Finite Element Method and Computational Structural Dynamics
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Lecture - 14
Finite Elements of C^O Continuity in 1-D-II

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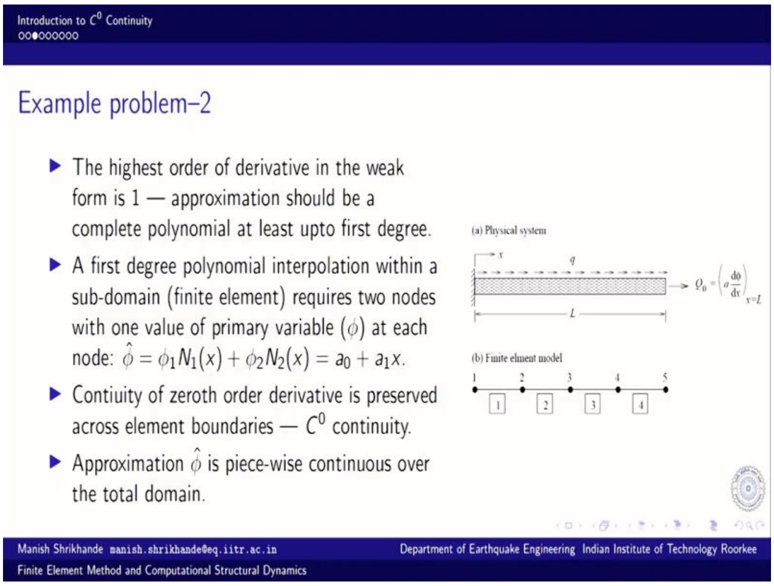

Hello friends. So, last time we discussed the solution of example problem this particular physical system and we decided that we will solve it using finite element method as an introduction I mean how to solve; how to solve this particular class of problems using finite element approximation.

And the first step to developing finite element approximation is of course, the development of the finite element mesh or the process of discretization as we call it. So, the physical domain from 0 to L that is the domain of the governing differential equation of the problem and along with two boundary conditions at x = 0, and at x = L.

So, this particular problem we have essential boundary condition specified at x =0 and natural boundary condition specified at x = L. Now the domain we decompose the domain into four finite elements so to say. So, these numbers in the square boxes they indicate the element and as you can see this particular element is defined by two exterior nodes. So, element 1 is defined by node 1 and node 2, element number 2 is defined by node 2 and node 3, element number 3 is defined by node 3 and node 4 and element number 4 is defined by node 4 and node 5.

Now, for simplicity we consider all elements of same size there is no such constraint that they have to be all of same size they can be of any size any length depending on I mean usually the location of nodes is dictated by the problem; if there is any discontinuity then we look at a node at the point of discontinuity we will see more on that little later.

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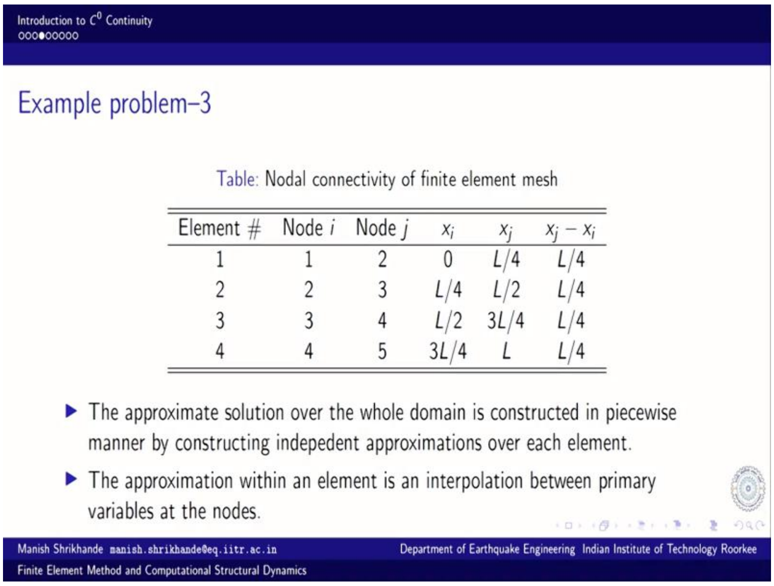

So, this is the discretization data. So, this is the what we call as nodal connectivity of finite element mesh so element number 1. Since there are 2 nodes so it is better it is effective or more efficient to talk in terms of a generic element. Let us say e any element number e which is defined by node i and node j. So, we refer to left side node of element e. So, for the example; so this is the left side node of element 1 and this is the right side node of element 2. So, we refer to left side node as node i and right side node as node j for this 1 dimensional 2 node element.

