Finite Element Method and Computational Structural Dynamics
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Lecture - 18
Finite Elements of C^1 Continuity in 1-D-II

Hello friends. We were discussing about the Finite Element approximation for Euler Bernoulli beams. We discussed about the necessity to ensure continuity of not just the transverse displacement, but also it's derivative across the inter element boundaries.

So, the primary variables are transverse displacement as well as the slope or the first derivative of the transverse displacement. Now, the interpolation function for that higher degree of smoothness is given by Hermite interpolation. And, we have already discussed Hermite interpolation in our discussion of interpolation theory.

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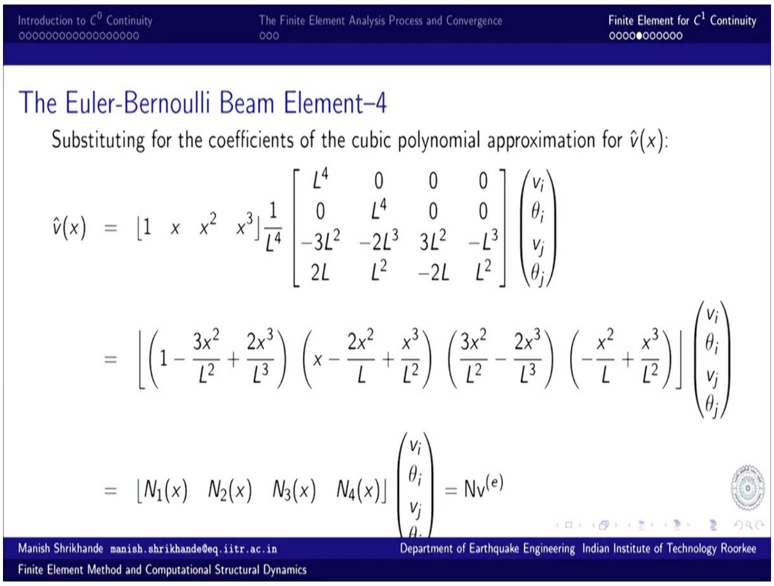

But, in this case we derived this from the basic requirement of the interpolation theory, that approximation has to be consistent, has to yield the primary variable at the nodes or the specified function values at the nodes. And, this is what we get as the interpolation functions.

Four interpolation functions or four primary variables in the element - two at each node - also referred to as the degrees of freedom. So, 2 degrees of freedom at each node -

transverse displacement and slope and collectively they are referred to us Nv . So, that is shape function matrix or interpolation function matrix N and the vector of nodal values of primary variables that is referred to as v .

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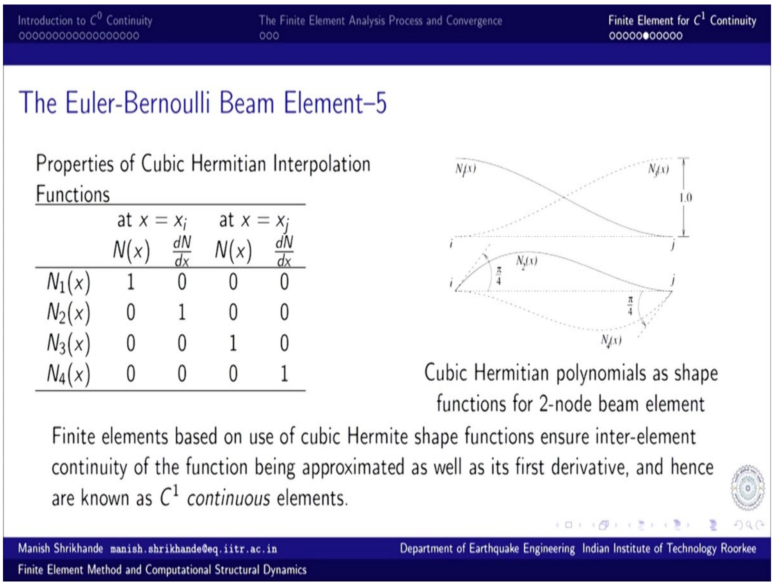

Properties of the Cubic Hermitian Interpolation polynomials functions - N_1 when evaluated at x_i is 1, but it's derivative when evaluated at x_i is 0. Similarly, N_1 when evaluated at x_j is 0 and it's derivative also vanishes at x_j . N_2 vanishes at x_i , but it's derivative is unity. Slope of N_2 is unity at x_i . And, it vanishes at x_j and it's derivative also vanishes at x_j . N_3 vanishes at x_i and it's derivative also vanishes at x_i , but it takes unit value at x_j and it's derivative is 0 at x_j . Finally, N_4 has unit derivative at x_j and the function value and it's derivative at x_i is 0 and it also evaluates to 0 at x_j . So, this property is encapsulated in this plot of these interpolation functions. So, N_1 starts at 1 and the slope is of course, 0 and it gradually goes to 0 at node j and with 0 slope.

