Finite Element Method and Computational Structural Dynamics

Prof. Manish Shrikhande
Department of Earthquake Engineering
Indian Institute of Technology, Roorkee

Lecture - 19
Finite Elements of C^0 Continuity in 2-D and 3-D-I

Hello. So, we completed our discussion on the Finite Elements in one dimension and we saw two application areas - one dealing with second order differential equation in one dimensional domain and another one dealing with the beam bending problem which is defined by fourth order differential equations in linear domain.

These two examples allowed us to explore two different classes of finite element approximations i.e. finite elements with different degree of continuity requirement or smoothness across the inter element boundaries C^0 continuity and C^1 continuity elements.

Now, we extend our discussion to more general problems of interest that is 2 and 3 dimensional elasticity problems and as all problems are essentially 3 dimensional and continuum modeling allows us to model all the problems using governing differential equations of motion in 3 dimensional continuum.

But with some simplifying assumptions the dimension can be brought down and a reasonably good approximation can be obtained by using a 2 dimensional idealization by some simplifying assumption in the third direction about the behavior either because of the loading condition or the deformation condition and so on.

That is the only difference otherwise conceptual point of view there is really no difference between finite elements of 2 dimensions or 3 dimensions except that one dimension is extra in case of 3 dimensional elements.

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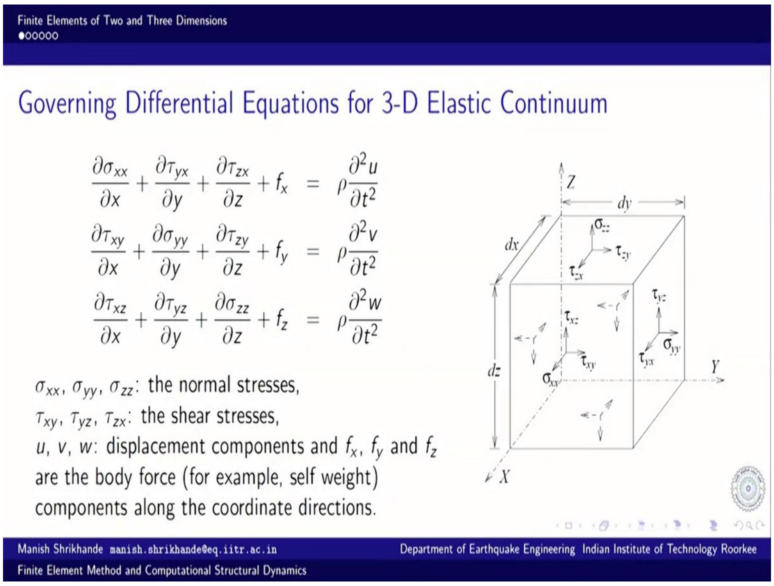

So, the beginning of the discussion of course, starts with the governing differential equation and this is the familiar stress block that all of us are familiar with. A infinitesimally small cuboid of dimension $dxdydz$ and the stress components, which are marked on each of the 6 faces and on the positive phases we have also mark the names of the stress components are shown in the figure.

So, σ_{xx}, σ_{yy} and σ_{zz} are the normal stresses along respective directions and τ_{xy}, τ_{yz} and τ_{zx} these are the shear stresses. And of course there are complementary shears and if we establish moment equilibrium of the forces acting on these infinitesimal block we can come to this standard result that complementary shears are equal that is $\tau_{zx} = \tau_{xz}$ and $\tau_{yx} = \tau_{xy}$. Similar for other shear components. So, governing differential equation is essentially a force equilibrium on a body of infinitesimal volume enclosed by dx, dy, dz and the governing differential equations are again defined by force equilibrium.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = \rho \frac{\partial^2 u}{\partial t^2}$$

f_x is the body force component along x direction. And this is equal to the rate of change of momentum along x . So, u is the displacement along x direction, v is the displacement of the body along y direction and w is the displacement along z

direction and $\frac{\partial^2 u}{\partial t^2}$ is the acceleration or rate of change of momentum . ρ is the density and for unit volume we have, $dxdydz$. Hence $\rho dxdydz$ will be the mass of this infinitesimal volume and multiplied by the acceleration. So, that is the rate of change of momentum assuming that mass of the system mass of the body is invariant.

So, that is the basic governing differential equation and the classical methods of solution in theory of elasticity of course, involve use of strain displacement equation. And then there are stress strain relationships and then of course, there are compatibility equations because these are not enough constraints to ensure unique valued deformation solution.