So, in this particular mesh we have which is defined by 4 finite elements. The entire domain from 0 to L is discretized using 4 finite elements with 2 nodes each. And the node connectivity data is represented in this table in this tabular form. So, element 1 is defined by node 1 as ith node and node number 2 as jth node. So, right side node and; obviously, these nodes are defined by the coordinates. So, node 1 has coordinate x coordinate of 0 and node 2 has the x coordinate of L / 4. And the difference total length of the element is; obviously, L/ 4. And similarly for node 2 defined by element number 2 defined by node 2 and 3 and similarly coordinates of node 2 and 3 and length of the element 2 and similarly for 3 and 4.

Now, the approximate solution over the whole domain we are interested in finding out the approximate solution for the entire domain. So, for the whole domain the approximate solution for the whole domain is constructed in piece wise manner. We consider the approximation over element 1, we consider approximation over element number 2, we consider the approximation over element number 3 and then we consider approximation over element number 4 independent of each other.

So, what we do in element 1 has no effect whatsoever on what we do in element 2. So, this approximation over an element 1 is completely independent of approximation over element 2 and so on. The only thing that we ensure that we insist upon is that the nodes of interpolation. So, primary variable is the basic unknown of the problem. So, the approximation is constructed as an interpolation between the primary variables at the nodes. So, the approximation within element one will be defined as interpolation of primary variable in this case ϕ. So, primary variable at node 1; so that is ϕ_1 and primary variable at node 2 that is ϕ_2 which are as yet unknown. So, but those unknown become the basic coefficients and we define approximation in terms of interpolation between these values at these two points. Similarly for approximation over element 2 is defined as interpolation between ϕ_2 and ϕ_3. So, the primary variable at node 2 and primary variable at node 3.

So, let us see how this thing works. So, approximate solution over the whole domain is constructed in piece wise manner by constructing independent approximations over each element. And approximation within an element is an interpolation between primary variables at the nodes. Now how does the element geometry look like and how do we interpolate.

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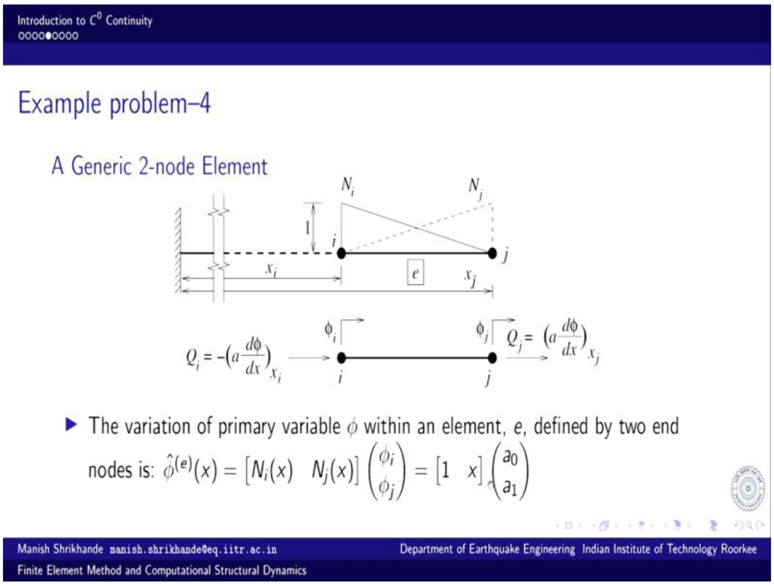

So, this is the geometry of a generic 2 node element that we have. So, element number e is defined by two nodes left hand side node is i and right hand side node node j and the interpolation is defined in terms of $N_i \phi_i + N_j \phi_j$. So, that is the interpolation model of variation of ϕ within the element and N_i; obviously, as the interpolation condition requires it has to take unit value at when evaluated at node i and it has to vanish at node j. So, it will linearly vanish to 0. So, from 1 at node I, N i will linearly vary to have to vanish at node j. Similarly N j will be 0 at node i and it will linearly increase to take unit value at node j. So, these are the interpolation functions also known as the shape functions of the problem; shape functions for this linear element.

