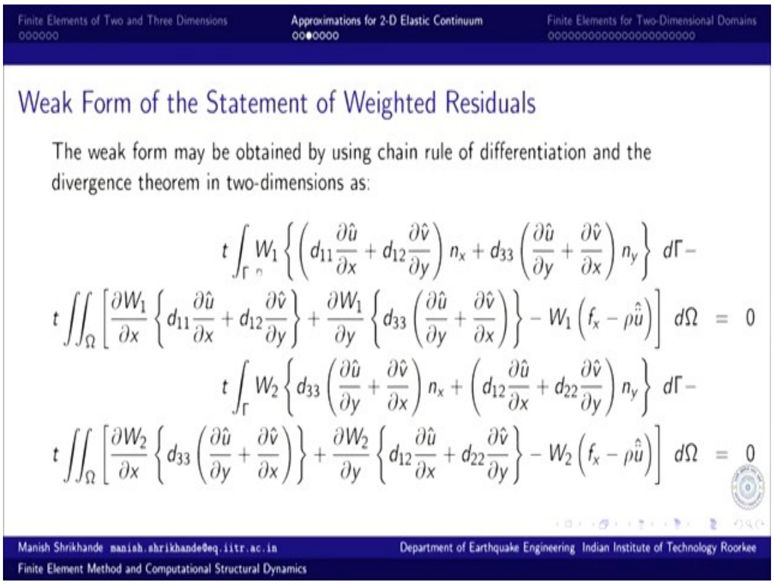

This is the boundary integral. So, Γ is the integral of the element that we have. So, how do we evaluate these boundary integrals?

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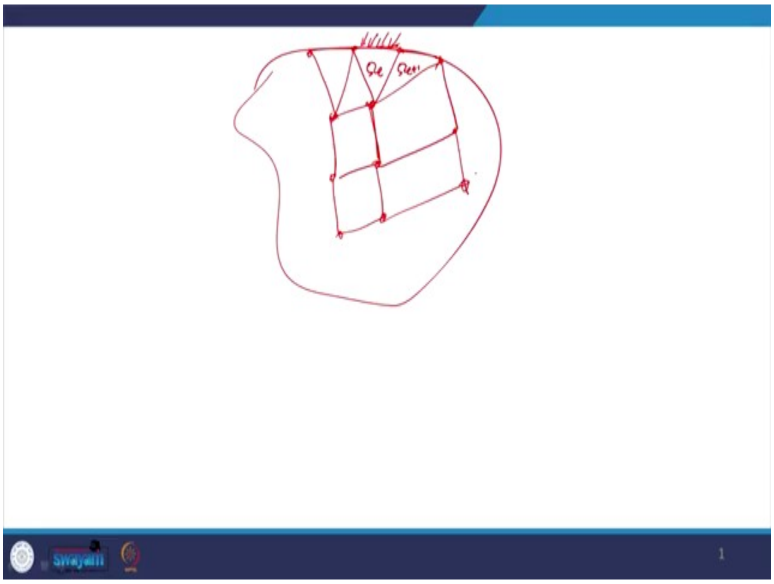

First thing is we need to appreciate that if I have some domain which is modeled using finite elements and there are some tractions on the edge of the element. So, the entire domain is discretized using 3 noded triangles as shown in figure above.

So, this is Ω_e and let us say this is Ω_{e+1}. So, for Ω_e the boundary integral would involve this edge, this edge and this edge and Ω_{e+1} boundary integral would involve this edge, this edge and this edge. So, let us look at the common boundary between Ω_e and Ω_{e+1}. Note that the boundary integral positive direction are all outward positive.

Now when a boundary is shared between two adjacent elements then this outward normal fluxes would cancel each other out. So, this is same thing as what we had in the case of first one dimensional element the common nodes between two elements when they add together. So, secondary variables tend to cancel out each other because of their opposite signs and this is the same thing.

So, secondary variables on two different adjacent elements on the common boundary turn out to be of opposite signs and they would cancel out and therefore, we do not really need to evaluate these boundary integrals when the boundary is in the interior of the domain. For example, this boundary is in the interior of the problem domain. So, we need to evaluate this boundary term only when the element boundary coincides with the problem boundary, any boundary of the element in the interior of the domain, we do not really need to evaluate this boundary term, because those contributions would be cancelled by corresponding contributions from the adjacent elements. So, we do not really need to evaluate these interior boundary terms.

