Finite Element Method and Computational Structural Dynamics

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Lecture - 17
Finite Elements of C1 Continuity in 1-D-l

Hello friends. We have seen the finite elements of one-dimension using C^0 continuity; where only the variable of the governing differential equation was required to be continuous across the entire element boundary. That variable was the primary variable. This typically happens when the governing differential equation is of second order and the highest derivative available in the weak form is first order.

We also have another class of problems in one-dimensional finite elements where not only the variable of the governing differential equation, but also its first derivative needs to be continuous across the element boundaries. So, not only the variable, but its derivative as well are the primary variables of the problem.

Therefore, the approximation has to be sufficiently smooth such that not only the basic variable of the differential equation, but also its first derivative is continuous across the element boundary. So, the first order derivative continuity is also required i.e. not just the function value, but the slope of the function also needs to be continuous.

So, a higher degree of smoothness is required in the approximation and that is covered under what we call as C^1 continuity finite elements. So, what we are referring to is the beam bending problem.

Just to rejig the memory or the concepts, it is based on Euler-Bernoulli hypothesis. Although any beam is a three-dimensional system and deformation is of course in three-dimensions, we can note that in the case of beam, the length is very large in comparison to the cross-sectional dimensions of the member.

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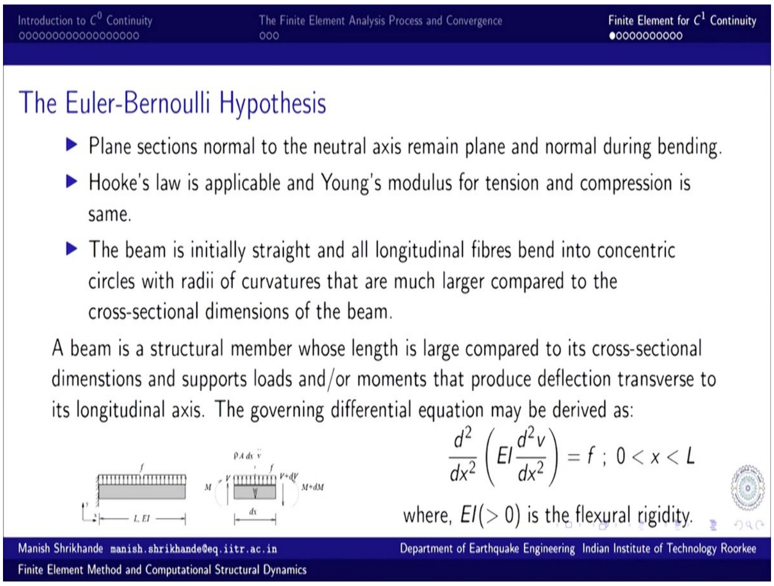

So, a beam is typically a structural member whose length is large compared to its cross-sectional dimensions and it supports loads and moments that produce deflection in the transverse direction.

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So, consider the longitudinal axis of the beam along the axis of the pen as shown in above figure. The deformation would be either in vertical direction or in the direction normal to plane of the figure. So, there are two different planes of bending. We can also note that the cross-sectional dimension are very small compared to the length of the beam and the loads are in the transverse direction. So, the deflections are transverse to the axis.

Now, this has a complete three-dimensional state of deformation. It is of course, too complex and the problem becomes too unwieldy if we try to go into continuum-based 3D modeling. Instead very good results can be achieved by imposing certain kinematical constraints on the deformation and developing simplified expressions and those kinematical constraints are what we refer to as the Euler-Bernoulli hypothesis.

Please mark the word hypothesis, it is an hypothesis because it is not a theory and why it is not a theory? I will come to that after in a while. So, the first hypothesis is plane sections normal to the neutral axis remain plane and normal to the neutral axis during bending. This happens when the Hooke's law is applicable.

So, linear Hooke's law is applicable and Young's modulus for tension and compression is same. It does not make a difference whether it is tensile strain or compressive strain, as Young's modulus is same in both the directions. Moreover, Hooke's law is for linear elasticity so that obviously implies that we are looking at small deformation domain.

