Finite Element Method and Computational Structural Dynamics
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Lecture - 13
Finite Elements of C^0 Continuity in 1-D-I

Hello, so today we start with formal introduction of Finite Elements per se using finite elements to solve problem of engineering analysis. And we begin with the simplest of all the finite elements of one-dimension, that is only one coordinate dimension is enough to describe the problem at hand. So, many problems of engineering in analysis can be adequately modelled with only one independent variable and that can be described by one coordinate direction.

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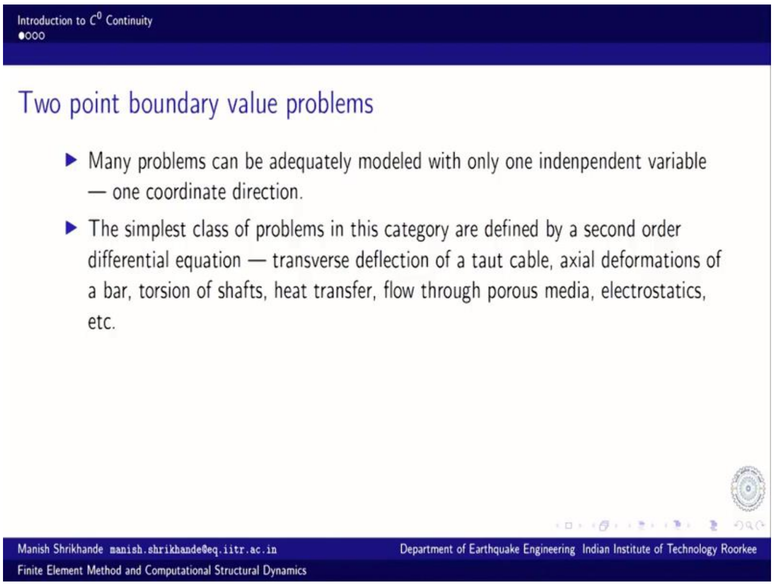

Now there are so many problems; the simplest class of this problem is defined by a second order differential equation. Similar to what we had seen earlier regarding describing the axial deformation of a bar or heat conduction in a rod or it can also describe transverse deflection of a taut cable or it flow through porous media or even electrostatics and several other problems.

The general structure of the problem is same and it is only the interpretation of various terms various elements of the equation that are; that keep on taking on new interpretation depending on the field of application. But mathematically the problem remains of same class, second order differential equation.

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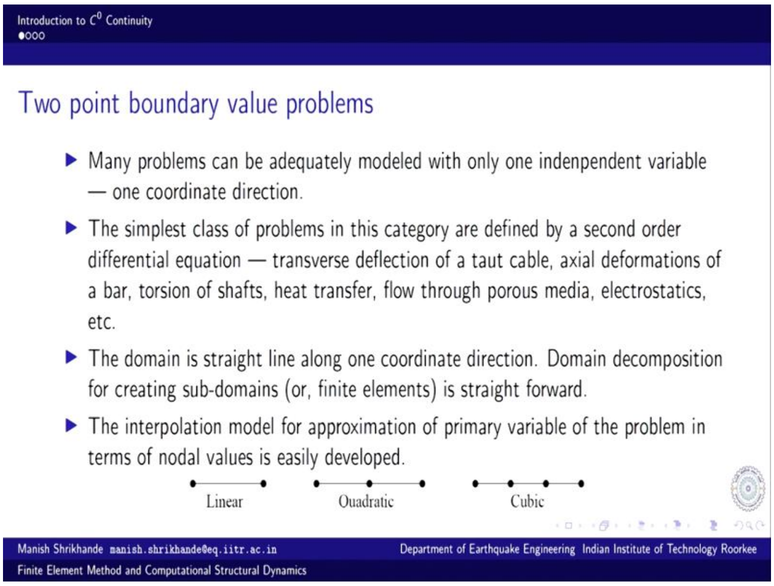

The domain is a straight line along one coordinate direction. And therefore, the domain decomposition as we discussed about the problem of finite element; the concept of finite element that we define the total domain of the problem as assembly or as union of several non-overlapping sub-domains or finite elements.

