Finite Element Method and Computational Structural Dynamics

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Lecture - 12
Introduction to the Finite Element Concept

Hello friends. So, last lecture, we discussed the variational principles and how the approximate solution can be constructed using variational principles. And if we work out the entire thing, then we will find that variational principle approximate solution derived using variational principles as long as the same admissible basis functions are used, are exactly identical to those obtained by weak form of method of weighted residual function, and method of weighted residuals using Galerkin approach.

Now, we look at how this concept can be developed into finite element method and why there was a need for development of finite element method per se. As we saw that these approximate solutions have the potential to capture the true solution; as we keep on increasing the number of terms in the series, it becomes more and more accurate and it has the potential to capture the true solution in its possibilities.

So, the true solution is indeed captured in the form of the approximation at least in theory. But in practice it is difficult to achieve, particularly first reason being the trial functions have to be admissible functions; that means the essential boundary conditions have to be satisfied, only then the convergence, monotonic convergence can be guaranteed.

And this basic essential constraint is difficult to satisfy in case of irregular domains. If the domain problem domain is not so simple as a straight line and the boundary only being the two end points; then it is easy to satisfy, easy to construct the functions. But if they even for one dimensional problem, if the domain was not a straight line, but a curved line; then it becomes that much more difficult to construct approximation admissible functions, which will satisfy the essential boundary conditions.

So, even for one dimension instead of being straight line; if it is a curved, curvilinear problem domain, then it becomes a problem. And another complication can be the

domain itself may be non-uniform; for example, in the bar axial deformation problem, we discussed the actual rigidity was uniform throughout the domain.

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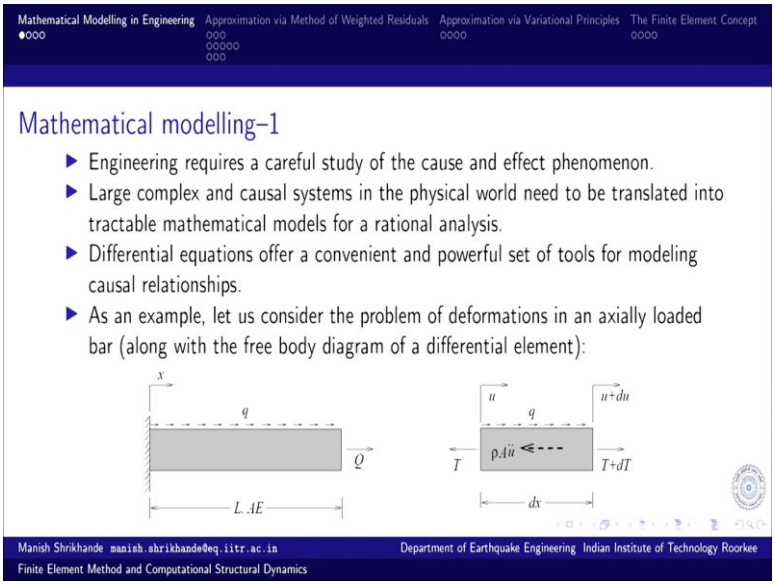

For example, in this case. So, axial rigidity A E is uniform and that is what allows us to construct the same approximation over the whole domain. If some of these things are not uniform; let us say cross section is not uniform or different materials appear in different parts of the domain, so Young's modulus might change. So, in that case it becomes difficult to construct an approximation, which will hold over the entire domain.

So, although in principle it is possible to develop approximate solution and true solution is in principle contained in the possibilities, various possibilities admitted by the approximate solution; it is very difficult to implement for a general case, where the domain may be very irregular or the boundaries may be very irregular or the material may be heterogeneous, not non homogeneous, non uniform.

