Finite Element Method and Computational Structural Dynamics
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Lecture - 11
Approximation via Variational Principles

Hello, friends. So, in our last lecture we discussed about the Weak form of method of weighted residual – how it allows us to reduce the continuity requirement from the approximation that we construct for the approximate solution by transferring the derivative on the weighting function.

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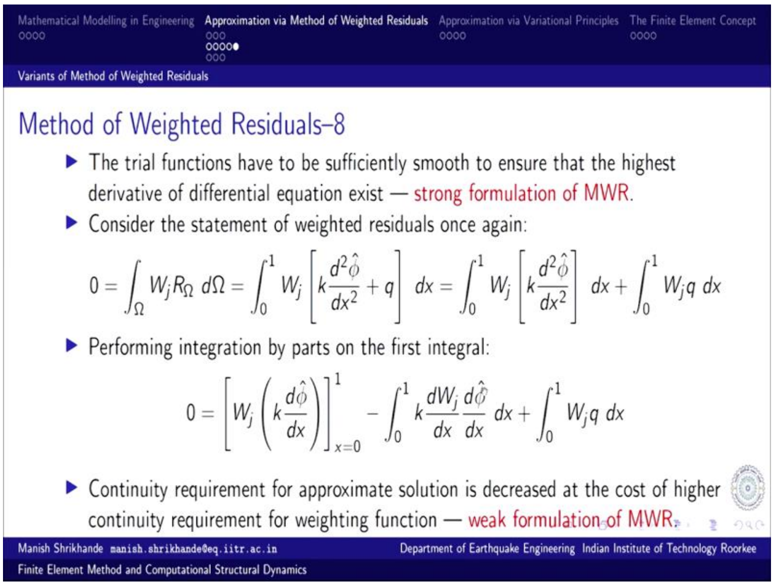

So, now although in the strong form the unknown had to have continuity of at least twice differentiation. In the weak form, the continuity requirement is reduced by one order by at the cost of increasing the continuity requirement on the weighting function. So, this derivative here on the approximation is transferred to the derivative on the weighting function.

So, this is what we call as weak formulation of method of weighted residual because a weaker continuity in approximation is implied and a working approximation can still be constructed by even if the approximate solution is not sufficiently smooth as dictated by the governing differential equation.

Now, this weak formulation also has some interesting interpretations and particularly we look at this boundary term. So, weak formulation weak form of method of weighted residual is distinct from the strong formulation in the sense that we have a boundary term. And boundary term is always in pair in a set of pairs.

So, it is a product of two terms. So, one term involving the weighting function and there is some other term that we can say coefficient of the weighting function term. So, weighting function term can involve either the weighting function itself or a derivative of the weighting function that depends on the what is the original order of differential equation of the problem.

So, this particular case is a second order differential equation. So, the boundary term in the weak formulation involves only the weighting function without any derivative.

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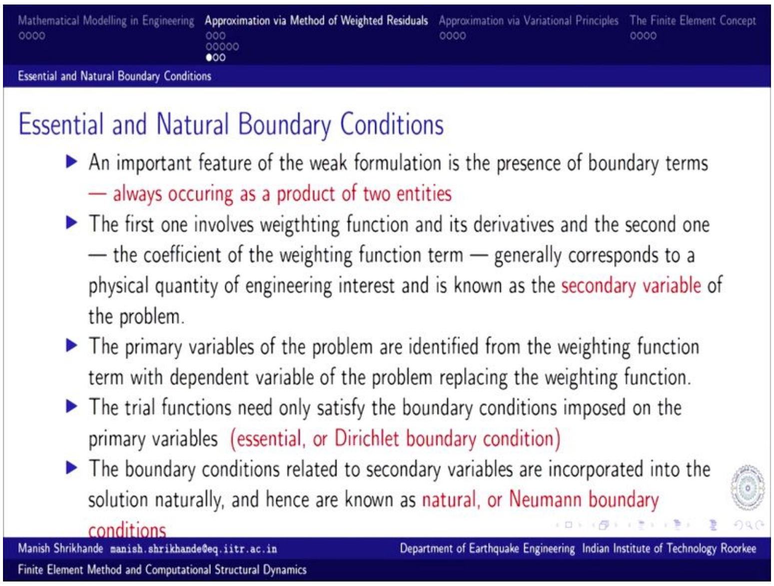

So, the presence of boundary terms that always occurs as a product of two entities and the first one involves weighting function and its derivatives and the second one, the coefficient of weighting function term it generally corresponds to a physical quantity of engineering interest and it is referred to as the secondary variable of the problem.

So, in this particular case we say here $k\frac{d\emptyset}{dx}$. So, if we relate it to axial deformation problem then this particular term k is equivalent to the axial rigidity and this total quantity, then represents the axial thrust in the bar and similar interpretations exist for other domains of application wherever this second order differential equation governs. So, that is the secondary variable the which is physical quantity of engineering interest, we need to design the system such that the axial thrust does not exist or does not exceed the yield force in the system in the material and that is why it is said to be of engineering interest, it is directly relevant to the problem of design.