So, when the domain itself is a straight line, then sub-domains are merely line segments of part of those, part of the given straight line. So, if the length of the bar that we need to model is of L with x coordinate ranging from 0 to L, then we can define we can divide this domain problem domain into several smaller sub-domains of different lengths as the need maybe.

And the interpolation problem model, that we discussed earlier. The primary variable of the problem needs to be approximated and we adopt interpolation model and interpolation model for primary variable can be easily developed for one dimension and knowing the geometry. How that is done, we will see in a short while.

Now, polynomial interpolation; obviously, it degree of polynomial depends on how many terms are there, how many independent terms are there. So, a linear polynomial requires two terms, constant term and first degree term and it requires there are two unknown coefficients.

And obviously, for to know to define two unknown coefficients we need two constraints that can uniquely define this variation. And therefore, there are two nodes of interpolation that will be required to define a linear variation.

Similarly, a quadratic polynomial will have three unknown coefficients constant term, linear term and a quadratic term. And these three unknown coefficients can be evaluated by using three constraints and those three constraints are provided by the function values for interpolation at the node. So, there will be three nodes for quadratic interpolation and similarly for cubic.

So, these are the possibilities that we can have for one-dimensional interpolation. So, for a linear interpolation we will need two nodes for interpolating between these two values at the nodes; function values at the two nodes.

For quadratic, we will need function values at three nodes and for cubic variation we will need the function values at four nodes. And a unique polynomial variation of a desired degree can be determined by imposing these constraints at these nodes of interpolation. So far so good. So, nothing better than introducing a concept through an example.

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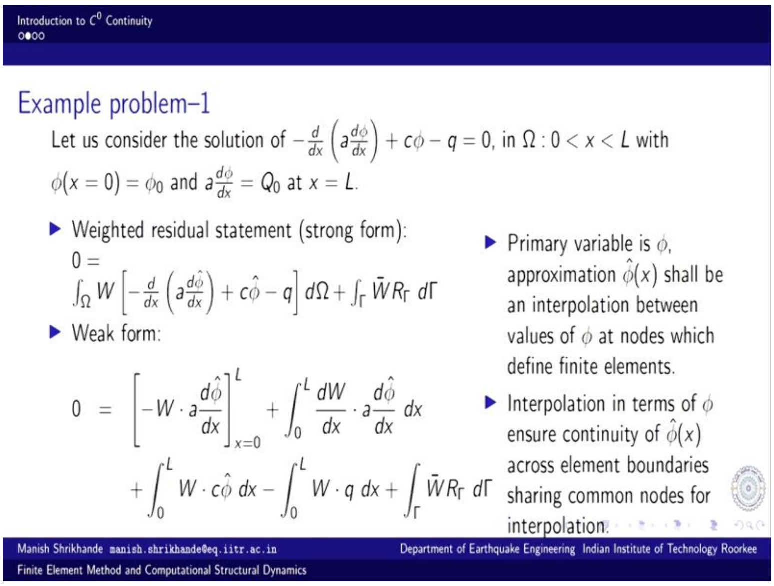

So, let us consider the solution of familiar differential equation; second order differential equation which can define as we said axial deformation in a bar or transverse deflection of a cable under tension or heat conduction and so on.

So, we have this second order differential equation the domain given by 0 to L total domain. And there are two boundary conditions one is at x is equal to 0, we have essential boundary condition, the function value is defined to be phiϕ_0 and at x is equal to L we have boundary condition that is prescribed on the secondary variable of the problem. That we will see, and that is at the other end of the domain. So, the domain is defined by two-end point. So, it is a two point boundary value problem that we are looking at. Now if we have to find the approximate solutions of basic unknown function we need to find the solution for this unknown ϕ.