And solution of those is of course, very tedious and simpler approach is often used by making use of Airy stress function and trying to find solution of this governing basic elasticity problem. Now that is of course, very much involved and feasible only for very simple cases with regular geometries and simple loading cases.

So, we will explore how we can use finite element method and how simple the entire solution process becomes when we use finite element method when we apply finite element method to this problem.

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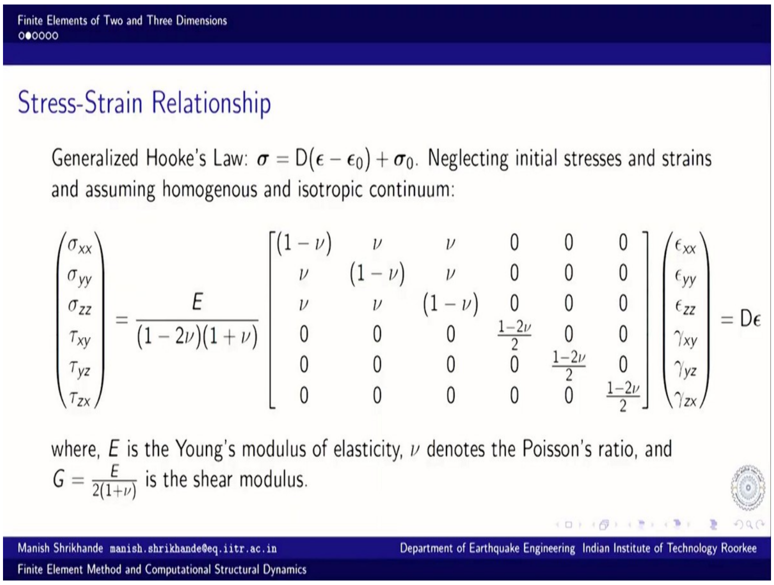

To take our discussion towards finite element formulation let us have a look at how these stresses are related to the stresses mentioned in this governing differential equation. They are related to the deformations that are produced in the body.

So, how these stresses are related to the deformations? By using the generalized Hooke's law, that is stress is proportional to strain and defined by σ that is the stress tensor proportional to the strain tensor, ϵ and D is the constitutive matrix.

ϵ_0 is the initial strain, one common example is the strain due to temperature difference. Those strains are not originating because of the forces and the loads imposed on the body. So, this initial strain has to be removed for considering the stresses because of the loads. σ_0 is of course, the initial stress. Similar to initial strain, the body may be prestressed. So, those have to be added in addition to these stresses because of the deformations. This is of course, a very general form of stress strain relationship and σ_0 and ϵ_0 are known quantities. So before the problem is handled, they can be straight away incorporated in our formulation without a much ado.

So, we will drop these two initial values from the discussion and assume that we are not dealing with initial strains and initial stresses just to keep our discussion simple. But you will see that its not really much of an issue incorporating them in the formulation except adding two more terms in our process. Further, we also assume for simplicity that the material is homogeneous and isotropic.

So, the continuum that we are dealing with is homogeneous; that means the same elastic properties apply throughout the domain. So, whether it is $\left(x_1,y_1,z_1\right)$ point or $\left(x_2,y_2,z_2\right)$ point or $\left(x_3,y_3,z_3\right)$ point, any point within the domain same elastic properties hold.

Now, the elastic properties can be different in different directions for some cases. For example, the timber or some composites. They have different elastic properties in different directions or more importantly in different orthogonal directions. So, those are called orthotropic materials where the elastic properties, which may be same at every point, but in orthogonal directions the properties may be different. In general case for different orientation we might have different properties.

So, those are in general referred to as unisotropic material. But to keep our discussion simple and at the macro level when we look at the broad picture the material properties do not really change with changing direction unless the material is spatially engineered and the fibers are oriented in a preferential way. But in general we see random crystallization and randomly oriented grains. So, the properties at macroscopic level are more or less identical in whichever direction we look at. That we call as isotropic condition. So, this is our of basic framework. We are looking at homogeneous and isotropic elastic continuum and we are neglecting initial stresses and initial strain.

And since second order stress tensor will have nine elements - 3 diagonal, they will be the direct stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ and then there will be 6 off diagonal terms called shear stresses. Now we just discussed little while back that these off diagonal terms are equal i.e. the stress tensor is symmetric. So, τ_{xy} is equal to τ_{yx} .

So, instead of considering 9 elements of stress tensor we can only consider 6 unique components and we can arrange it in a simple vector. And similarly we can also have similar kind of arrangement for strains. So, 3 direct strains and 3 shear strains corresponding to the shear stresses.