This is of course, a cubic polynomial. N_3 is the compliment of N_1 . So, it is 1 minus N_1 . And, similarly for N_2 , it starts with 0, but with unit slope. So, the tangent angle is

$\frac{\pi}{4}$ and it goes to 0 at N_j with 0 slope. N_4 starts with 0, with 0 slope at i^{th} node and it has 0 value at node j with unit slope.

So, finite elements based on these interpolation models, based on the use of these cubic Hermite shape functions, ensure inter element continuity of the function being approximated as well as it's first derivative that is $\frac{dv}{dx}$ or θ, slope of the deflected shape.

And, therefore, these elements are known as C^1 continuous elements, C stands for Continuity and 1 superscript denotes the derivative order. So, the function with first order derivative is continuous.

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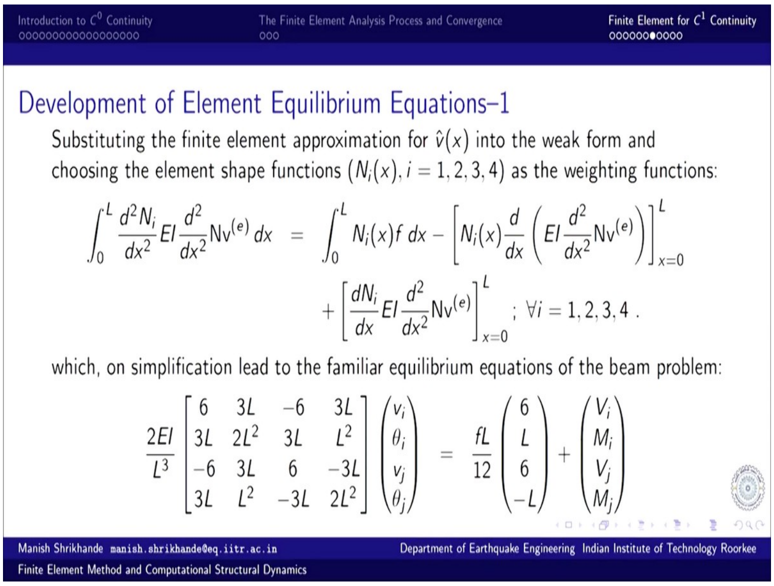

So, with these development, we now have our weighting functions in the weak form available with us.

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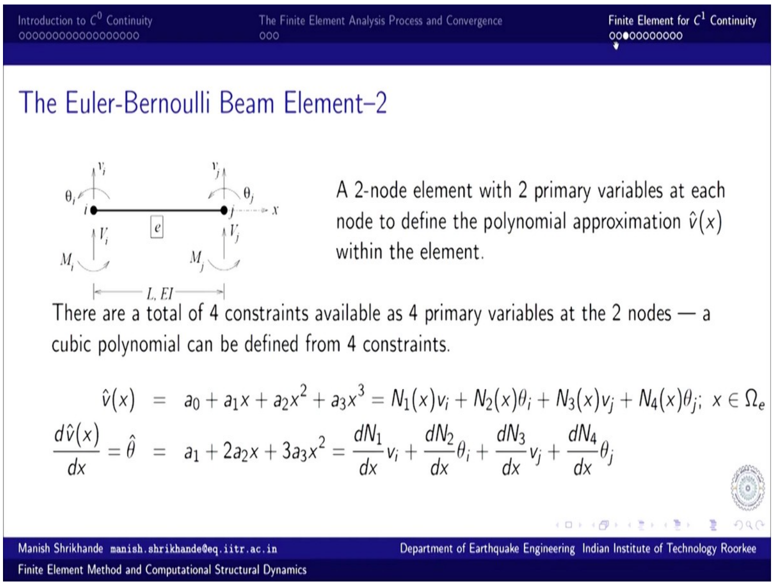