So, our frame of reference is small deformations and beam is initially straight and its longitudinal fibers bend into concentric circles which are much larger compared to the cross-section dimensions of the beam. This is the kinematic constraint that is imposed on the deformation of the beam and based on this, very simplified expressions for the transverse deflection can be obtained. This deflection can be related to the bending moment and shear force or stress resultant in the beam. Hence useful design quantities such as maximum bending moment, maximum shear force, maximum strain and maximum stress that will develop in the beam, can be calculated.

So, why is it then a hypothesis? First thing is as I said that beam itself is a three-dimensional body. Now, any three-dimensional state of deformation will have Poisson's effect. If I stretch the beam in one direction, it has to undergo compression in other two orthogonal directions.

Now, this Poisson's effect is not captured by Euler-Bernoulli hypothesis and therefore, the deformation field that is actually predicted by Euler-Bernoulli hypothesis is not really the exact deformation field that will be satisfying the governing differential equation in true elasticity problem.

But still it is a working approximation in the sense that we get the values of the stress resultants, the bending moment and shear force, which are very close to what would be there in a more rigorous analysis and they can be computed with very little effort. So, the Euler-Bernoulli hypothesis still provides a very good working approximation, and we continue with that.

Consider case of a fixed cantilever beam subjected to uniform cantilever beam of length L and uniform flexural rigidity EI subjected to uniform distributed load transverse to the beam axis and the x is oriented along the longitudinal axis of the beam and transverse deformation are along y direction as shown in the figure.

If we draw free body diagram of an infinitesimal element of the beam, that will establish the equilibrium of forces and using these equilibrium of forces, we can derive the governing differential equations as the fourth order differential equation. Moment is proportional to second derivative - the curvature - and the second derivative of the moment i.e. the rate of change of shear force is related to the transverse applied load. So, this is the governing differential equation

$$\frac{d^2}{dx^2} EI\left(\frac{d^2 v}{dx^2}\right) = f\,; \qquad 0 < x < L$$

EI obviously, greater than 0 is the flexural rigidity of the beam and this differential equation is applicable over the domain 0 to L that is the length of the beam.

Now, this is the governing differential equation and let us say I am looking for an approximate solution $\hat{v}$ as an approximation of v.

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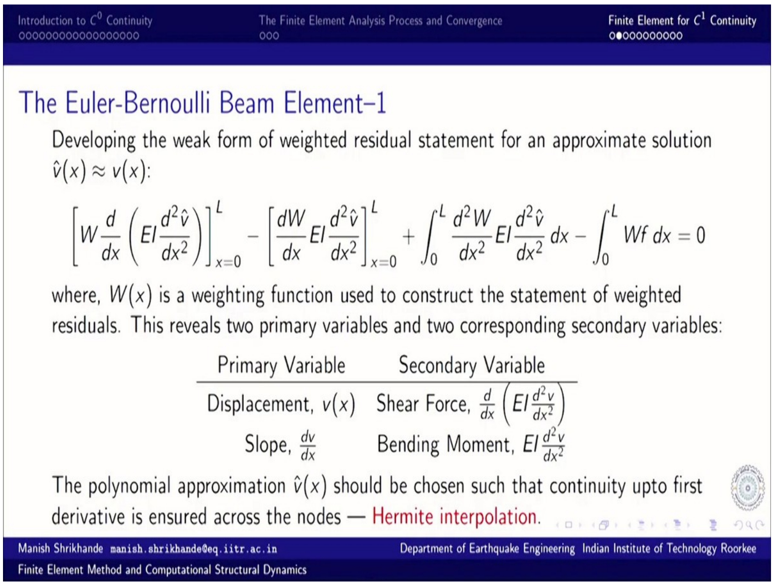