So, these are the primary variables ϕ i and ϕ j and these are what we will come shortly are the secondary variables of the problem and these terms will arise from the weak form as we substitute. So, nature of variation of primary variable within an element e is defined by 2 end nodes i and j this is the approximation $\hat{\phi}$ within element e as a function of x is defined as interpolation between ϕ i and ϕ j the primary variables at node i and node j.

And since there are only two points so there can be only a linear polynomial that can define the function value between two points right. So, that is the unique linear polynomial. So, that polynomial generic term representation is $a_0 + a_1 x$, a_0 and a_1 these are the coefficients that are yet to be determined.

Now, we always we know that because of this polynomial approximation should yield ϕ_i when evaluated at ith node and this polynomial approximation should evaluate to ϕ_j when evaluated at x is equal to x_j and these constraints they allow us to find out what are these what these unknown coefficients of the polynomial should be.

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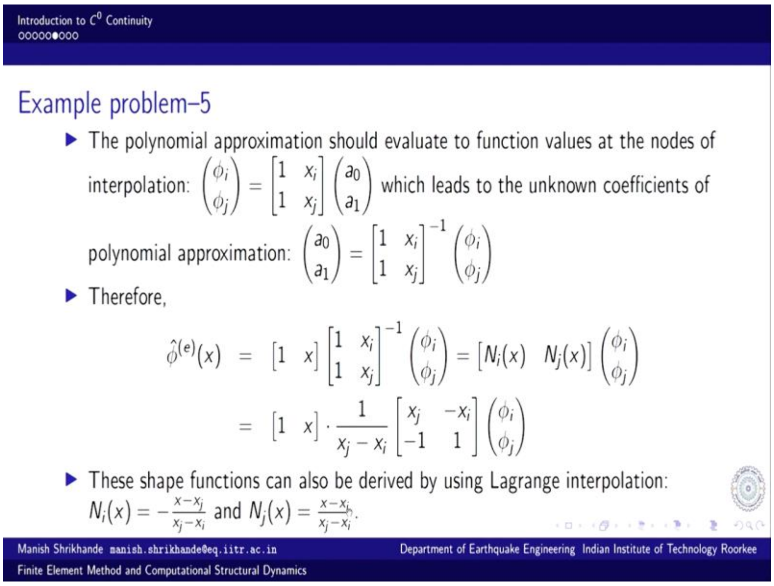

So, the polynomial approximation should evaluate to function values at the nodes of interpolations. We can invert this coefficient matrix of coordinates and a_0 a_1 can be evaluated as simply inverse of this coordinate matrix and the nodal values of primary variables.

And once we have this, this can be substituted a_0, a_1 can be substituted in our polynomial approximation and that gives us we substitute a 0, a 1 as this evaluation. So, 1 x and then a 0, a 1 we combine these matrix product as define this as n_i n_j by analogy with this is what we had started with the approximation has to be an interpolation between ϕ_i and ϕ_j.

So, that will be $\hat{\phi}$ within an element x is equal to $\phi_i N_i + \phi_j N_j$. And this is what it is so ϕ i ϕ j. So, this product; obviously, the two terms of this product should naturally be N i and N j and N i and N j can be evaluated once we evaluate this inverse and this is what it will lead to. Now, this N i N j as I we can multiply this we will find that N i is equal to x j minus x divided by x j - x i and N j will be equal to x - x_i divided by x_j - x_i. Now this can be also evaluated very simply very easily by using Lagrange interpolation formula;

for linear polynomial its really trivial and we get the same result by using Lagrange interpolation formula as N i = to (x $_j$ – x) / x $_j$ - x $_i$ and (x - x $_i$)/ (x j -x i) that is N j.

So, for so good so we have defined these interpolation functions in terms of the primary variable at the nodes and with this, the approximate solution that we need within the element is now available with us. Now once we have this approximate solution we can look at what is the weighted residual statement what happens to the weighted residual statement this.

