Finite Element Method and Computational Structural Dynamics
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Lecture - 16
Finite Elements of C^0 Continuity in 1-D-IV

Hello friends. We have seen the finite element modelling of bar element or elements which can ensure the continuity of the basic variable of the differential equation. We refer to it as the Element of C^0 Continuity that is continuity of zeroth derivative is maintained across the inter element boundaries.

And eventually we have gone through the complete formulation of the element and then the assembly how the domain is discretized using different finite small elements and then develop finite element approximation over each element.

We took the example with respect to two node element. Interpolation model of primary variable between these two nodes will be a straight line, so, linear polynomial. These linear polynomials ensure the continuity of the primary variable across the inter element boundaries, that will be through the shared node.

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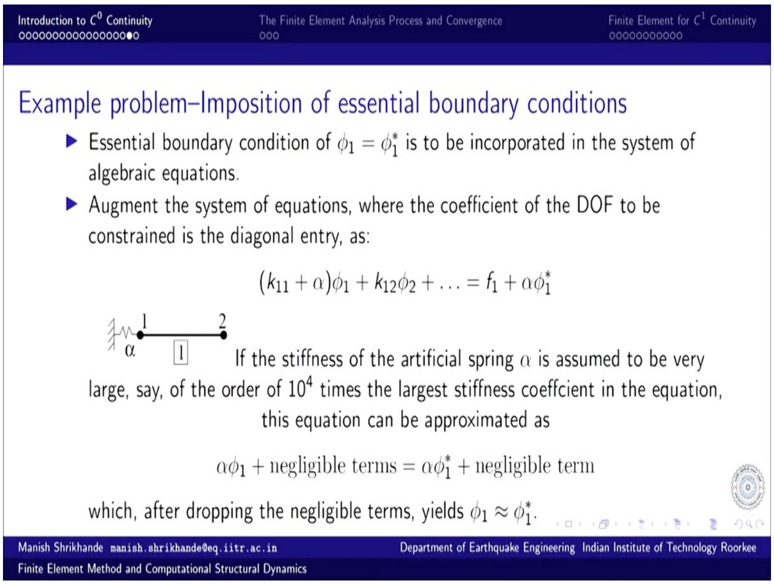

And once the element level equilibrium equations are established those are assembled into a global system of equations. After the global assembly the specified essential boundary conditions are imposed using this very simple penalty function approach or the stiff spring approach.

The system of equations thus obtained can be solved and we can find the unknown variables. We can then go back to element level equations and find out the secondary variables that are of our interest for the problem.

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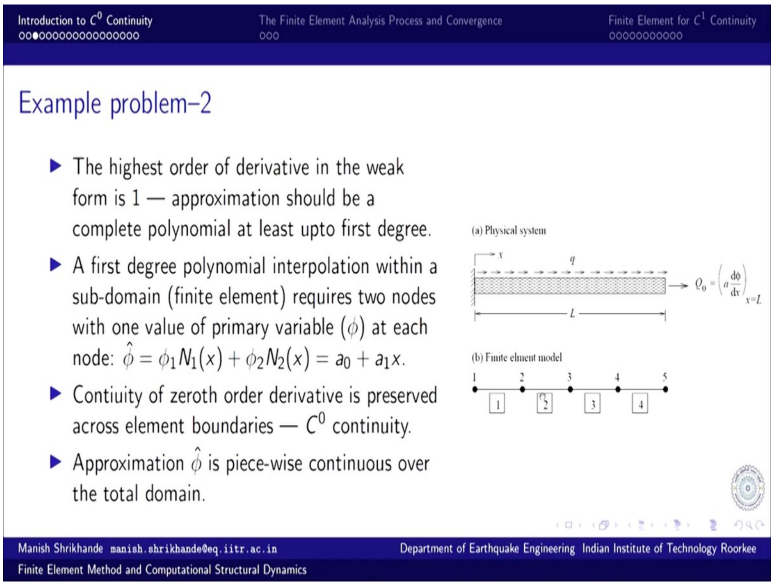

Now, for these simple problems that we have seen here we only referred to the finite element mesh, which was only straight line. The entire problem entire domain was a straight line. All the elements that we used (four elements) were in the same line. So, the same frame of reference could be used for defining the element level equations for all the elements, but that is not true in general.