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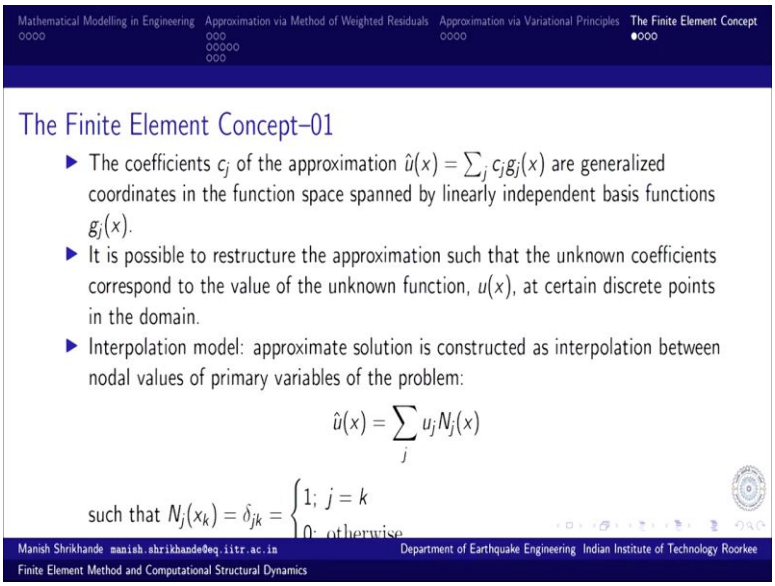

So, we look at another issue that the approximation that is of the form c_j g_j. The coefficient c_j they are generalized coordinates of the function space spanned by the basis functions g_j. So, g_j they are scaled by these coefficient c_j. So, individual importance or contribution of individual basis function is scaled by this coordinate c_j.

Now, it is possible to restructure this, I mean as long as once we have this linear combination; the function the terms can be rearranged and some factors are taken common and some factors, some terms are combined with others, such that the approximation can be converted into a form that these unknown coefficients they may correspond to the value of the unknown function, let us say u_x at certain discrete points in the domain. And when we do that, so what we are looking for is, trying to replace this generalize coordinate by the value of the function itself at a certain point in the domain. So, some point x_j in the domain of the problem and then we have this c_j replaced by u_j. So, u_j is the value of the unknown function at x is equal to x_j. So, if we do that, I hope you are getting the hint that I am trying to develop here.

Once I have this unknown coefficient as the function values, then immediately I identify the possibility of constructing approximate solution as interpolation problem. I can define the approximate solution as an interpolation between the nodal values, I mean the nodes of interpolation which are distributed over the entire domain; different points in the domain, so I define the unknown value of the unknown function at those nodal

points. And then define interpolation functions, which will interpolate between these nodal values. So, the approximation function can be represented as an interpolation model, that is summation over all the node points j. So, nodal values u_j multiplied by interpolation function corresponding to node j. Now, interpolation function we discussed earlier, they have to satisfy this interpolation condition that $N_j (x_k)$ should be equal to $\delta_{j\,k}$.

So, N_j will evaluate to 1 only when x is equal to x_j at all other nodes, only at the nodes; at all other nodes, it has to vanish. So, if j is not equal to k, then $N_j (x_k)$ should evaluate to 0. In between nodes, it can have anything; there is no restriction, it can be 0, it can be non zero, it can be greater than 1, it can be less than 1, it can be less than 0 whatever. There is no restriction on the variation of N in between the nodes; but at the nodes it can have, it should evaluate the unity at its native node. So, N_j should be equal to 1 at x_j and N_j should evaluate to 0 at x_k that is at any other node other than j. So, that is the basic definition of interpolation function.

Now, this interpolation model conceptually very simple and very appealing also; because c_j they do not give us any meaningful idea, unless I compute c_j after the process of approximation and then substituting this approximation form, only then I will get the the interpretation of approximate solution. But if I somehow translate the problem by replacing c_j in terms of u_j, then immediately after solution I get the solution of the unknown. So, immediately I know the solution at certain discrete points in the domain; what is the value of the unknown. If I need values at some point in between again, then I need to go back to the interpolation model; otherwise there is no need, I already have the function values known at several points within the domain and that is usually good enough for our purposes many times.

