Finite Element Method and Computational Structural Dynamics
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Lecture - 20
Finite Elements of C^0 Continuity in 2-D and 3-D-II

Ok friends, so we saw the governing differential equation for three-dimensional homogeneous and isotropic elastic continuum. Initial governing differential equation is in terms of the force equilibrium, so obviously, it is in terms of the stresses and body forces and the rate of change of momentum or the so called inertia force.

And once we take into account the generalized Hooke's law to relate stresses to strains, and then the definition of the strain components, relating different strain components to the deformation we can transform the governing differential equation in terms of the deformations.

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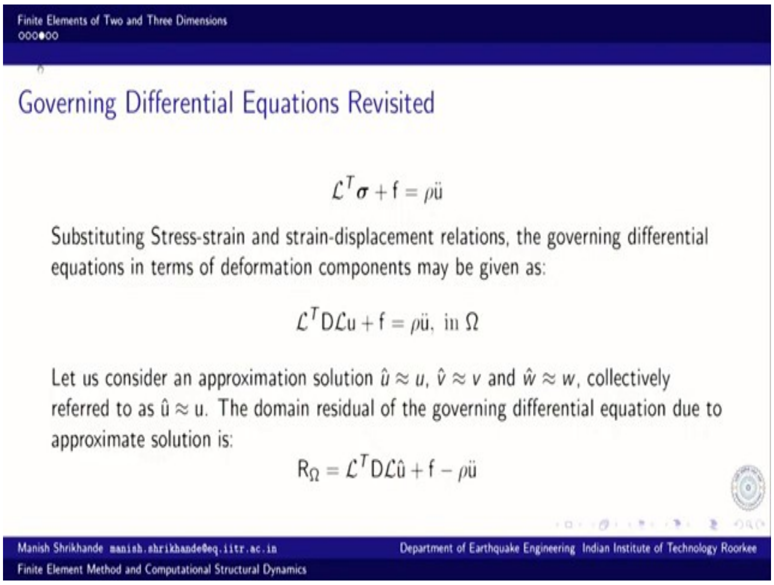

And that governing differential equation in terms of deformation can be given in terms of a differential operator that relates strains to displacement L. So, $L^T DLu$ is the vector of deformations plus f is the vector of distributed body forces along each component oriented along respective coordinate directions, and this is equal to the rate of change of momentum. So, that is essentially the statement of Newton’s second law, all the forces acting on the body, are equal to rate of change of momentum of the body.

Our objective is finding an approximate solution. So, approximation $\hat{u}$ as an approximation for u, $\hat{v}$ as an approximation for v, and $\hat{w}$ as an approximation for w. So, collectively we can refer to this as the vector $\hat{u}$ as an approximation to vector $\hat{v}$.

And when we substitute this approximation in the governing differential equation that is what leads to the domain residual. So, the domain residual of the governing differential equation due to approximate solution is given by

$$R_{\Omega} = L^T DLu + f - \rho \ddot{u}$$

L is the familiar differential operator which relates strain to displacement.

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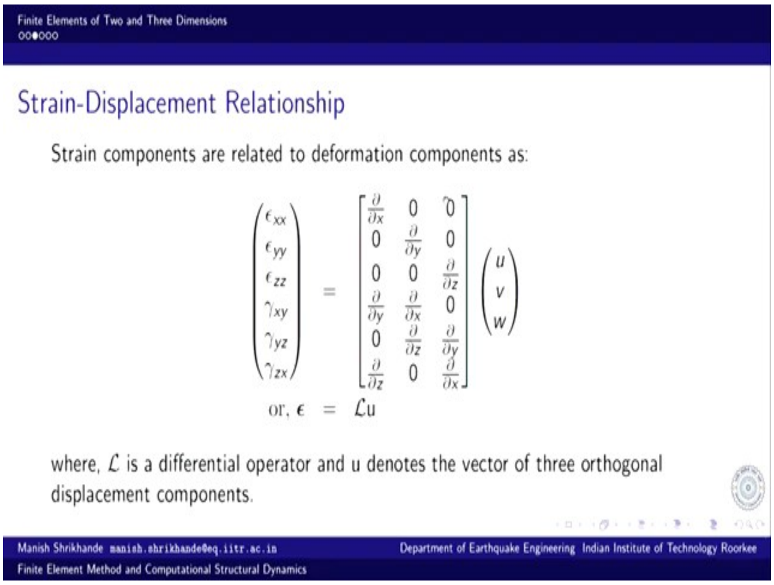

So, $L^T D$, D is the constitutive relation matrix; and L strain displacement operator; and $\hat{u}$ is the displacement plus f the body forces minus $\rho \ddot{u}$. So, that is the second derivative of the displacement with respect to time i.e. accelerations. We are of course assuming that mass of the body does not change with time within the domain. So, this is the domain residual.