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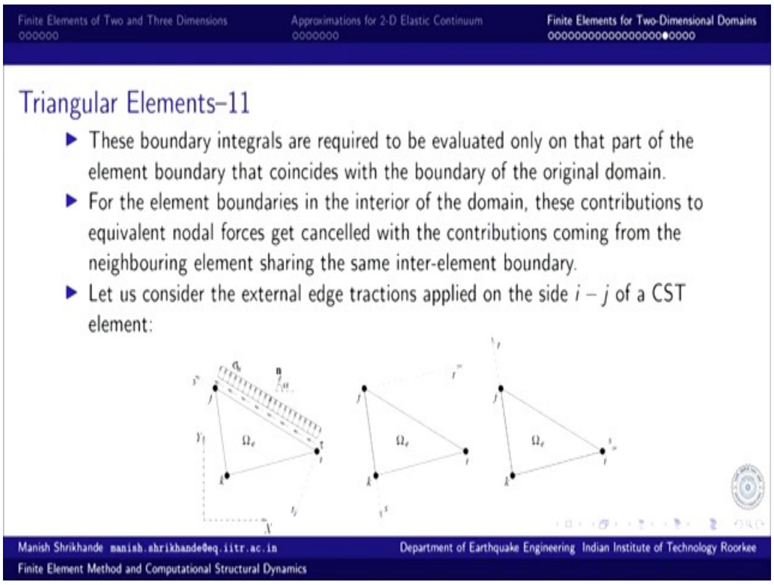

And only the boundary term has to be evaluated when it is coinciding with the domain boundary. So, the boundary integrals are required to be evaluated only and on that part of the element boundary that coincides with the boundary of the original domain. So, element boundary is the same as the part of the original domain boundary. Of course, one element cannot cover the entire domain boundary.

So, one element boundary can have only can part of the original domain boundary. So, for the element boundaries in the interior of the domain these contributions to equivalent nodal forces get cancelled with the contributions coming in from the neighbouring elements. So, the secondary variables are of the opposite signs and they would cancel out along the inter element boundaries.

So, effectively it is like saying some value minus some value is an approximately equal to 0. As 0 cannot be exactly computed on digital computers, why approximate 0 by something nearly 0 when we know that in practice it should be 0. So, now, we focus only on the edge tractions only on the boundary which coincides with the original problem boundary.

So, in 3 noded triangle let us say i-jth edge i-j edge coincides with the boundary of the problem. So, this is where we need to calculate the secondary variables and they will be evaluated for there are 2 nodes i and j. So, they will be evaluated as the forces that are going to equivalent nodal loads on the element on the node i and j. So, how are they going to be there?

So, there can be two possibilities - any kind of tractions can be resolved into two orthogonal direction normal and tangential. So, σ_n is the normal stress on this edge and τ is the shear along this edge i-j.

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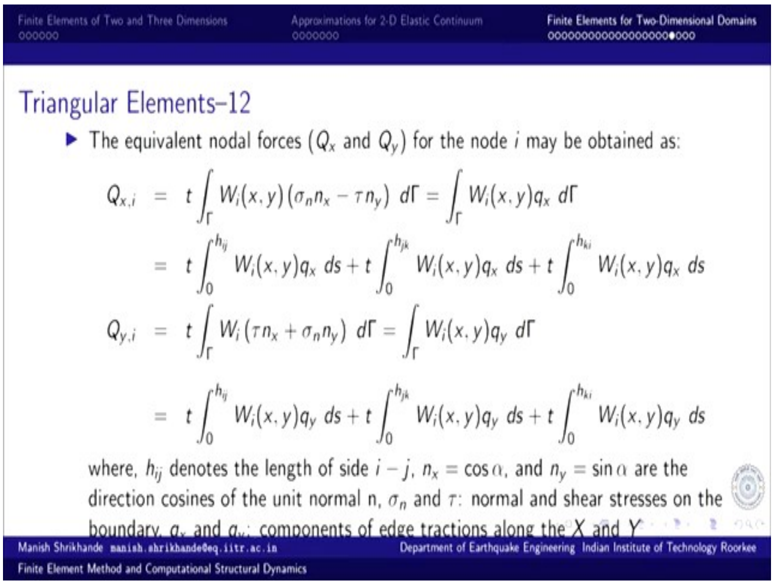

So, once we do that then multiplied by this is what naturally comes out during that evaluation. So, $\sigma_n n_x - \tau n_y$ multiplied by the weighting function that will be same as the shape function and that has to be evaluated along this boundary. So, this is of course, Qx if I resolve along X and Y direction. So, these are the tractions that we have.