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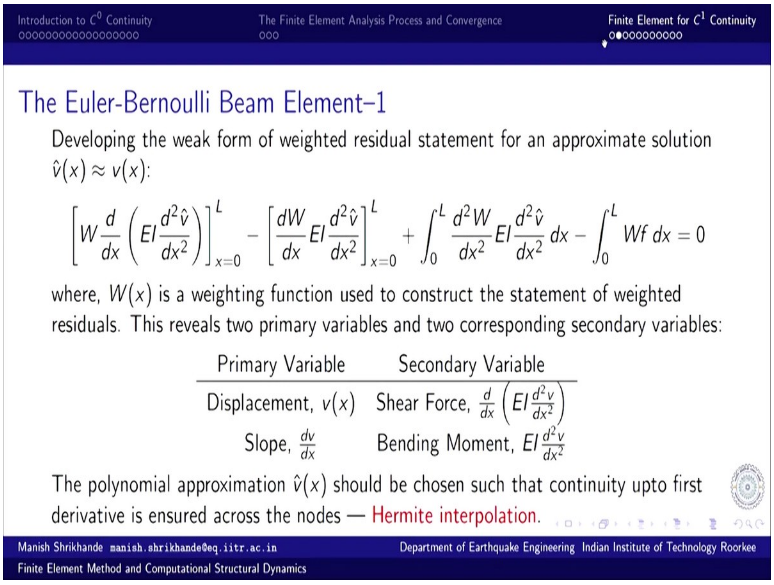

This is weak form of the weighted residual statement. All that we do now is replace W with respective interpolation functions N_1, N_2, N_3 and N_4 four weighting functions one after the other. so, substituting finite element approximation for $\hat{v}$ into the weak form that is Nv and choosing the element shape functions N_i , i ranging from 1 to 4 as weighting functions. So, in sequence we use weighting function as $W=N_1$, $W=N_2$, $W=N_3$, $W=N_4$ and these will be the 4 equations that we will have.

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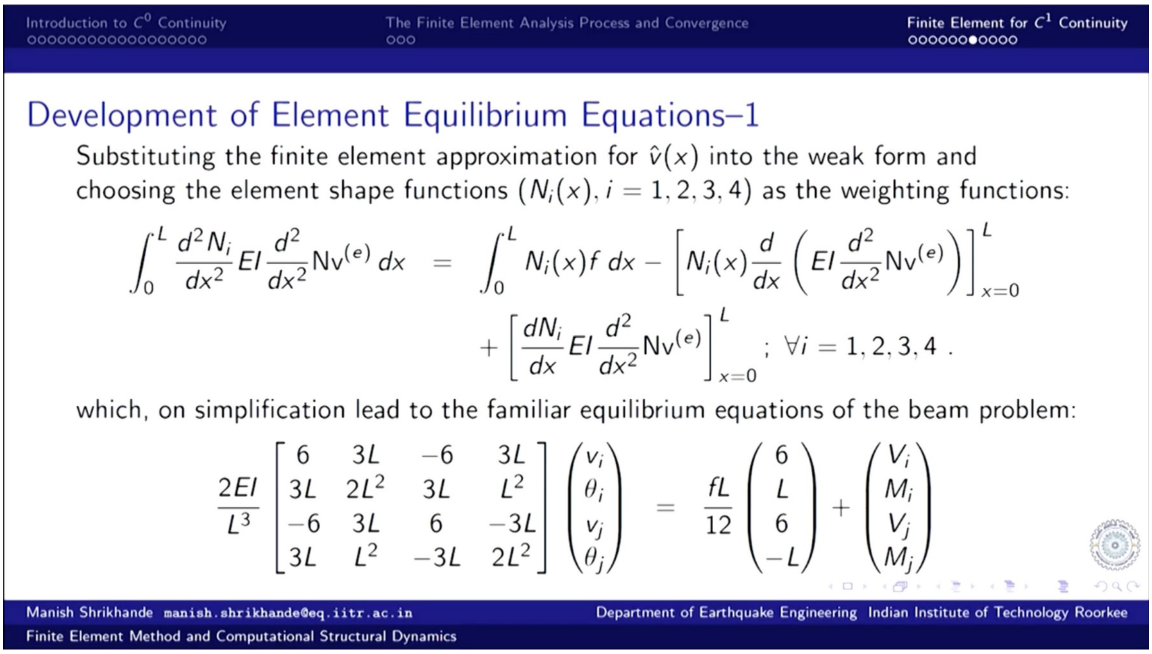

So, these are the 4 equations that we have once we substitute it in the weak form. In each equation we have 4 boundary terms. Two each at $x=0$ and $x=L$. Now, because of this particular variation of the interpolation function, you can see that the weighting functions will vanish for three of the four terms and only one term would exist at that time at any point.