Now, you may find the equivalence with the way we develop finite element formulation for one-dimensional problem. The whole idea was to find out what is the error in the approximation. If we are looking for an approximate solution there will be some error in the solution, and that error or residual within the domain is to be minimized for getting a good quality of solution. And that is what we are trying to do here; we first estimate the residual or the error of approximation.

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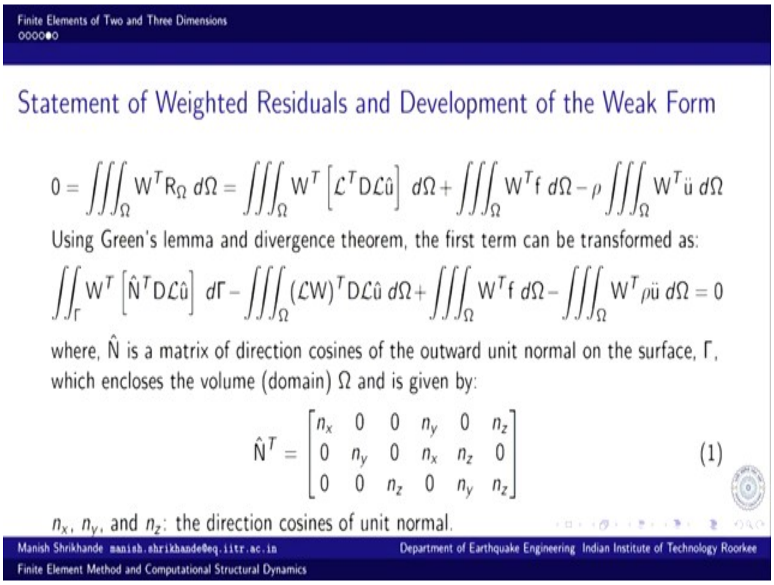

And then we develop the statement of weighted residuals and develop the weak form of the statement of weighted residuals. So, this is the weighted residual statement. W is a weighting function. Please note that R_Ω here that we are looking at is a vector with three elements; there are three relations in each coordinate direction. So, this is in a sense a set of simultaneous differential equations.

So, there are three differential equations, and here the residual that we are looking at are three residuals – $R_{\Omega x}, R_{\Omega y}, R_{\Omega z}$, i.e. residual along x direction, residual along y direction, and residual along z direction. So, accordingly there will be three weighting functions. So, W is a diagonal matrix with W_x, W_y, W_z along the leading diagonal of this 3 by 3 matrix.

Transpose is something that I put intentionally although for a diagonal matrix transpose really does not matter, W is same as its transpose. So, $W^T R_\Omega d\Omega$. So, I substitute for

R_Ω the residual. So, $L^T DL\{\hat{u}$¿ that is the first term of the domain residual, and then the two terms $W^T f$ and $\rho W^T \ddot{u}$, and they are all under domain integral.

Domain Ω here is a three-dimensional domain enclosed by some boundaries in x,y,z space. So, it is a triple integral. And this is the complete weighted residual statement in the strong form.

Now, you may recall that the first step towards the development of a finite element model is to develop the weak form of the weighed residual statement. The whole point of developing the weak form is that we need to look at the boundary term of the weak form in order to identify what are the primary variables of the problem, and what are the secondary variables of the problem.

So, look at the first term of this governing differential equation. And let me just take one row of the equation, this operator form is sometimes not so obvious to understand. So, let me just look at this first term of this equation.

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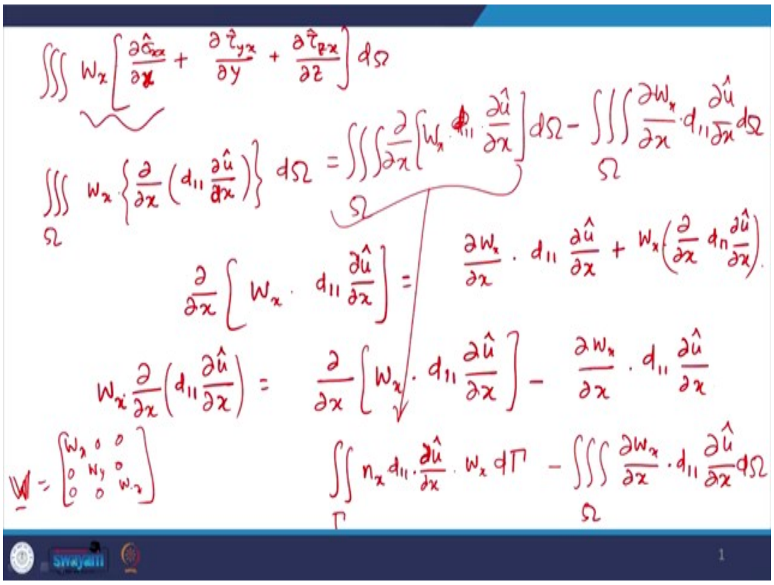

$$\iiint_\Omega W_x\left[\frac{\partial \hat{\sigma}_{xx}}{\partial x}+\frac{\partial \hat{\tau}_{yx}}{\partial y}+\frac{\partial \hat{\tau}_{zx}}{\partial z}\right]d\Omega$$

These are all approximate stresses because they are derived from the approximate displacements. So, we look at only this particular term just to illustrate how we develop the weak form. The process is similar for each term.