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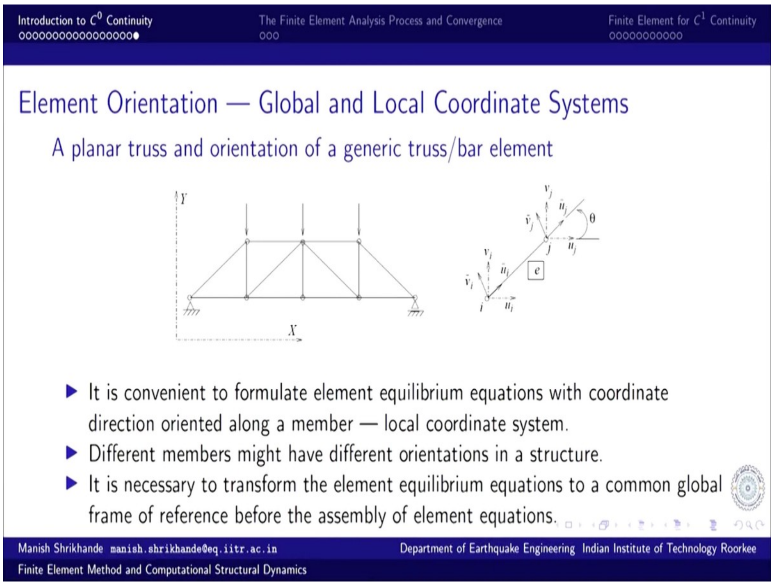

And we may have elements with arbitrary orientation. For example, a simple truss bridge as shown in figure above. This planer truss comprises of several bar elements which are interconnected through hinged connections and they together support the transverse load. These kind of structures are very common in the truss bridges. These different elements have different orientation and obviously, it is easier to define the element level equations when the axis is oriented along the member length.

There is only one variable which defines the axial deformation. Transverse to the axis, the displacements are 0 because all the loads are along the axis. So, it is easier to define the element level equilibrium along the individual member axis. These are called local frame of reference or local coordinate system.

But if I use different coordinate system for each element, the assembly cannot take place immediately as elements are in different frames of references. So, before we can assemble the contributions or element equilibrium equations for different elements into making a global representation of the total structural system, it is necessary that these element level equilibrium equations in the local frame of reference are transformed to a new frame of reference which is held common. Such a common frame of reference is shown as X-Y frame of reference in the figure and often referred as a global frame of reference.

So, all element level equilibrium equations are transformed with respect to this X-Y frame of reference. This is same as the change of basis that we had earlier seen during our discussion of vector spaces. So, the element level equations are transformed to this global frame of reference and then they are used for assembly of equations.

For example, in this case a generic element e oriented at element angle θ . $\bar{u}$ and $\bar{v}$ are the local degrees of freedom. So, $\bar{u}_i$ and $\bar{u}_j$ are the deformations along the local member axis and $\bar{v}_i$ and $\bar{v}_j$ are transverse to the member axis.

Now, these are to be related to the global degrees of freedom along the global frame of reference. So, that is u_i and v_i at node i and u_j and v_j at node j . The member itself is oriented at an inclination of theta with the global X axis as you can see in the figure.

So, the whole problem is relating these local degrees of freedom defined along the local coordinate system to the degrees of freedom at each node along the global coordinate system.

In other words, there are 4 total degrees of freedom in local coordinate system ($\bar{u}_i$, $\bar{u}_j$, $\bar{v}_i$, $\bar{v}_j$) and those are mapped to four degrees of freedom in all for two nodes in the global coordinate system (u_i, u_j, v_i, v_j). Once this is done the equation of equilibrium for each element are referring to the same frame of reference and then the element assembly can take place. After the assembly, we impose the essential boundary conditions and after the imposition of essential boundary condition the system of linear simultaneous equations are solved for the unknown variables or the primary variable at the nodes of the structure.