The first term is symmetric term and we transfer remaining terms to the other side. So first term on RHS is the weighting function multiplied by external force. And remaining two are the boundary terms.

So, for any choice of N_i, N_1 would exist only at $x=0$, derivative of N_2 would exist at $x=0$ all other terms would vanish. N_3 would exist only at $x=L$ and all other three terms would vanish, derivative of N_4 will exist only at $x=L$ and all other 3 terms would vanish.

And, that is what leads to after substituting for these interpolation functions, and evaluating these integral eventually as it is just a 1 dimensional integrals in x. So we get this particular element level equilibrium equations, which are very familiar, this is $\frac{12EI}{L^3}$, $\frac{6EI}{L^2}$ terms and this is $\frac{-12EI}{L^3}$, and this is $\frac{6EI}{L^2}$ again. So, these are standard stiffness coefficients for the fixed fixed rigid moment transfer moment connections.

And, similarly the RHS. So, $\frac{fL}{2}$, $\frac{fL^2}{12}$, $\frac{fL}{2}$ and $\frac{-fL^2}{12}$. Plus, these are the secondary variables. Shear force bending moment vector at node i shear force bending moment at node j from the boundary terms that we evaluate.

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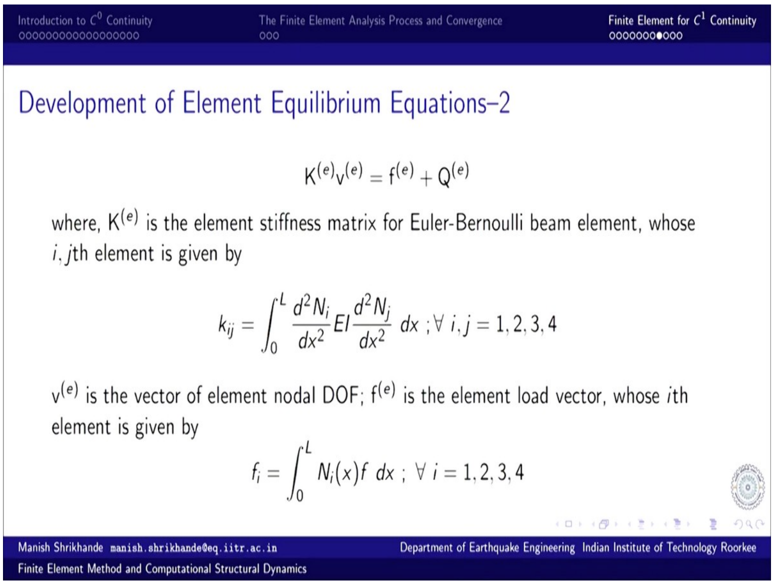

So, collectively we can refer to these equations as $K^{(e)} v^{(e)}$ is equal to $f^{(e)}$ plus $Q^{(e)}$. So, $K^{(e)}$ is the element stiffness matrix for Euler Bernoulli beam element, whose ij th element is given by the expression

$$K_{ij} = \int_0^L \frac{d^2 N_i}{dx^2} EI \frac{d^2 N_j}{dx^2} dx ; \quad \forall \quad i,j = 1,2,3,4$$

So, ij th element would involve product of second derivatives of i th shape function and j th shape functions. And, of course, the structural rigidity, which is quite often uniform. And, for all i,j varying from 1 to 4 and this is of course, symmetric operator. So, that leads to symmetric stiffness matrix as you can see. And, this is of course, by virtue of using Galerkin approach.

If waiting functions were chosen anything other than the interpolation functions, this symmetry would not have been there. But for structural mechanics problem, this symmetry is not only desirable it is essential to be consistent with the mechanics of the problem to satisfy the Maxwell Betti Reciprocal Theorem.

So, $v^{(e)}$ is the vector of nodal displacements - nodal primary variables. So, v_i, θ_i, v_j, θ_j are the primary variables, f_i is the i^{th} element of nodal equivalent force So, the entire distributed load f is being converted into the nodal loads.

So, f_i is the nodal equivalent of that distributed load f. And, that is done by using the work equivalent, it is the same amount of work f_i - the forces, which will do in moving through displacement v_i as would be done by distributed force f. So, that is the work equivalence of this nodal force.