So, the entire boundary Γ, we can break it down into three parts. So, along the i-j edge, along the j-k edge, along the k-i edge. So, I go through each of these one at a time. So, from i to j the length is h_{ij}. So, this boundary is defined by some local variable starting from 0 and going to h_{ij} at this point, similarly j-k it goes from 0 and comes to k at value of h_{jk}. So, instead of working with x y it is easier to define local frame of reference such that one particular axis coincides with the edge of the triangle that we are looking at.

So, we define s-t frame of reference and map it to global x-y frame of reference such that s variable i.e. one of the local coordinate variable varies from 0 to length of the edge and that is what we are doing here. So, the element boundary i-j is defined as s varying from 0 to h_{ij} element boundary j-k is defined by s variable going from 0 to h_{jk} And element boundary k-i is defined by s variable going from 0 to h_{ki} and similarly for tractions in y direction.

Once we have this then we can evaluate the integral very easily if we can somehow convert this also into function of s.

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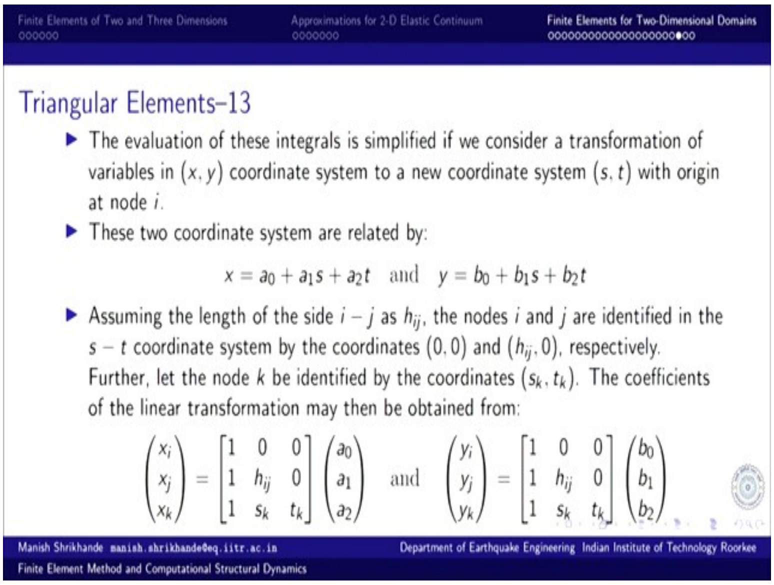

And that we will do by defining the linear mapping. So, evaluation of these integrals is simplified if we consider a transformation of variables in x, y coordinate system to a new coordinate system s, t with the origin at node i. So, for each edge we take it one at a time.

So, for edge i-j we consider the origin at node i. So, I define

$$x = a_0 + a_1 s + a_2 t \text{ and}$$

$$y = b_0 + b_1 s + b_2 t$$

This is the general linear transformation that we can have. So, we have not missed out any lower order terms and so, lower order term would actually capture the constant shift of origin and linear terms of course, define the linear mapping.

Once we substitute the values of x and y for nodes i and j and k, we get the complete mapping defined by a_0, a_1, a_2. So, I define x_i as [1 0 0]. So, s is 0 at node i and t is also 0. That is the origin. and then for x_j again t is 0 along that line i-j, but s value is equal to the length of the element and k we need to define. what is the value of node k along with respect to s t coordinate direction right with using s t coordinates local coordinates s t and once we have this mapping then a_0, a_1, a_2 can be directly computed by taking the inverse or even the forward substitution. It is a lower triangular matrix.

So, it can be solved easily even the inverse is not required similarly b_0, b_1, b_2 for coordinate y and direct solution for a and b can be computed by using forward substitution and by solving this triangular system of equations.

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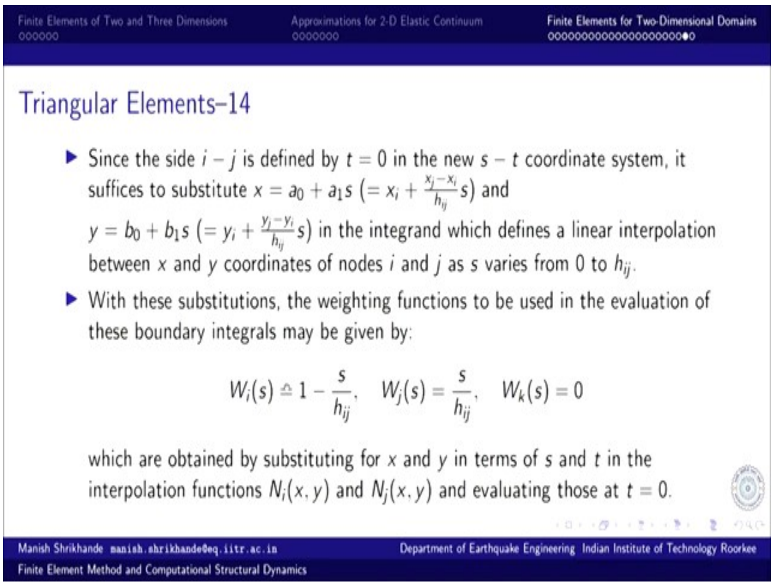