And if I substitute the approximate solution in the governing differential equation, I will get the domain residual as

$$R_{\Omega} = \frac{d^2}{dx^2}\left(EI\frac{d^2\hat{v}}{dx^2}\right) - f$$

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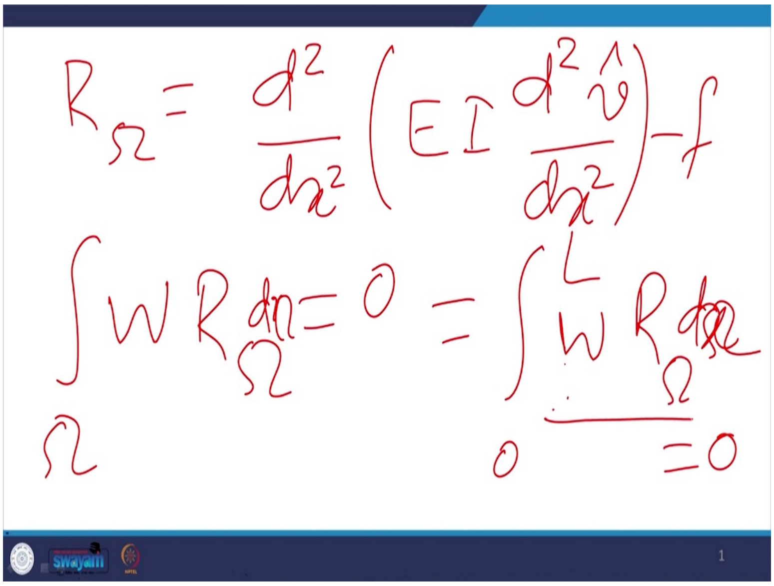

So, the statement of weak form of method of weighted residual would be the domain integral of this domain residual should is equal to 0. So the statement can be written as given in the figure.

After this, I have this weighting function unknown as yet. But it is a function of x, so integrate the whole term by parts in a sequence twice and that is what leads me to this particular expression.

$$\left[W\frac{d}{dx}\left(EI\frac{d^{\hat{v}}}{dx^2}\right)\right]_0^L - \left[\frac{dW}{dx}EI\frac{d^2\hat{v}}{dx^2}\right]_0^L + \int_0^L \frac{d^2W}{dx^2}EI\frac{d^2\hat{v}}{dx^2}dx - \int_0^L W\,f\,dx = 0$$

So, this is the weak form of the weighted residual statement, W is the weighting function. First two terms are the boundary terms, of which, second term is from second application of the integration by parts. Third term is the symmetric operator term. And the last term is due to applied loading.

Notice that there are two boundary terms here. Let’s identify the primary variables in these boundary terms. As there are two boundary terms, there are going to be two primary variables and two secondary variables. First primary variable is the weighting function term, what is the form of the weighting function? 0th derivative in this particular boundary term.

So, replace the weighting function by the unknown of the problem, v - transverse displacement. So, the first primary variable becomes transverse displacement v. Associated secondary variable is the third derivative of displacement which is shear force.

Now, looking at the second boundary term the weighting function appears as the first derivative. So, the appropriate primary variable would be replacing W by the primary basic unknown of the problem so, that is the transverse displacement v. Hence the primary variable becomes $\frac{dv}{dx}$, which is the slope of the transverse displacement. The corresponding term which is secondary variable is obviously, the bending moment. Now, you can have a look at it, the primary variable, secondary variable, they appear together, and it is always like a work quantity.

So, the weighting function can be interpreted as the virtual displacement. So, this is like boundary shear force or the boundary force in moving through the virtual displacement at the boundary points. So, the work done by the shear force at the end point in moving through the virtual displacement at the end points.

Similarly, second term, the bending moment at the end point in moving through the slope at the end point, that is the work done. So, both of these boundary terms, they are always work quantities. The last term is W f, if I am interpreting W as a virtual displacement so, this is again the virtual work done by the external force in moving through virtual displacements.

When we interpret this weak form using the Galerkin approach, the W we use is the same thing as the approximation function, then this becomes entirely identical to the statement of principle of virtual work that is so commonly known in the field of structural mechanics.

So, coming back to the problem of finite element approximation. Essentially, there are two primary variables, the transverse displacement of the beam and its slope - the first derivative of the transverse displacement. So, the approximation has to be chosen such that continuity up to first derivative is ensured across the nodes.

We have already discussed one such problem during our discussion of interpolation theory and that is Hermite interpolation. Hermite interpolation, that interpolate not just the function values, but also the required order of derivatives of the function values.