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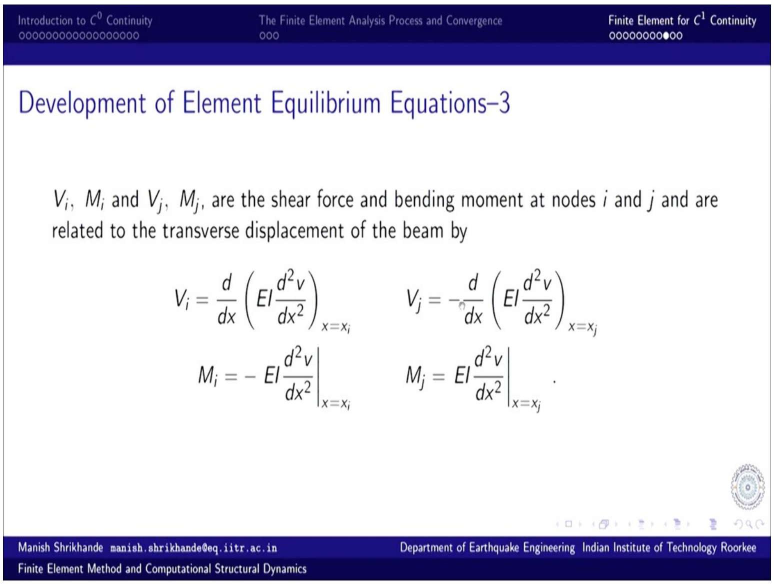

V_i, M_i, V_j, M_j are the shear force and bending moments at node i and j and are related to transverse displacement of the beam by the boundary term expressions. So, V_i is proportional to third derivative of the transverse displacement at x_i, V_j is called proportional to third derivative of displacement at j^{th} node. And, M_i proportional to second derivative of displacement at x_i at i^{th} node and M_j is proportional to second derivative of the transverse displacement at j^{th} node.

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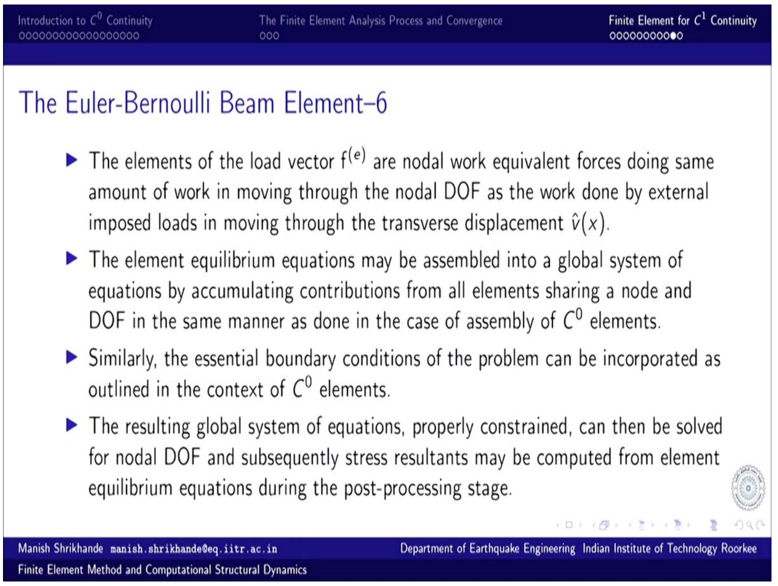

So, as I said the elements of the load vector $f^{(e)}$ are the nodal work equivalent forces doing the same amount of work in moving through the nodal degrees of freedom. As the work done by external force; external impose loads in moving through the transverse displacement $\hat{v}$. Now, $\hat{v}$ is a continuous function of x, external distributed load is distributed over the domain. So, whatever is the work done by the external force, we impose equality.

So, the nodal equivalent loads, do the same amount of work in moving through that particular respective degree of freedom. And we add it up all of them and that will be the total work done by the force in moving through transverse displacement.

So, once we have this the element level equations, which is $K^{(e)} v^{(e)} = f^{(e)} + Q^{(e)}$ for each of the elements, we can discretize the domain into several smaller elements and for each element we can have similar kind of element level equations.

And, these element level equations would be assembled into global system of equations, comprising of all the degrees of freedom of all the nodes in the structure of the finite element mesh. And, that global system of equations would be accruing contributions from different elements.