And since side i-j is defined by t is equal to 0 in the s-t coordinate system it is sufficient to substitute x is equal to linear function of s and y as a linear function of s; and t is obviously 0. So, no point in adding that term no additional benefit. So, it is a linear interpolation as you can see it is a linear interpolation between x y coordinates of i and j as s varies from 0 to h_{ij} and once we have this mapping for x and y in terms of s and t and t is of course, is to be substituted as 0.

So, x and y values can be substituted in the expressions for W or weighting function or the interpolation functions as a function of x and y. And we can obtain these expressions which can be evaluated along the edge as s ranges from 0 to h_{ij} along the i-j edge. So, these are the two interpolations.

So, N_i and N_j can be replaced by W_i as a function of s and W_j as a function of s and these can be used for evaluating. So, the boundary term now only needs to be evaluated with respect to one variable s and that can be integrated very easily and analytically or numerically using quadrature rules and the equations can be solved.

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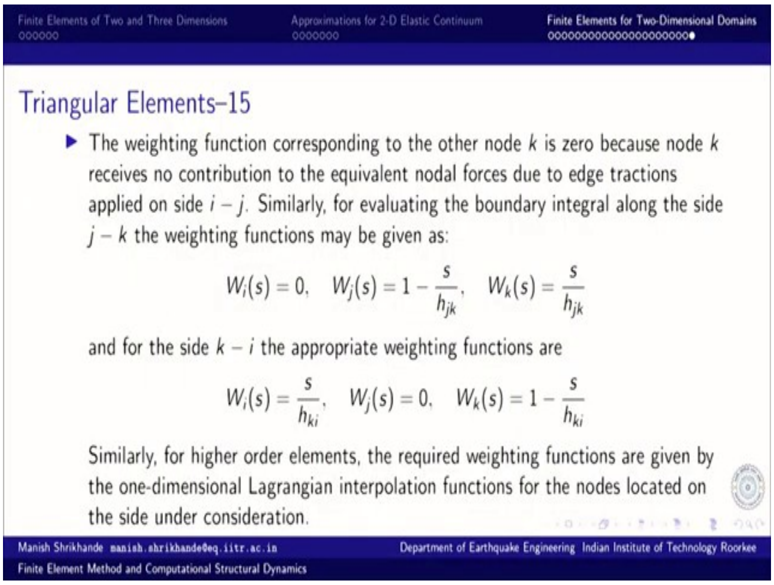

So, the weighting function corresponding to other node k is 0 because node k receives no contribution from the equivalent nodal forces due to edge traction because edge tractions are only coming on the node i, j. So, the other node does not really matter.

So, similarly for the next boundary j-k again similar exercises can be done and local coordinate axis can be defined and weighting function or interpolation functions x and y can be substituted with corresponding variations with respect to s and t and then evaluating it at t is equal to 0 we get one dimensional variation in local coordinate s which can be easily evaluated using a one dimensional integral.

And once we understand this process it is easy to develop just one dimensional Lagrangian interpolation and get on with it without going through the complete intermediate details of calculation. And with that we can compute the complete domain integrals and boundary integrals and we can work out our element level equilibrium equations for one element then we go to the next element and so on.

Then we can start assembling the element level equilibrium equations, impose natural boundary conditions at the element level equilibrium equations and then assemble in the global equations and then impose the essential boundary conditions on the primary variable and then solve the system of equations. That is the complete finite element solution.

So, this is all for triangular family. In our next lecture we will start with a rectangle domain. Obviously, for rectangle I need at least 4 nodes – i.e. one extra node and variable - but the advantage is that it is possible to cover larger domain with fewer elements. So, that is more economical from computation point of view. But there is a penalty to be paid for that and that we will see in the form of additional terms coming from the higher order polynomial approximation which do not contribute to the convergence or rate of convergence of the approximate solution.

But that is a necessary fallout of having larger number of nodes than what is required by the complete polynomial of certain degree. So, we will continue with our discussion on rectangular family in our next lecture and after that we will try to explore the use of these elements in solving a real sample example problem that will kind of put all pieces of the jigsaw puzzle in one whole picture.

Thank you.