So, in this case we require continuity in not just the value of the transverse displacement, but we also want continuity with respect to the slope of that displacement function at the nodes. So, there are two primary variables at the nodes.

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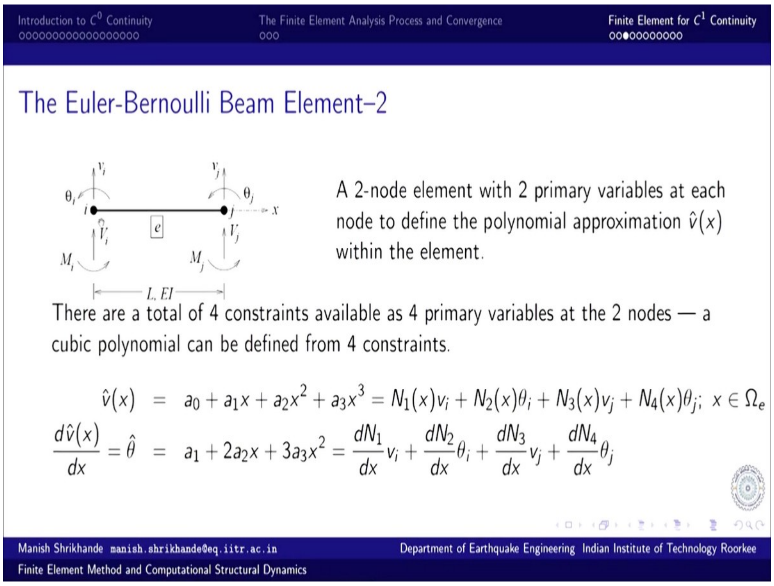

So, this is what the elementary finite element model looks like. So, element e defined by two nodes i and j, i left side node, j right-hand side node with two degrees of freedom at each node. So, primary variables are the transverse displacement v_i at node i and slope θ_i.

And similarly at the node j, transverse displacement v_j and the rotation θ_j . Then there are associated secondary variable which are the shear forces V_i , V_j and bending moments M_i , M_j , at the respective nodes. So we have 2-nodes with 2 primary variables at each node to define the polynomial approximation. Hence, there are 4 constraints.

The approximation has to satisfy the limiting conditions of these variables at these nodes. So, whatever approximation we choose, when it is evaluated at node i, the displacement should evaluate to v_i , its derivative should evaluate to θ_i . When the function approximation for v is evaluated at node j, it should evaluate to v_j and its derivative should evaluate to θ_j . So, these are the 4 conditions, 4 constraints that are required.

So, with 4 constraints available, we can fit a polynomial for v . So, with the 4 constraints available, we can have a 4 term approximation and a 4 term complete polynomial starting from the constant term would be

$$\hat{v}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

So, there are 4 unknown coefficients of the problem in this approximation. And these 4 unknown coefficients can be evaluated uniquely by imposing these constraints of the approximation that has to satisfy at node i and node j.

So the 4-term interpolation model can be written as :

$$\hat{v}(x) = N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2$$

And similarly, derivative of this polynomial approximation can be written as :

$$\begin{aligned} \frac{d\hat{v}(x)}{dx} = \hat{\theta} \quad &= \quad a_1 + 2a_2 x + 3a_3 x^2 \\ &= \quad \frac{dN_1}{dx} v_i + \frac{dN_2}{dx} \theta_i + \frac{dN_4}{dx} v_j + \frac{dN_4}{dx} \theta_j \end{aligned}$$