So, $W_x \frac{\partial}{\partial x}{}_{xx}\sigma$ is of course, proportional to d_{11} so that is the element of constitutive relation matrix multiplied by ϵ_{xx}. So, σ_{xx} is equal to d_{11} times ϵ_{xx} ; ϵ_{xx} is equal to $\frac{\partial \hat{u}}{\partial x}$, so that is the first term. So, domain integral of the first term of the residual of the first row.

Now, by using the chain rule of differentiation, I will have

$$\frac{\partial W_x}{\partial x} d_{11} \frac{\partial \hat{u}}{\partial x} + W_x \left(\frac{\partial}{\partial x} d_{11} \frac{\partial \hat{u}}{\partial x} \right)$$

So, I can replace first term by the above term. So, essentially what this means is

$$\frac{\partial}{\partial x}\left[W_x d_{11} \frac{\partial \hat{u}}{\partial x} \right] - \frac{\partial W_x}{\partial x} d_{11} \frac{\partial \hat{u}}{\partial x}$$

This can be converted into a boundary integral by using Gauss theorem or divergence theorem in vector calculus. So it is a three-dimensional domain enclosed by the two-dimensional boundary and the derivative is with respect to x. So, appropriately it will be

$$\iint_{\Gamma} n_x d_{11} \frac{\partial \hat{u}}{\partial x} W_x d\Gamma$$

This is what the boundary term looks like.

And then second term is of course

$$\iiint_{\Omega} \frac{\partial W_x}{\partial x} d_{11} \frac{\partial \hat{u}}{\partial x} d\Omega$$

This is what you can refer to as three-dimensional analog of integration by parts that we did in the one-dimensional problems by which we traded the derivatives from the residual term to the weighting function term and that is what we have done here.

So, the derivative from the residual term is transferred to the weighting function term. And then there is a boundary term that comes by invoking the Gauss theorem or Green's lemma. So the domain integral is transformed into this particular integrand multiplied by the respective direction cosine. So, for derivative with respect to x, it will be the direction cosine of the outward unit normal with respect to x

This we can do term by term. For the second term it is derivative with respect to y. So, it will be n_y in the similar term here. In the third term, it is derivative with respect to z direction, so it will be n_z in the third term.

We can continue with this term by term and then the weak form can be developed as the weighting function multiplied by the matrix of direction cosines. So, this is the matrix of direction cosines n_x, n_y, n_z, and that multiplies with these stresses – stress vector.

So, the first term is boundary integral and minus this second term is the domain integral, which contains LW, so that is the differential operator L from the stress terms is transferred to the weighting function term.

And then the third and fourth terms are of course the domain integral with respect to forces and inertia term. And $\hat{N}$ is the direction cosine matrix transpose, so that comes from the respective differential operator. And n_x, n_y, n_z, these are the direction cosines of the unit normal.

So, once we are done with that, we look at the boundary term of this weak form. We can see that this is the weighting function W, which is a diagonal matrix of three weighting functions – W_x, W_y, W_z along the diagonals.

So, it appears in its natural form on the boundary term without any derivative. So, that implies the primary variables are the unknowns of the problem. So unknowns of this equation are u - the displacement.

So, the primary variable of this problem are the displacements. And the multiplier here, these are the surface tractions. So, these are the stresses along the boundaries. These are the corresponding secondary variables.

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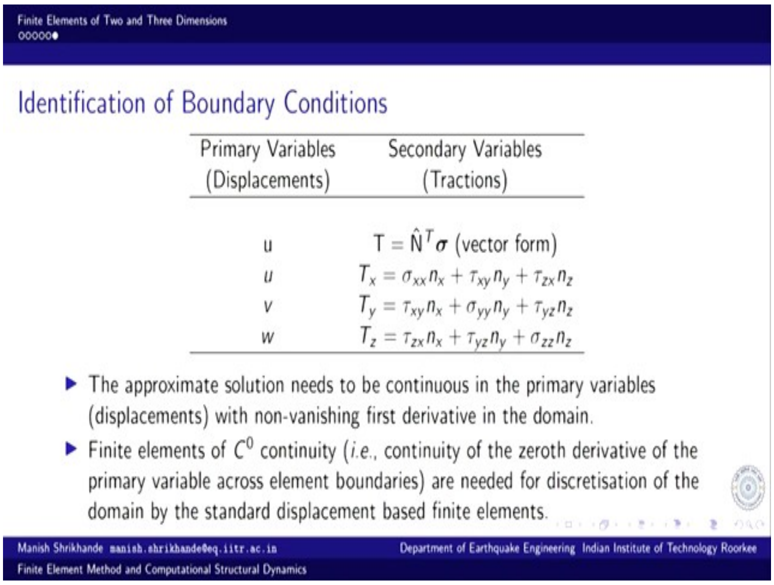