So, this in effect completes our discussion of finite elements of C^0 continuity in structural mechanics. It can be used to model these truss or rod kind of problems or even torsion of a circular shaft that can be modelled using this element because the governing differential equation is second order differential equation and highest order of derivative in the weak form of weighted residual statement is the first order.

And therefore, a linear interpolation between primary variables defined at two nodes is sufficient to provide a working approximation of the problem. The weighted residual statement exist and we can get the system of equations from the weak form of weighted residual statement.

Similarly it can also be used for other fields of application. For example, electrostatics or heat conduction or even the flow through porous media. They are all governed by the similar second order differential equation and thus C^0 elements can be used for modelling the problem and finding useful approximate solutions.

Now, that is a crucial point. Finite element solution is an approximate solution to the governing differential equation. The moment we talk about approximate solution the natural question that arises is how good is the approximation and can I believe in this approximation, can I believe in the quality of approximation that I am getting.

So, we will see what makes it a working approximation and what are the steps to ensure that the approximation is good and it is converging to something meaningful. But before that let us recapitulate the entire process of finite element analysis the way we had done.

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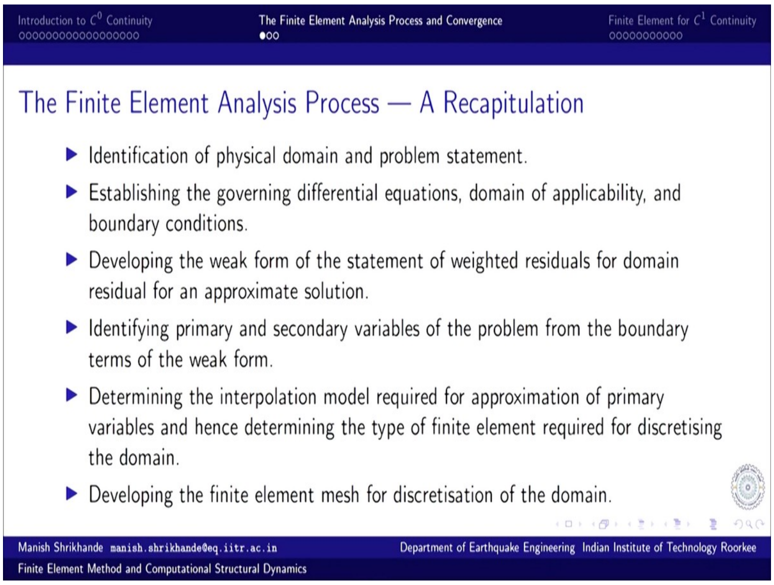

We identified the physical domain and a problem statement. Then we find out what is the applicable governing differential equation for the problem, what is the domain of applicability and what are the associated boundary conditions for the problem. Once we have these, we develop the weak form of the statement of weighted residual for some approximation. And from the boundary term of the weak form we identify what are the primary variables and what are the secondary variables of the problem. Now, this is very important because the finite element approximation has to be an interpolation model. Polynomial approximation has to be in the form of interpolation between the primary variables.

So, it is absolutely crucial to identify what are the primary variables and this is not always the same as the unknown function of the governing differential equation as we will see in the next part of this lecture very soon. So, identification of the primary and secondary variables of the problem from the boundary terms of the weak form and knowing the identifying the interpolation primary variables.

We can find out what is the type of interpolation that is required and hence the type of finite element that will be required for discretising the domain. And once we find out what is the kind of interpolation model that is required and what is the kind of finite element that will be required for finite element modelling of the problem then we develop the finite element mesh for the discretisation of the domain. So, the entire domain is divided into a number of finite elements interconnected through nodes.