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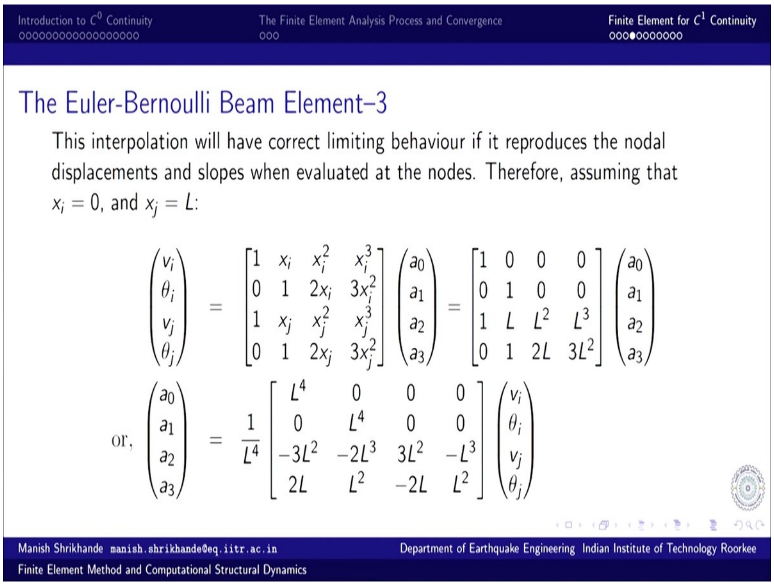

This interpretation will have correct limiting behaviour if it reproduces the nodal displacements and slopes when evaluated at nodes and therefore, assuming without any loss of generality that the coordinate of node i is 0 and coordinate of node j is L.

So, then this a particular approximation that we have, which can be again written as

$$\hat{v} = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}; \quad \text{and} \quad \frac{d\hat{v}}{dx} = \hat{\theta} = \begin{bmatrix} 0 & 1 & x & x^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Upon substituting values of x_i and x_j and evaluating, it becomes,

$$
\begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = \begin{bmatrix} 1 & x_i & x_i^2 & x_i^3 \\ 0 & 1 & 2x_i & 3x_i^2 \\ 1 & x_j & x_j^2 & x_j^3 \\ 0 & 1 & 2x_j & 3x_j^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}
$$

Hence coefficients $a_0,\ a_1,\ a_2,\ a_3$ in terms of $v_i,\ \theta_i,\ v_j,\ \theta_j$ can be obtained from the above equation as :

$$
\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{L^4} \begin{bmatrix} L^4 & 0 & 0 & 0 \\ 0 & L^4 & 0 & 0 \\ -3L^2 & -2L^3 & 3L^2 & -L^3 \\ 2L & L^2 & -2L & L^2 \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}
$$

Substituting the coefficients into the expression for $\hat{v}$ and simplifying results into an expression given in the following slide.

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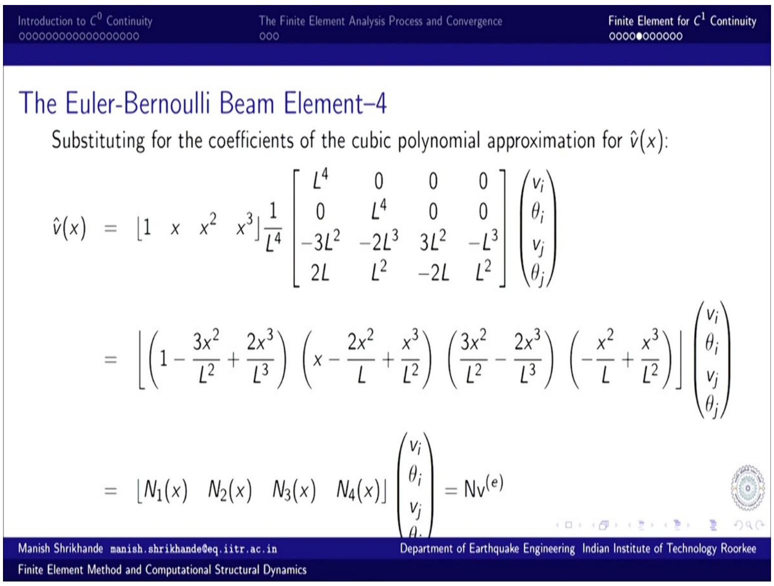

This is the interpolation model for Euler-Bernoulli beam, which is the cubic Hermite interpolation polynomials.

We derived this from the first principles, but as an exercise, you can possibly go back to our discussion of the interpolation theory and look at the expression for Hermite interpolation and there is a generic expression that is given there and you can straight away derive the equations, interpolation functions from that generic expression and you will get the same expression. So, this is the basic finite approximation.

How does this lead to our finite element equations? That is what we will discuss in our next lecture.

Thank you.