So, that is very attractive proposition, but that hits the hurdle of difficulty of constructing interpolation function, which will be valid over the whole domain, particularly if the domain is non uniform or very irregular boundary. So, it is easy to construct interpolation for a very nice regular rectangular kind of domain or circular kind of domain. So, coordinate system is very well defined and it is possible to develop interpolation functions, if the node points are located within the at the grid points.

So, I can locate the nodes at the grid points and it is possible to derive the interpolation functions very simple by Lagrange interpolation formula, very trivial thing to do. But the

moment you have, the moment we look at more complicated domains; the problem becomes more and extremely challenging, it is not possible, it is not so easy to construct interpolation functions, which will be valid over the entire domain.

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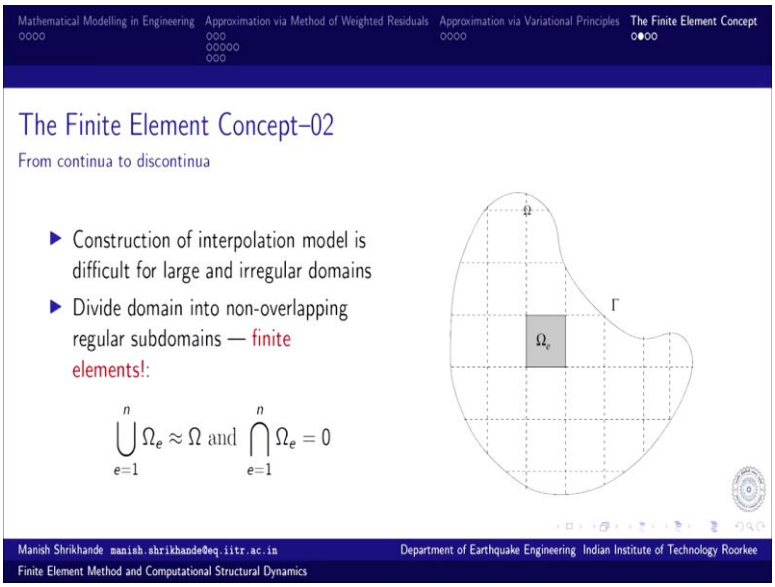

So, what to do? So, the solution is very simple, we divide the domain. So, divide and conquer that always works, this strategy never fails. So, since the construction of interpolation model is very difficult or may well naively impossible for large and irregular domains; we look at the possibility of approximating the total domain by superposition of smaller sub domains and these sub domains are of regular geometry.

So, it is same thing like saying any arbitrary curve no matter how crooked the curve may be; if I look at small enough portion of the segment, I can always approximate it by a straight line. So, it is similar thing to that, whatever may be the domain; if I can break it down into smaller part, sufficiently smaller parts and look at the assembly of the smaller parts, each of the smaller parts can be a regular shape, whatever shape I may need, it can be square, it can be triangular or whatever.

So, the basic idea is we divide this arbitrary domain which is the area or domain encompassed within the boundary gamma and divide this into smaller sub domains, non overlapping sub domains that is the important part. So, each of these sub domains is we denote as Ω_e. So, that is the sub domain or representation of the sub domain.

So, total union of these individual sub domains is taken as an approximation for the total domain omega. And this union or this subdivision partition has to be in a way that there is no overlapping. So, the intersection between individual sub domains is 0. So, there is no intersection, there is no boundary; there is no common area or common portion, which is covered by two different sub domains.

So, then the problem suddenly becomes much more amenable, much more comfortable to deal with.