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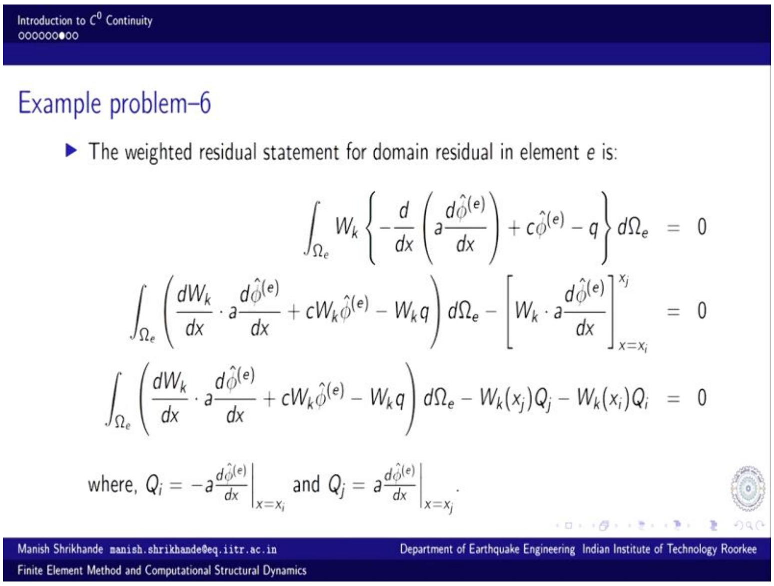

And we go back to the weighted residual statement only the domain residual. We will come to the boundary conditions when we assemble the contribution from the all elements when we look at the complete assembly. So, at the element level we only look at the domain residual in the weighted residual statement. So, when we say domain residual weighted residual statement for the domain residual.

So, this is our familiar some weighting function W $_k$ and this is the residual once we substitute this approximation phi hat e and this becomes the domain residual within the element and domain of the element is denoted by Ω_e; in this particular case it ranges from x $_i$ to x $_j$ for element e is defined by nodes i and j. So, domain Ω_e ranges from x $_i$ to x $_j$.

So, carrying out the weak form developing the weak form this derivative will get transferred on to W $_k$ and eventually this is what we end up with. So, first derivative of W $_k$ multiplied by this derivative term and then the simple integration of W $_k$ and other terms. And in addition we have this boundary term coming up and once we substitute these evaluate these boundary terms at x i and x j this boundary terms at the two ends and this is what we end up with. So, we denote this secondary variable as Q j and Q i when valuated at x j and x i. So, Q j is equal to a times d ϕ / d x and Q i is equal to -a times d ϕ / d x at when evaluated at x i. So, with this notation this is what the weighted residual statement looks like. And now all that we need to do is invoke Galerkin approach and consider interpolation functions N i and N j one after the other as weighting functions and then once we have this. I mean you can see that the highest order derivative is 1. So, a first degree polynomial should be should have some representation in this equation. So, that is a minimum requirement and that is why we develop this two node approximation or linear polynomial approximation for the unknown variable.

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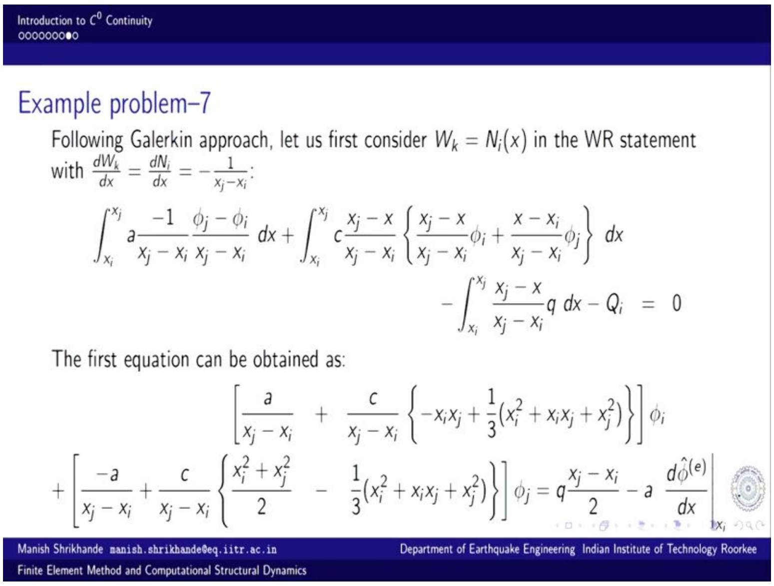

So, following Galerkin approach we first consider W $_k$ as N $_i$ in the weighted residual statement and; obviously, the first derivative d W $_k$ / d x is equal to d N $_i$ / x d x and that is equal to a constant term. If you look at the basic variation function variation so N i varies from unity to 0, a linear variation, so the derivative is going to be constant.