So, u is the primary variable vector. So, collectively I can refer to this as a vector u, all three displacement components. And T is the traction and that is given by $\hat{N}^T \sigma$, in the vector form. And these are the secondary variables if I look at the component wise, so u that is the displacement along x direction, corresponding secondary variable is the traction or the forces along x direction.

So, that would be $\sigma_{xx} n_x + \tau_{xy} n_y + \tau_{zx} n_z$. So, these are the surface tractions along x direction.

And similarly for y and z, so we have v and T_y, w and T_z as corresponding pairs of primary variables and secondary variables. And as we discussed earlier primary variables and secondary variables will always come together in the boundary term. So, that it is always a work like quantity, it is always the work done by the secondary variable tractions in moving through the primary variables.

So, now looking at the boundary term, we can infer that the approximate solution that we need to construct needs to be continuous in the primary variables. These are the displacements with non-vanishing first derivative in the domain. Because the domain integral that we have here involve first derivative of this approximation Lu; L is a first order differential operator. So, the first derivatives of the displacement should exist. So, non-vanishing first order derivatives are required.

And the element itself has to be C^0 continuity i.e. continuity of the zeroth derivative of the primary variable across element boundaries are needed for discretization, exactly same as what the inference was from the boundary term for one-dimensional problem. For two-dimensional problem also it is the similar kind of inference that is drawn.

So, we substitute the unknowns of the problem in the weighting function term in the boundary term, and we get the primary variables as u, v, and w, in place of W_x, W_y and W_z.

So that is the basic primary variables interpolation that is required to be continuous across the inter element boundaries. And the elements that we develop should have suitable interpolation model for this displacement components while preserving this C^0 continuity.

How do we construct these elements? This is the problem that we need to look at because this odd domain Ω needs to now be split into several sub domains that is finite elements.

Now, looking at this weak form, we identify the finite element model. Now, we will look at what kind of finite element model would be suited. The domain is three-dimensional, so the element has to be a volume element; it has to enclose a volume.

Now, the simplest regular figure which can enclose a finite volume is a tetrahedron with four vertices. So, that is the simplest finite element in three dimensions, four noded tetrahedron. Two-dimensional analog is the domain is an area domain, plane domain bounded by a curve. So, the simplest element that we will have in case of two-dimensional problem should enclose an area – finite area. And the simplest element which can enclose finite area is a triangle, so three noded triangle.

So, three nodded triangle in two dimension, four noded tetrahedron in three dimension, are the simplest elements in two and three dimension, but of course, we can go further and develop other elements. For example, I can use four noded rectangle that also encloses area in a two-dimensional plane. And similarly analog of rectangle in three dimension is a cuboid. So eight noded cube can be used for three dimensional domain.

So, triangle and tetrahedron belong to one family of elements that we can develop, then we can have another family of elements rectangle and cube. And of course, there are other possibilities such as triangular prism or even wedge element, so that will also be enclosing volume.

Now, the treatment of the finite element formulation remains the same. It now boils down to defining these respective elements which will satisfies following requirements. They have to enclose area and they also have to ensure the continuity of the approximation across the element boundaries.

So, we will see how that is achieved. Once we take care of that, then all that remains is the basic interpolation model. How are the unknowns interpolated within the element in terms of primary variables defined at the nodes of interpolation?

For a substantial portion of our engagement will be devoted to just this discussion of development of interpolation function for different element geometries. And subsequent analysis of the finite element model would be straightforward.

So, in our next lecture, we will simplify these three-dimensional equations to two-dimensional cases because fewer terms to work with and the diagrams are easier to draw and easier to understand. So, the basic philosophy of finite element approximation is more easily understood when we discuss in two dimension, three dimensions bring in lot more complications of visualization.

So, in our next lecture, we will take the similar equations for two dimensions, and then develop finite elements for two dimensions first, and see how things work out in different form. And from there we will return to this discussion of three-dimensional finite elements, and see how the discussion of two-dimensional finite elements can be extended to three-dimensional finite elements in a straight forward seamless manner.

Thank you.