So, by accumulating contributions from L elements, all elements sharing a node and degree of freedom, in the same manner as done in the case of assembly of C^0 elements.

The only difference being in case of C^0 elements particularly when we are looking at all the elements aligned in 1 line, 1 coordinate frame, so, there are only 1 degrees of freedom at each node. But for arbitrary orientation there are 2 degrees of freedom and it becomes similar kind of an exercise. So, the accumulation from all contribution from all elements at sharing a node and degrees of freedom happens in the same way as in done in the case of assembly of C^0 elements. Just have to be consistent with the degrees of freedom and the common degrees of freedom and common nodes, and accumulate respective contributions from different elements.

And, then the essential boundary conditions of the problem can be incorporated as outlined in the context of C^0 elements using stiff spring approach or penalty approach. And, any imposed point loads etc in the problem can be defined by using secondary variables.

If there is any discontinuity, we place a node there and then we take care of those discontinuous point loads or point moments applied and include them in $Q^{(e)}$ term.

If, there is any secondary variable that has been prescribed, we place a node at that particular point where the point loads are prescribed. Hence the secondary variables are defined appropriately in the element level equations, and then it is taken up for the assembly.

And, after the assembly of the global system of equations, essential boundary conditions of the problem can be incorporated. And, the resulting global system of equations properly constrained with respect to imposition of essential boundary conditions can be solved for nodal degrees of freedom i.e. the unknown primary variables at the nodes of finite element mesh.

Subsequently stress resultants which are the secondary variables of the problem in different points of the domain, can be computed from element level equilibrium equations, during the post-processing stage. Because element level equilibrium equation, we will have once we have this complete displacement, then we can compute the moment and shear force by the appropriate derivative.

So, we get the displacements as a function of x in terms of interpolation of the primary variables, which we will get from the solution of global system of equations. And, once we have this displacement within an element, we can find it's second derivative for finding out the moment, and third derivative for finding out the shear force in the beam evaluated at different points.

That is what we call as post processing stage. So, for finite element solution the main processing stops with the computation of primary variables. After that, whatever any secondary results are required; those are referred to as the post processing of finite element analysis.

Now, as in the case of bar element, even the beam elements can be of any arbitrary orientation. For example, moment resistant frame structures.

So, there are columns, beams and inclined members also possible. So, there is no guarantee that all beams in a structure would be at the same orientation. For example, the skeletal structure, frame structure, or the staging of elevated water tanks. So, framed staging of the elevated water tanks, you can see beams interconnecting each other and at different elevation at different orientation. So, we can do the same thing with the similar operation as in the case of arbitrary orientation.

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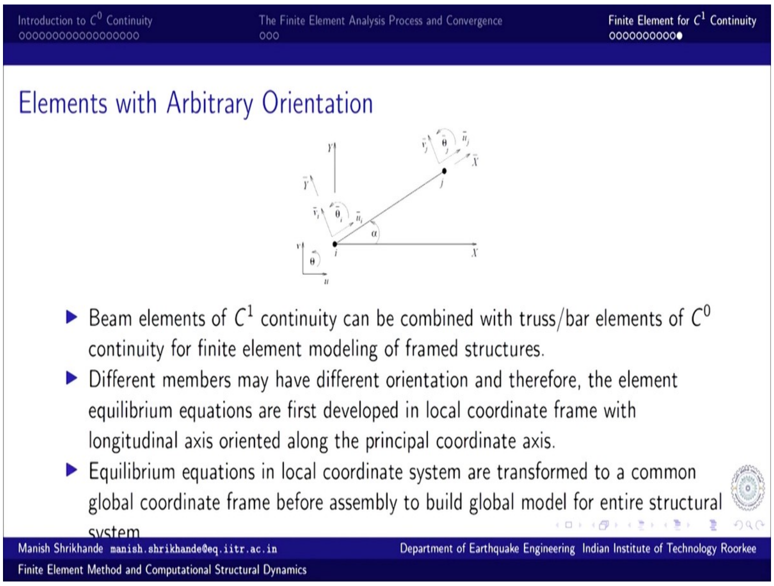

And, of course the Euler Bernoulli Beam element can only deal with transverse bending. So, axial deformation will have to be accounted for by using the truss elements. So, beam elements of C^1 continuity, can be combined with truss or bar elements of C^0 continuity, for finite element modeling of the frame structures.