Similarly N j and derivative is going to be this is negative gradient. So, the derivative is going to be negative value slope. And similarly N j is on the ascending radiant and that is also going to be constant. So, that is what leads to. So, d N_i /d x is a negative constant derivative and once we substitute this in the weak form this statement d W_k / d x as d N_i / d x and then of course, substitute for approximation $\hat{\phi}$ as N i ϕ_i + N_j ϕ_j and this is what you get.

So, d ϕ / d x is actually going to be d N_i / d x multiplied by ϕ i and plus d N_j / d x multiplied by ϕ_j right. So, this is what yields to this thing.

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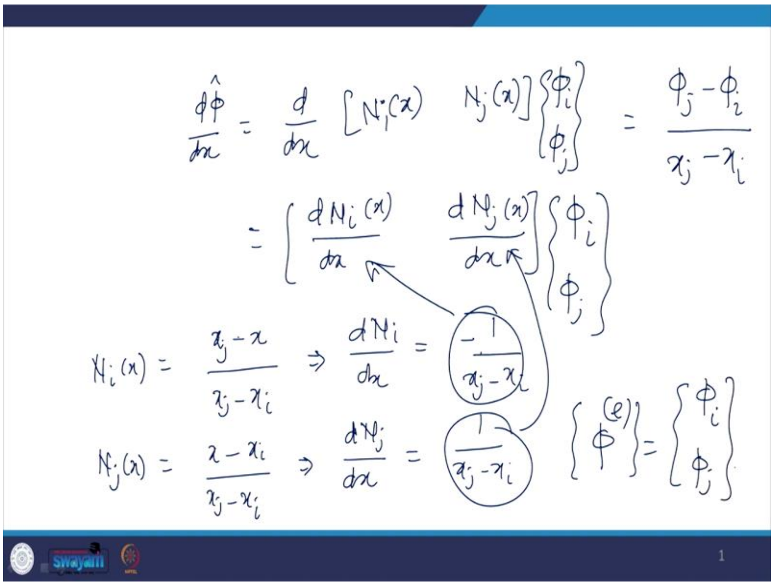

Now as we saw d N i / d x is a constant negative slope and d N j / d x is same positive constant slope and this is what is referred to I mean if you look at it N i x is given by x j - x / (x j -x i). So; that means, d N i / d x is equal to -1 / (x $_j$ - x $_i$). And similarly, N_j x is equal to x -x_i / x $_j$ - x i and therefore, the derivative is equal to 1 / (x $_j$ - x $_i$). So, when we talk about this d N i / d x. So, this term goes here and this term goes here and therefore, this term d $\hat{\phi}$/ d x is simply equal to ϕ_j - ϕ_i / (x j - x i) and this is what is written here ϕ j -ϕ_i / x $_j$ - x $_i$. So, that is the term d $\hat{\phi}$/ d x and then we have this W $_k$ representing N $_i$ and then again the complete $\hat{\phi}$ approximation.

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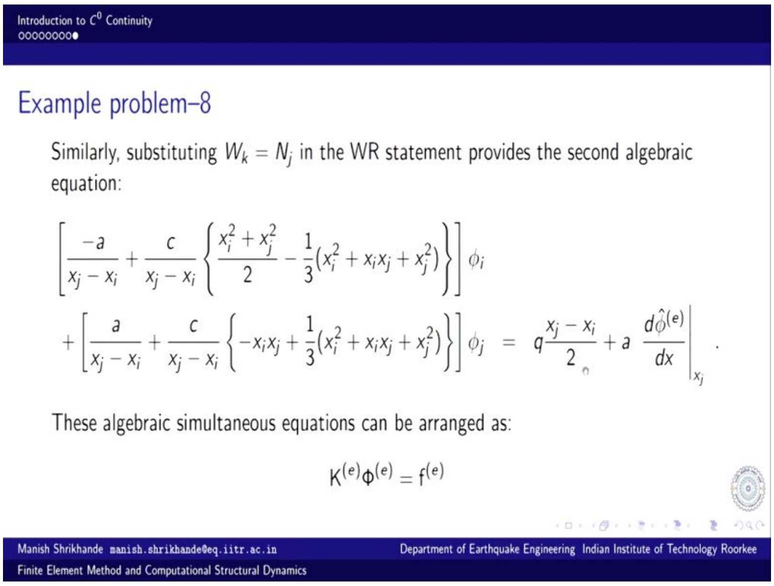