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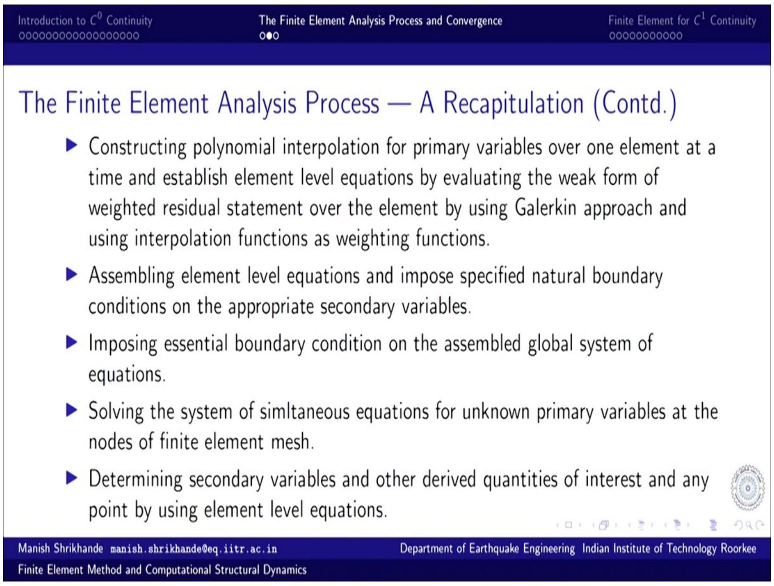

Then we construct polynomial interpolation for primary variables over one element at a time. Establishe the element level equations by evaluating the weak form of weighted residual statement over the element by using Galerkin approach and using interpolation functions or the shape functions as the weighting functions. Once we have these element level equations we impose the natural boundary conditions and then assemble them into the global level of equations. The essential boundary conditions are imposed on the assembled global equations. After this the simultaneous equations can be solved for unknown primary variables at the nodes of finite element mesh. And once we find the finite element solution for the primary variables the secondary variables and other derived quantities of interest may be computed by reverting back to element level equations with known values of the primary variable at those nodes of the element.

This in a sense is the entire finite element process. The difference only happens to be in the interpolation functions and different types of finite elements. But rest of the process remains the same.

So, it is a pretty straight forward very structured process well suited for automatic computation. And that has been the reason for the phenomenal growth of finite elements which coincided with the development of computer technology. Although the similar concepts had been known since the days of Gauss, but for the lack of automatic computational tools the method could not find much following.

But in the middle of the twentieth century when tremendous growth happened in the development of computing technology that provided the right platform for development of finite element method as well and from then onwards the finite element method has grown tremendously to the extent that it is now the de-fecto standard method for engineering analysis for any CAE problem.

As I said finite element solutions are approximate solutions. So, an approximation can be a good approximation, it can be a bad approximation and our results or any subsequent engineering decisions can be only as good as the results of the preliminary engineering analysis. If my analysis is wrong if my computations are wrong any subsequent decisions any subsequent designs on the basis of that are; obviously, going to be completely wrong.

So, it is absolutely essential to be very sure that our approximate solution including finite element solution is a good approximation and the error is within control. So, what is the basic requirement that we have to observe before we can say that the results of finite element analysis is acceptable? There are certain rules for construction of finite element approximation and we will go through those rules first.

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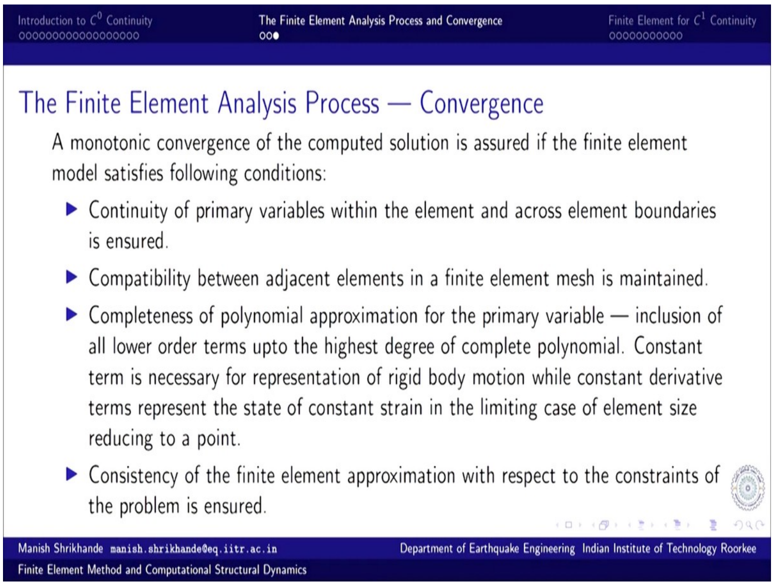

The first requirement is of course, continuity of primary variables within the element. Within the element the variable that we are approximating has to be continuous. This is automatically achieved by using interpolation model. And because we are using interpolation model and different elements connect with each other through shared nodes and the primary variables at those nodes are also shared. So, the continuity of the primary variable is also ensured across element boundaries.