So, these 6 stresses are related to 6 independent strains through a matrix comprising of elastic coefficients and this matrix is known as constitutive relation matrix denoted by D .

So, in a sense what we are looking at is $\sigma = D\epsilon$. So, stress is proportional to strain and D is the constant matrix of proportionality as given in figure. E here is the Young's modulus v denotes the Poisson's ratio and G is of course, the shear modulus which can be related to Young's modulus and Poisson's ratio.

So, that defines the stress strain relationship and if we want we can substitute for stresses in this governing differential equation and we will have the governing differential equation in terms of strains and elastic constants. Still not very useful form as of now. How are the strains related to deformations? Because primary motion is of course, the displacement along x , displacement along y , displacement along z , 3 orthogonal directions.

So, the direct strains and shear strains are related to the deformations along Cartesian coordinates through a differential operator.

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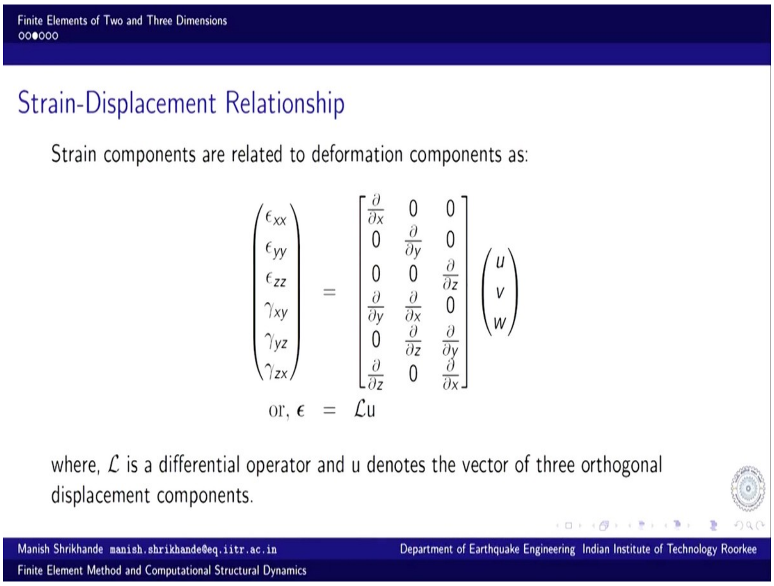

So, you can see here $\epsilon_{xx} = \frac{\partial u}{\partial x}$, that is the standard definition of direct strain along x , ϵ_{xx} is equal to rate of change of deformation with respect to x and ϵ_{yy} direct strain along y direction.

So, ϵ_{yy} is rate of change of v - the deformation along y direction with respect to coordinate direction y and similarly for ϵ_{zz} - direct strain along z direction is rate of change of w with respect to z .

So, these are the 3 direct strains related to deformation components. Now we come to the shear strains. Shear deformation would be distortion from 2 sides, $\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$. So, that is the distortion in xy plane. Similarly $\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$. So, that is the distortion in the yz plane and then the distortion along in the zx plane $\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$. Now this can be encapsulated in the operator notation that we had seen earlier in our discussion in one dimensional problems.

So, the ϵ is given as a differential operator times u . So, u is the vector $[uvw]^T$. So, these three components we are representing as a single vector notation u . And this L is the differential operator which operates on this displacement components to provide us the respective strain components.

So, this is the strain displacement relationship. We started with governing differential equation of motion in terms of stresses. Now after we defined the stress strain relationship, we can formulate the governing differential equation or recast the differential equation in terms of strains. We just have to substitute for stresses with strain components using stress strain relationship.

Now, strains are in turn related to the deformation and the displacement relationship for strain can be substituted into that differential equation. Now if you look at it very closely the governing differential equation (in stress components form) can also be written in the form of an operator equation. So, you can see the operators here. First derivative with respect to x , with respect to y , with respect to z and they operate on different components and so its a similar operator.

So, if you look at the arrangement of these differentials you will find that this operator is going to be exactly similar. If I write it in terms of some operator times the vector of stresses, then the operator that I will need is exactly same as the transpose of operator L because I will have 6 stress components and there are 3 equations.

So, this L operator is of course, 6 by 3 size. So, I will have transpose of that and that would be 3 by 6 operator size. So, let us look at this.