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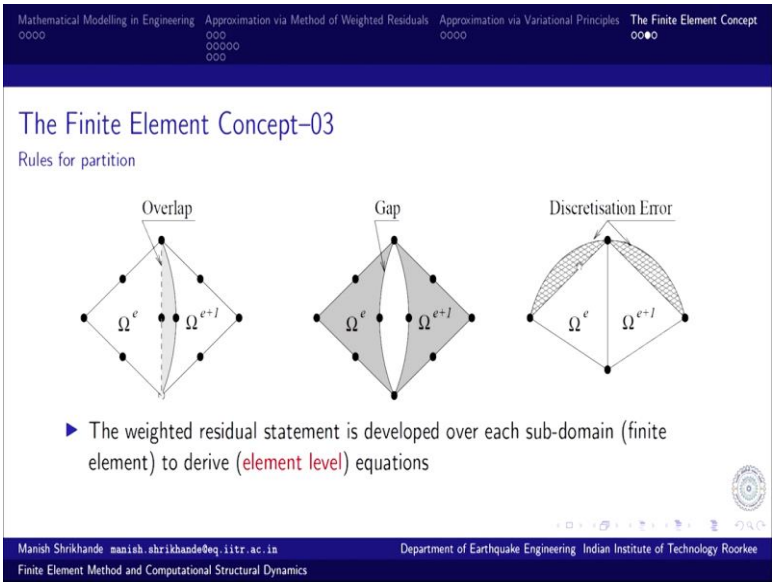

And that is what we have. So, there are certain rules of partition, as we say there are I mean we divide the entire domain into smaller sub domains and these sub domains this is what we call as partition of the domain, domain partition. And there are rules of partition, such that two adjacent domains, they cannot have any overlapping portion.

That is what we mentioned mathematically also, there cannot be any overlap between these two any of the domains; but in addition there cannot be any gap between thetwo adjacent domains. So, if this was the total domain, the domain partition rule requires that this gap should also be represented; we cannot leave this gap out of the consideration, because then no matter what I do, union of Ω^e and Ω^{e+1} will never be close to omega, the total domain of the problem.

So, this is not allowed, overlap is not allowed, gap between two elements two sub domains is not allowed; but whenever we approach the boundaries, the curved boundaries that we see here. So, in this case there is no way, I can model these curved boundaries as you can see; I cannot have this domain, I cannot have this rectangular domain over the square domain at this boundary.

So, I need to model this by a smaller rectangle here and a triangle here. So, that way I can represent this and when I do that. So, no matter what I do, it will always have some amount of discretisation error; I cannot exactly represent a higher degree curve by a straight line, right. So, if it is a higher degree curve and I am representing it by a straight line, some part of the domain is being left out.

Now, this discretisation error as you can see, as we call it; this can be reduced if I reduce the size of this particular segment. So, if I model this by several smaller straight lines, then it is possible to reduce this discretisation error. So, discretisation error can be reduced, but it can never be eliminated in these straight edge kind of configuration. We will come to more on this straight edge and modelling this or how to minimize this discretisation errors in finite element formulation as we go along.

But this is essentially to explain the concept of finite elements that, the entire the need to develop interpolation model of approximation is very intuitive and very appealing for analysis; because the solution will give us the values of the unknown function that we are looking for at different points in the domain. And for that we need interpolation model, interpolation model is difficult to construct over the entire domain at one go. So, we look at possibility of approximating the whole domain by a superposition of smaller sub domains or addition of sub domain. So, the one large whole is divided into smaller and smaller parts. So, we first divide whole into part and then construct solution over each part and then construct the solution for whole from the assembly of each part. So, two way process; whole to part and then part to whole, borrowing the term from triangulation as in survey.

So, the weighted residual statement. So, as I say, we have these different sub domains. So, we develop the weighted residual statement for the differential equation, approximate solution of differential equation over each of these sub domains and evaluate those,

evaluate the integrals and transform those weighted residual statements into algebraic equations and those are called the element level equations.