So, if that is approximated by let us say $\hat{\phi}$, then obviously there is going to be some error in the differential equation and boundary condition. So, the weighted residual statement, the strong form of weighted residual statement that is we just substitute the approximation in the governing differential equation and the boundary condition and collect the terms. So, that is the strong form. So, this first term is the domain residual weighted statement of domain residual and this is the boundary residual term. I am not expanding on the boundary residual term at the moment, we will be concentrating on the domain residual term for development of finite element approximation.

So, if we perform the develop the weak form, weak form of the weighted residual statement from this expression it is just one step of integration by parts of this first term. If we integrate by parts by this first term then we will have this boundary term and the transfer this derivative will get transferred on to the weighting function. Rest of the terms are exactly identical, just expanded it. So, we are not touching any of these.

So, now, we look at this weak form. Now obviously, we discuss that the reason why this is called a weak form is the highest order derivative is decreased. While the strong form statement of the strong form had the second order derivative as the highest order of derivative on the approximation, approximating function approximate solution $\hat{\phi}$,; the weak form, the highest order derivative on the approximation is only first order.

So, the approximate solution need only be differentiable up to first order and it can still be considered as a valid approximation. So, obviously, that will have some implications on the smoothness and the derivative requirement etcetera, but we will see it can lead to very useful approximation and the approximations can be improved as progressively.

Now, looking at the boundary term, we again take request to this boundary term and interpret or try to identify what are the secondary variables of the problem and what are the primary variables of the problem. Now primary variables of the problem the weighting function term that is W. So W presents itself in its own form without any derivative on the boundary term, in the boundary term of weak form. And we replace W by the unknown of the problem that is ϕ. So, that ϕ becomes the primary variable. So, primary variable of the problem is the unknown function ϕ and this is the coefficient of weighting function in the boundary term that is the secondary variable. And therefore, boundary condition on the secondary variable this is the natural boundary condition at x is equal to L. Boundary condition imposed on the primary variable that is ϕ at x is equal to 0, this is the essential boundary condition of the problem. Now, as we saw the primary variable is ϕ and the approximation $\hat{\phi}$ has to be developed as an interpolation between the values of phi, the values of primary variable at the nodes which define finite elements.

So, finite element we will come in a little while. We just saw in the previous slide there are different types of finite elements, so one-dimensional finite element, which will have linear variation, quadratic variation and cubic variation. There will be for linear variation there will be two nodes which define the extremes nodes at the extremes of the element and that define the sub-domain which we refer to as finite elements.

For quadratic variation, there will be one mid side node in addition to extreme nodes. And for cubic there will be two inter internal nodes in addition to two extreme nodes right. So, interpolation in terms of ϕ. So, if we interpolate in terms of primary variable, then the approximation $\hat{\phi}$ that we are developing over each of the sub-domains as we will see. The continuity is ensured, continuity of ϕ is ensured because there is a node which is common, the elements adjacent elements they share common nodes, they share common boundaries. So, the variable of interpolation is common in that case. And at interpolation model of approximation as we know the function value is retained.

So, for one element on the left hand side of a node we have some function value and the same function value is enforced for the element on the right hand side of the node, at the common node. And therefore, the function value is common and therefore the approximation remains continuous.

So, the and that is what we call as 0 continuity, C $_0$ continuity. So, 0th order derivative is continuous, is guaranteed to be continuous across the element boundaries. And therefore, these elements that we will be discussing they are also referred to as finite elements for C_0 continuity. C_0 continuity refers to the case where the 0^{th} order continuity on the function of approximation, the primary variable of approximation ϕ is guaranteed.