And once we evaluate this and Q i is of course, the boundary term that we have here. So, boundary term because W_k N_i; N_i will evaluate at to 0 at x_j. So, this term will vanish. So, this term will vanish and we will be left with only this term. So, N_i x_i is equal to 1 - Q_i. So, this is the term that we will have Q_j term will not figure when the weighting function is N i. Similarly, Q i term will not figure because N j will evaluate to 0 at x_i.

So, that is why this term remains q_j term does not figure and once we evaluate this is simple integral in variable x and this can be evaluated straightforward way. And once we evaluate little bit of integral evaluation of three integrals and we end up with this equation; say algebraic equation this coefficient of ϕ_I, this another coefficient of ϕ_j and then there is load term corresponding to the surface tractions q and the end term Q i.

So, this is the first algebraic equation that is obtained from the weak form of the weighted residual statement domain residual by taking considering waiting function as the interpolation function N i or shape function N i for the ith node. Similarly, if we repeat the whole exercise for W_k as N_j and again going back to the same weighted residual statement instead of W_k as N_i we now take W_k as N_j. Then go through this entire process again and this is what we will end up with the second equation by considering W_k is equal to N j in the weighted residual statement and we end up with a second equation second algebraic equation. And we can consider these two algebraic equation in as into as a set.

So, this is first algebraic equation and this is second algebraic equation we can combine these and they can be arranged as a system of algebraic simultaneous equations for element e and that can be arranged as K_e ϕ_e is equal to f_e; these are the right-hand side vector elements of right-hand side vectors ϕ_e represents a vector between. So, ϕ_e essentially denotes ϕ_i ϕ_j; the two primary variables of the problem at the in the node nodes defining the element e. So, i and jth nodes defining element e. So, primary variables at node i and node j. So, these are the basic definitions and K_e are these coefficients. So, this is the coefficient what we call as stiffness coefficients.

So, if I arrange these so we will see that there are two coefficients in the second row, there are two coefficients in the first equation. So, this is one coefficient this is another coefficient and this is one coefficient and this is another coefficient. So, together they form a 2 by 2 matrix which we call as element stiffness matrix. Stiffness matrix is a term borrowed from structural mechanics and if we consider this as an axial deformation problem then this becomes the axial stiffness of the element.

So, this is the element level equation and for element number e. Now the same exercise can be repeated now for next element let us say e +1. So, let us say if we did this exercise for element number 1 we obtain these equations we arrange these equations in some placeholder and then we move on to element number 2; again evaluate the same go through the same process and that will lead to similar set of equations.

And once we have that those can be arranged along with the equations for the first element and so on. So, this process what we are saying is known as the process of assembly. So, once what we have discussed so far is developing element level equation. So, the first step in finite element approximation finite element solution is the domain discretization, how do we discretize the domain into individual sub domains, non-overlapping sub domains contiguous. And then looking at the weak form of the solution weak form of the weighted residual statement we derive or we infer what kind of approximation is required. And depending on that we define the interpolation model and the primary variables are defined at the nodes of the interpolation nodes of the element.

And the approximation within the element is defined as interpolant of the primary variables at the nodes. And once we have the interpolation model those interpolation functions or shape functions of the nodes which define the particular element they are used as weighting functions in the weighted residual statement.

And subsequently it is the evaluations the integrals are evaluated definite integrals are evaluated over the element boundaries particular domain element. And then we can arrive at these element level equations and subsequently these element level equations are taken together to combine the approximation for the whole. That is something that we will discuss in our next lecture.

Thank you.