So, in this particular stiffness matrix that we computed for the Euler Bernoulli Beam element, we will have two more rows and columns corresponding to v_i and v_j.

And, respective elements would be decoupled from these fractional degrees of freedom, and there would be appropriate AE/L diagonal terms and $-AE/L$ off diagonal terms in the respective rows and columns, of the stiffness matrix. So, that becomes the element level stiffness matrix for the frame element.

So, combining C^0 bar element and C^1 Euler Bernoulli beam element, we can get a frame element and that can be used for modeling of frame structure, with elements oriented in different direction. Now, in the same way as we did for the case of truss problem, these need to be transferred from local frame of reference to global frame of reference.

So, these degrees of freedom the equations are derived in terms of $\bar{x}-\bar{y}$ frame of reference for each element. And, these equations need to be transformed to $x-y$ frame of reference that is global common frame of reference, before the element level equations are assembled into global system of equations.

And, once that is done then of course, the boundary conditions could be imposed and equilibrium equations in local coordinate system are transformed to a global coordinate frame, before assembly to build global model for entire structural system.

And, once this global structural system model is ready then, we impose the essential boundary conditions of the problem wherever there are constraints on the primary variable and remaining system of equations is then solved for remaining unknowns of the problem.

And, then subsequently going back to individual element level, we can find out the stress resultants bending moment and shear force at any point at in any element of the structure and that is how it works. This completes our discussion of C^1 continuous element.

Now, C^1 continuous elements as I have been emphasizing, ensures a higher degree of continuity, not just the unknown function of the differential equation, but also it's derivative is required to be continuous across the element boundaries.

Now, section rotations are same as the slope. That is the fallout of natural consequence of Euler Bernoulli hypothesis - plane section remain plane, but they can rotate after the bending. So, this rotation of the plane is same as the slope of the deflected shape. And, that happens because of neglecting the shear deformations, it is pure flexure.

So, if there are shear deformations present, which is the case in case the beam depth is much larger than width and not very small in comparison with the length - the deep beams. So, for deep beams shear deformations are not negligible and we cannot completely ignore them, they may not be as large as flexural component of the deformation, but they are not negligible either.

So, in that case the section rotation is not really same as the slope of the deflected shape, because the deflection transverse deflection is not just contributed by flexure, it is also contributed by the shear deformation. So, one to one relationship is lost. So, in that case we need to look at decoupling the two fields.

So, the slope or the deflected shape of the transverse deflection and the slope of the transverse deflection, they are different, they are treated differently. So, they are taken as independent variables. So, they are interpolated separately. So, in that case there is no need for the interpolation for transverse displacement to ensure continuity of the first order derivative.

Because slope is entirely different, it is being interpolated independently, which will be independent in case there are significant shear deformations.

So, they are interpolated independently and then the beam problem can be modeled using C^0 elements, the transverse displacement is interpolated independently and the slope is interpolated independently. And, we work accordingly and there will be two set of differential equations, second order each.

So, we will again have first order as the highest order derivative in the weak form. So, C^0 continuous element will do the job. Hence, C^0 continuous elements can be used for approximation of those shear flexible beams. Such theories are also known by the name of Timoshenko Beam Theory and it is possible to derive beam equations using just C^0 model. But, the problem becomes that is where the consistency requirement of finite element equations come into picture.

This is fine as long as the beam being modeled is deep and there are significant shear deformation, but if this model is used for modeling thin beams; I mean where Euler Bernoulli hypothesis works beautifully as shear deformations are negligible. In that case there would be inconsistency, if we use the same order of interpolation for displacement and the slope.

Because, the mechanics of the problem required that the slope has to be like the first derivative of the transverse displacement. And, if I use the first degree approximation, first degree interpolation for both the slope and the transverse displacement, then there is mutual inconsistency and that leads to locking problem. That leads to very very absurd results.

The solution here is to have consistent interpolation. So, the displacements have to be interpolated at 1 degree higher than, the slope interpolation. And, once we do that all the problems are sorted out and this consistent interpolation leads to very good results actually consistent with what is observed experimentally.

So, that completes our discussion on 1 dimensional finite elements. We will start with discussion of continuum finite elements i.e. 2 dimensional and 3 dimensional continuum problems and we will see how that is done. Again the basic process remains the same the only thing that needs to be looked at is the kind of elements that are used and how the interpolation functions are developed. Rest of the things is exactly identical; it is very mechanical and automated process.

Thank you.