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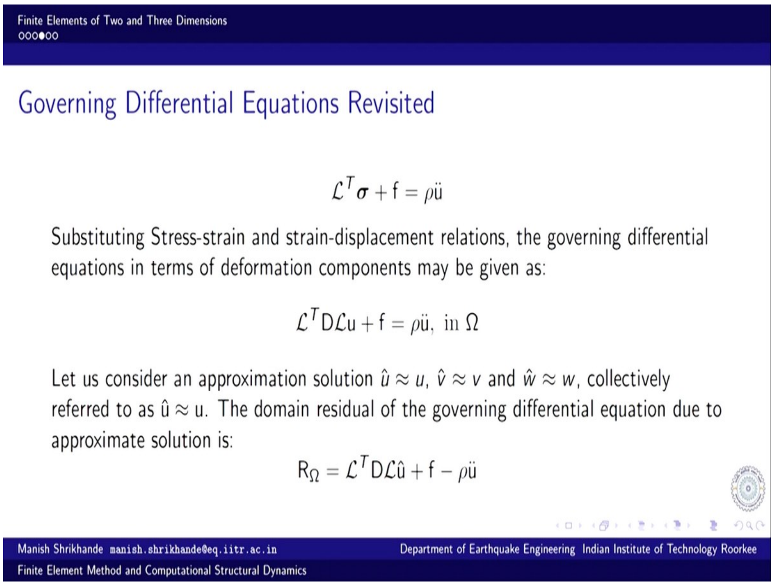

So, the governing differential equation can be written as shown in figure. So, we will see if we rearrange this equation then we can expand this into L^T the differential operator that we have here and multiply it with the stress components that we have and that will lead us to the basic same equations as this.

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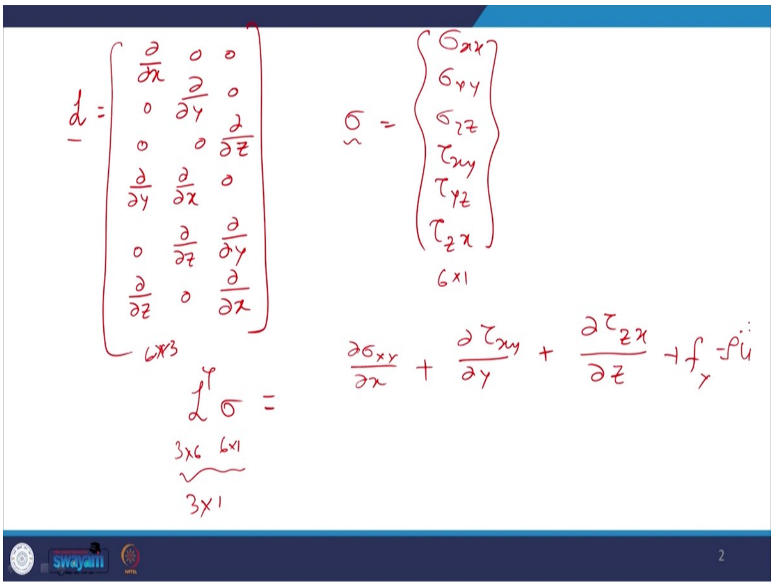

So L is given as :

$$L=\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

So, this is the basic differential operator and the stress components are of course, $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}$ and τ_{zx} . So, if I pre-multiply the σ with differential equation operator L^T , we will get governing differential equation. So, L is 6 x 3 dimension, σ is 6 x 1. So, L^T is going to be 3 x 6 and 6 x 1. So, multiplication is going to be 3 x 1. So, these are three different equations in orthogonal directions. So, let us look at the first equation.

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \gamma_{xy}}{\partial y}+\frac{\partial \gamma_{zx}}{\partial z}+f_x=\rho \ddot{u}$$

And similarly by operating on second row and third row we will obtain other rows of the governing differential equation. So, the point is that whatever differential operator we have, which relates deformations to strain, transpose of that operator is used for deriving the governing differential equation of motion in terms of stress components.

$L^T \sigma+f=\rho \ddot{u}$, that is the governing differential equation and we already discussed that stress is proportional to strain and strain is proportional to deformation. So, $\sigma=D\epsilon$ and $\epsilon=Lu$. So, substituting that, the governing differential equation becomes in terms of deformation components.

$$L^T DLu+f=\rho \ddot{u} \qquad \text{in} \quad \Omega$$

So, once we make use of stress strain relationship and strain displacement relationship and substitute in the governing differential equation we will get the governing differential equation in terms of deformation. Now what to do with these deformations?

How does this help us in formulating finite element equation? We will discuss that in our next lecture.

Thank you.