So, this may not appear much of a problem in the case of simple one dimensional problem, which are only defined by using single nodes. Boundaries are defined by single nodes and nodes are shared. So, everything is shared so, the continuity is ensured. The problem is little more tricky when we talk of two dimensional and three dimensional finite elements. So, while modelling 2D and 3D continuum problems, continuity across the element boundaries is not as trivial as in the case of one dimensional finite elements.

Second thing is compatibility between adjacent elements. There should not be any gaps or even overlap in the finite element mesh. There should not be over or under representation of any part of the domain during domain discretization.

Then the completeness of polynomial approximation for the primary variable. When I say completeness of the polynomial approximation, in the two node element we said that it has first degree polynomial. You know two variables and interpolation can be defined by using a first degree polynomial. So, first degree is the highest degree of polynomial term. So, now, first degree can also be just first degree term x without the lower order term that is a constant.

A complete polynomial would be a constant term a_0 plus a first degree term that would be a_1 coefficient multiplied by x. So, the complete approximation would be a $a_0+a_1 x$. So, that is complete up to the first degree. Similarly, if I want a two term approximation, I cannot use $a_1 x+a_2 x^2$. This is not a complete polynomial as it does not have a constant term. Approximation has to be complete starting from the lowest order term and that in this case is the constant term. And why so much emphasis on this? Why am I dwelling upon on this completeness requirement?

See the point is, consider a bar element and if there is no restraint to it then any push to it would lead to the entire bar moving as a whole without any deformation or strain developing in the body. This is what we call as rigid body motion. Now, this rigid body motion of the bar is not possible in the finite element solution unless I include the constant term. Because that constant term is what allows the displacement to have a constant value all through the domain. So, if possibility of this constant displacement is not there in the approximation then there is no way I can include a rigid body motion in the finite element model.

And even if the body is under no constraint, there would be strain in the body if some force is applied because it has to move without any restraint. So, even in the case of no restraint the model would predict strain in the element which is wrong. That is why it is important for a rigid body motion to be captured.

Also let us say a finite element is of certain size. But I can keep on discretising and in the limit I can come to a very infinitesimally small length theoretically. So, infinitesimal length let us say approaching the point. What is the value of the strain at a point? Stain at a point is a constant. So, that constant state of strain should also be present.

So, these are the two bare minimum requirements for finite element approximation. The finite element approximation should be able to represent the rigid body modes and the finite element approximation should be able to capture the constant state of strain which would be the case in the limiting case of element size shrinking.

And when I say constant value of strain I am referring to structural mechanics problems. All my examples would be from the domain of structural mechanics.

The constant state of strain necessitates inclusion of linear terms. So, first the constant term, i.e. the zeroth degree polynomial and the first degree polynomial term these are the bare minimum that are required in any finite element solution for structural mechanics problems of C^0 continuity. If these two terms are included then the convergence will happen and it will be monotonic convergence.

As you can imagine as the element size is decreasing, its approaching constant state of strain and if the element approximation is capable of representing that constant state of strain we would be approaching the true result in the limiting sense. And the finite element solution would converge to true solution in the limit. Finally, the finite element approximation should be consistent with the constraints of the problems.

For example, if there are two different fields of displacements or deformation and there is required to be some relationship between these two fields then the approximation that we use should not contradict should not go against those relationships. The basic physics of the problem has to be honoured.

Consistency requirement is very crucial when we come to the problem of shear deformable systems or shear flexibility shear flexible models. We will talk about it in greater detail as we come across that problem.

Thank you.