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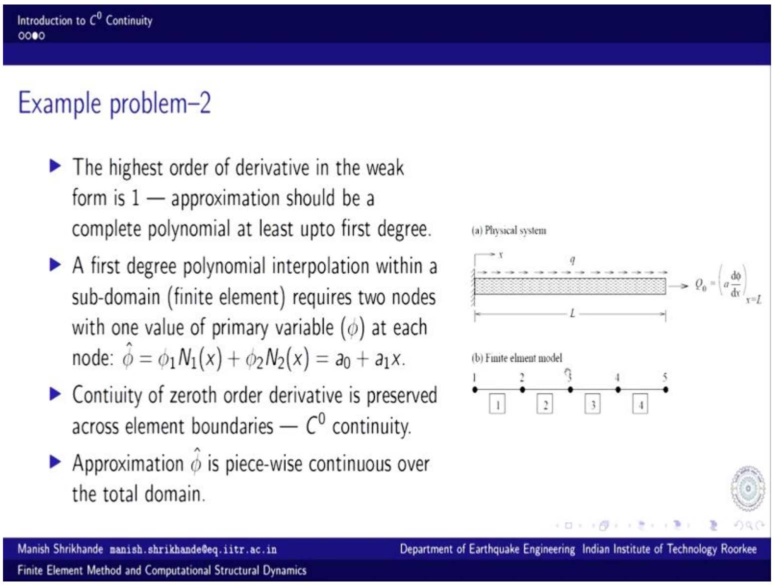

So, highest order of derivative in the weak form is 1 and that implies that approximation should be a complete polynomial at least up to first degree. So, this first order derivative should exist and the only way it can exist is that approximation should have at least one at least linear term, only then the first derivative will have some value. But that said it is required that approximation should be complete up to required degree. So that means, it is not enough, it is not acceptable just to have one term approximation of linear first degree. Approximation should keep on building up from the lowest order degree. So, starting from the constant term. So, constant terms are 0th degree and then the linear term. So, that makes the complete approximation complete polynomial. So, all the lower order terms lower order terms in the approximation they should have a representation in the approximation that we develop. So, if the highest ordered requirement is first derivative then first derivative should exist and that means, the linear variation is indicated and that implies a constant term should be there and a linear term should be there, at least at the minimum.

So, a first degree, polynomial interpolation within a sub-domain and that requires two nodes with one value of primary variable ϕ at each node. And then the interpolation model as we have seen becomes. So, $\widehat{\emptyset}$ within that element the variation of the unknown function ϕ can be given as interpolation model. So, $\phi_1 N_1 + \phi_2 N_2$. So, where one and two these are the nodes which represent the extreme nodes of the element; defining the element sub-domain. And N_1, N_2, they satisfy the basic interpolation requirement. That is N 1 evaluates to 1 at node 1 and it vanishes at node 2. Similarly, N 2 evaluates to 0 at node 1 and it evaluates to unity at node 2. So, that the function approximation when it is developed as this interpolation model, this function of this approximation evaluates to ϕ_1 at node 1 and it evaluates to ϕ_2 at node 2 and in between it is an interpolation linear variation between ϕ_1 and ϕ_2.

So, this linear variation can be general polynomial, it can be a general polynomial expression it can be represented as a two term polynomial that is one a constant term and a linear term So, that is the basic representation. So, these are two alternate representation of the same variation. A polynomial, first degree polynomial is a first degree polynomial, there can be various ways of representing it, but essentially the polynomial variation is unique once it is defined. So, the first degree polynomial is unique between two points.

So, continuity of 0th order derivative is preserved across the element boundaries. So, for one element we have this approximation and for the next element we will have similar kind of approximation. Those are independent approximation except that they will have adjacent elements will share node in common and once the node is shared the function value is preserved at that point. So, the continuity of 0th order derivative in this particular model of approximation is preserved across the element boundaries and this is referred to as C_0 continuity. And approximation $\widehat{\emptyset}$ is piece wise continuous. So, it is continuous over in element, it is continuous over the next element in between the boundary between these two elements the function value is continuous. Function value is compatibility of the function, compatibility of the approximation phi, primary variable is retained, but the gradient may not be; there is no constraint on the gradient. So, the gradients may differ.