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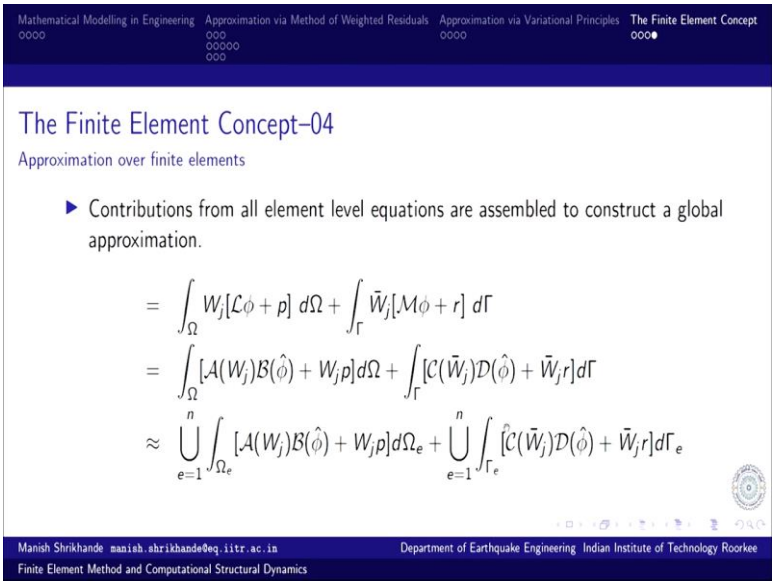

And once we have these element level equations, those element level equations are assembled to construct a global approximation. So, in operator form. So, the total domain represents this domain residual term, weighted residual of domain residual term and the weighted residual of the boundary residual term.

So, this is the total domain and this is of course, is equal to zero the weighted residual statement. And if I perform the integration by parts, develop the weak form; then this operator L changes to operator A and B, which are in general lower order operators than L. And operator M changes to operator C and D, which is again lower order operator than M if required or they may be same depending on the problem. And then there are these weighting function and the unknown function and terms. So, this is the domain integral and this is the boundary integral. Now, this can be represented. So, this domain integral can be represented as union of integral over each of the sub domains. So, union over each of the sub domains. So, Ω replaced by summation over Ω^e. So, Ω^e sub domain, each of these sub domain or finite element. So, I construct, I construct I develop these approximation over each of these finite elements and this also part of these finite elements individually, combine all of them, all these terms and I evaluate these integrals and I end up with algebraic equations in the in terms of the unknowns of the problem and

that and those unknowns will be the values of the function, unknown values of the function at discrete points, which are the nodes of the problem. And once I assemble, once I have these equations; I go from one element to another element element e is equal to 1, I develop these equations, algebraic equations. I go to next element e is equal to 2, I develop another set of equations and so on; I develop all these equations over the entire domain. So, Ω divided into several sub domains Ω^e, e ranging from 1 to n. So, n number of elements I develop these algebraic equations and then they are brought together by ensuring the compatibility conditions and compatibility and continuity conditions. And eventually we end up with a global system of equations, which can be then solved after imposing the boundary conditions of the problem, essential boundary conditions of the problem. And once we have that, we know the unknown function at discrete points in domain, distributed over the domain and if required, we can always go back to individual element and interpolation model for that element to determine the value of the unknown at any point in the domain, which is not coincident with the nodes of the problem.

So, that in a sense describes the concept of finite element and we will start with simple problems; I mean the we will take with the start with the one dimensional axial deformation problem, develop finite elements of one dimension. And we will go through the entire process what we have discussed here, weighted residual statement for the whole domain divided into weighted weak form, develop the weak form. And then evaluate these weak form statements over individual sub domains. And once we have done that, we look at what are these element level equations; assemble those element level equations to develop the global approximation or global system of equations and then impose the boundary conditions and solve the system of equations. So, basic recipe remains the same; irrespective of whether it is one dimensional problem, two dimension problem, three dimension problem. The only thing that changes is the interpolation model. So, as long as we are consistent, we know how to interpolate, how to develop the interpolation model in one dimension, two dimensions, and three dimensions; the rest of the process is entirely automatic and seamless.

So, we will try to, we will develop these finite element approximations gradually and in next lecture, we start with finite elements of one dimension and with particular reference to the bar problem; it can be also looked at as a heat conduction problem. So, that is all for today and next lecture, we will start with finite elements in one dimension.

Thank you.