So, the derivative of the function approximation at for one element might be something and for some other element just to the right of the node might be something else. So that is fine. So, that is why we call it piece wise continuous over the whole domain. So, total domain it is continuous over each element and it looks like an assembly of piece wise continuity and making up the whole domain. So, this is what the problem what we were hinting at.

So, total problem is defined by this domain 0 to x ranging from 0 to L and boundary conditions defined ϕ is 0 at x is equal to 0. So, that is the fixed support constraint. And ϕ is equal to ϕ_0, so it can be some constant at this point. And at of x is equal to L this is equal to this is the boundary condition imposed on the secondary variable of the problem.

Now, this is the physical system that mathematical model which defines a physical system. This first needs to be translated into a finite element model and finite element model requires domain decomposition. The first step to development of finite element model is to find out how many domains do we need to divide it into. And that is what we do here, we divide it into four elements so these are the in square boxes these referred to the element numbers. So, sub-domains or finite element numbers. So, this is finite element number 1. So, defined by nodes 1 and 2, these are the node numbers that they are numbered sequentially so that all nodes positions can be identified without any ambiguity.

So, each node is defined by its coordinate. So, node one is defined by co-ordinate x_1. So, that is x_1 is equal to 0 in this case node 2 is defined by the coordinate x_2, x_2 can be some value let us say L / 4 and node 3 is defined by coordinate x_3, which will be some value let us say L / 2 and node 4 is defined by the coordinate x_4 which is maybe 3L/ 4 and node 5 is the end point. So, that is x is equal to x_5 is equal to L. So, element 1 is defined by these two nodes; node 1 and node 2 that is all. Element 2 is defined by node 2 and node 3. Element 3 is defined by node 3 and node 4. Element 4 is defined by node 4 and node 5. So, now it is you can possibly see a highly structured arrangement here. If I know that it is all I am decomposing or this domain into smaller sub-domains of similar type of elements let us say two node element. So, each of these 4 elements is a two node element.

So, taken individually I only need to define two nodes which will uniquely define the element. So, and the nodes themselves are defined by the coordinates. So, it is a very structured system of information that makes it very convenient to computer programming and automatic computation.

So, this finite element model eventually leads to what we call as connectivity array or finite element mesh. So, that data, that connectivity array or the mesh data involves specification of which element is defined by which nodes and nodes are defined by their coordinates.

So, element 1 will be defined by x_1 and x_2. So, these are the coordinates of the nodes 1 and 2. Element 2 will be defined by nodes 2 and 3 and which are defined by which are specified by their coordinates x_2 and x_3 and so on. And in a generic sense it is the same type of element. So, that is being repeated all over.

So, the only thing that is changing here is the coordinate of the nodes, but otherwise the character of the element remains the same. It is two noded element and the variation is one degree first degree polynomial that is approximation.

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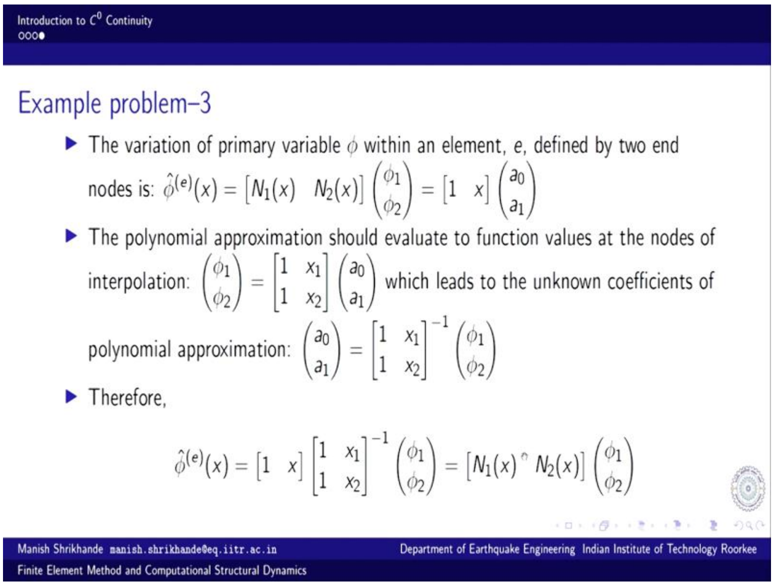

So, the variation of the primary variable phi within an element e; generic element that is what we are looking at if we isolate this one element here which is let us say generic element e which is defined by two nodes. Then the approximation is within that element e is given by interpolation model $\phi_1 N_1 + \phi_2 N_2$. So, this is what the interpolation model is and if I arrange it in matrix form this is what it will look like.

And this is of course, a complete linear polynomial $a_0 + a_1 x$. So, if I rearrange it into a matrix form and this is what that will look like. Now, this has to satisfy the interpolation condition. That is at x is equal to x_1, that is the first node of the element the approximation has to agree with ϕ_1, it has to yield I mean the approximation should evaluate to ϕ_1. Similarly at x is equal to x_2 the approximation should evaluate to ϕ_2. That is the condition of approximation as an interpolation model.

So if we impose that, polynomial approximation should evaluate to function values at the nodes of interpolation. So, ϕ_1 should be equal to $a_0 + a_1 x_1$. So, this polynomial variation when evaluated at x_1 I should get the value of function at node 1. Similarly ϕ_2 should be equal to $a_0 + a_1 x_2$. And this gives us the basic condition for evaluating the unknown coefficients a_0 and a_1 for the polynomial approximation within the element. And because this matrix is known the coordinates of the nodes are known a priory that is how we decided the nodes. When we discretize the domain converted the physical problem into an assembly of finite element. So, nodes are defined by their coordinates. So, these coordinates x_1 and x_2 are known to us and therefore, this matrix can be inverted. And as long as the points are distinct node 1 and node 2 there are distinct points then there is no way this will be singular, this inverse is guaranteed. And a_0 and a_1 can be identified by inverse of this coordinate matrix, and ϕ_1 ϕ_2 they are the function values.

Now this a 0, a 1 what we have evaluated we can substitute here back, in this equation polynomial approximation and that give leads us to the basic approximation of the problem that is the polynomial variation 1 x. So, the constant term and linear term multiplied by these coefficients. So, this is what we get it. And then when we compare this with the first structure ϕ_1 ϕ_2 then we would realize that the interpolation function N 1 and N 2 are essentially product of these two matrices. So, this is the polynomial term and this is the coordinate matrix; inverse of coordinate matrix. So, this is referred to as $p\, c^{-1}$. So, matrix of interpolation functions or shape functions as they are called they can be derived as a product of polynomial terms p and c inverse that is the matrix of coordinates.

So, this is how we can generate the interpolation model. Now these interpolation functions N 1 and N 2 they are evaluated, we know the functions given the coordinates x_1 and x_2. So now, all that we need to know is, what are the coordinates, what are the two points which define the element, what are the coordinates of those points and once I have the coordinates I am immediately can work out what is the interpolation function for each of these two nodes. And once I work out this interpolation function for each of these two nodes I can work out what is the approximation within that node.

So, how this works in individual elements, I mean this exercise we need to do it in individual elements and how this works out in the case of weak formulation. I mean we saw that in development of the weak formulation. Now we will work out this over each element we will take up this. In our last lecture we saw that the whole idea that this entire weighted residual statement which is defined over the entire domain. Now, we will consider it to be sum of residual statements over each of sub-domains. So, we will look at this weak form over each of the sub-domain. So, instead of talking about 0 to L maybe we will talk about x_1 to x_2 that is all.

So, the domain the boundaries of the individual elements and that is it. And from there we will develop the basic finite element level equations and we will see how it looks like. And then we will see how we can use these element level equations and assemble them to build the complete whole domain equations. So, that is what we will discuss in our next lecture.

Thank you.