And, the primary variable of the problem are identified from the weighting function term. So, the boundary term which is associated with the secondary variable so, this is the weighting function term. So, by replacing whatever form the weighting function is we if we replace this weighting function by the unknown of the problem let us say $\emptyset$ so, in this case the weighting function appears as it is. So, the primary variable of the problem would be just $\emptyset$. So, the basic unknown of the problem is the primary variable. So, primary variables of the problem are identified from the weighting function term with dependent variable of the problem replacing the weighting function.

So, in this particular case; the primary variable is the basic unknown of the problem $\emptyset$. The trial functions need only satisfy. So, what we call as I mean earlier we while constructing the trail approximation we talked about admissible set of functions, the basis functions should be admissible and we discussed that the trial functions they should satisfy the essential boundary condition. Now, the essential boundary conditions are those which are imposed on the primary variables of the problem. So, in order to identify which boundary conditions are essential and which boundary conditions are natural, we need to look at the boundary term of the weak formulation of method of weighted residual and then identify secondary variables of the problem and primary variables of the problem then the boundary conditions which are specified in terms of the primary variables of the problem those are the essential boundary conditions. And, any construction any approximation that we develop has to be based on using basis functions or trial functions which satisfy these essential boundary conditions which are imposed on in terms of the primary variable.

The boundary conditions which are related to the secondary variables are incorporated into the solution naturally during the process of solution as we will see in a short while and these are known as the natural or Neumann boundary conditions because they are incorporated naturally. So, we do not have to worry about it. They will be approximately satisfied and the quality of approximation we will keep on improving as we improve the degree of approximation.

So, let us look at this problem of essential boundary conditions and natural boundary condition.

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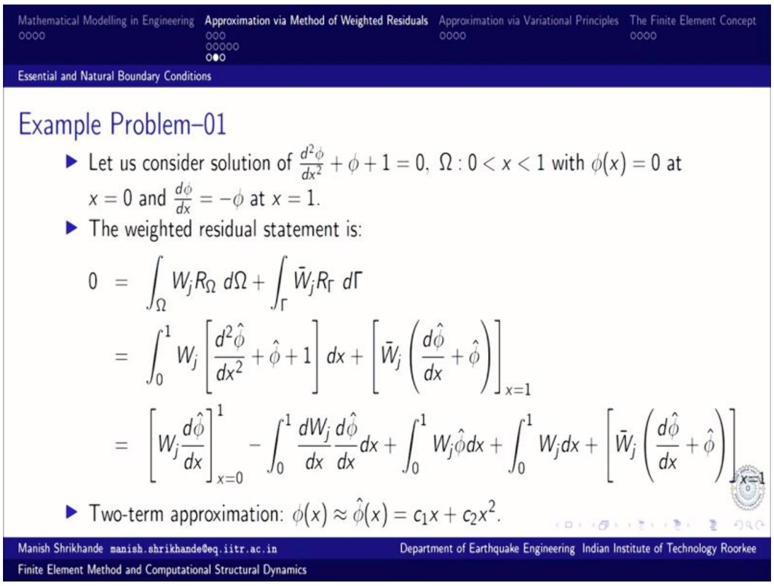

A simple problem to second order differential equation $\frac{d^2\emptyset}{dx^2} + \emptyset + 1 = 0$

Domain is linear domain from x ranging from 0 to 1 and the boundary conditions are specified at $\emptyset$ at x is equal to 0 is 0 that is the essential boundary condition as we will see and the other case we have $\frac{\partial\emptyset}{\partial x}$ is equal to -$\emptyset$ at x is equal to 1. So, at x is equal to 1 we have a natural boundary condition. So, the weighted residual statement if we can construct, so, we can work out our formulation, we can develop the weak state weak formulation of the method of weighted residual statement and look at the boundary term.

So, $\emptyset$ is the primary variable of the problem and $\frac{\partial\emptyset}{\partial x}$ is the secondary variable of the problem and therefore, this particular boundary condition is the essential boundary condition and this boundary condition is the natural boundary condition.

So, we have to satisfy the essential boundary condition. So, $\emptyset$ at x =0 has to vanish it has to be equal to 0 and therefore, we can choose a two-term approximation which is $c_1x + c_2x^2$. So, this approximation vanishes at x is equal to 0 and therefore, it satisfies the essential boundary conditions and therefore, it is an admissible, both of these basis functions they are admissible basis functions. Now, once we substitute this work out little bit expand these two terms boundary condition we have one boundary term here and another boundary term here coming from the boundary residual. So, we have one boundary term coming from the domain residual by carrying out integration by parts, developing the weak form and then we have boundary residual coming from the boundary term coming from the boundary residual term of the weighted residual statement.

Now, we identify that there is similarity of terms here; here we have $\bar{W}\frac{\partial \emptyset}{\partial x}$ and here we have $W\frac{\partial \emptyset}{\partial x}$ and this is at x is equal to 1; this is at x is equal to 1 and x is equal to 0. So, since W and $\bar{W}$we discussed that these are arbitrary weighting coefficients. So, they are arbitrary, it can be anything. So, if it can be anything I might as well choose the terms the weighting functions in such a way that some of the terms they cancel out. So, for example, we have this term $W_j\frac{\partial \emptyset}{\partial x}$ here and I have another term $\bar{W}_j\frac{\partial \emptyset}{\partial x}$. So, if I choose W j is equal to -$\bar{W}_j$, then these terms would cancel each other and I am allowed to do so. Because W and W bar they are arbitrary functions I, those can be anything.

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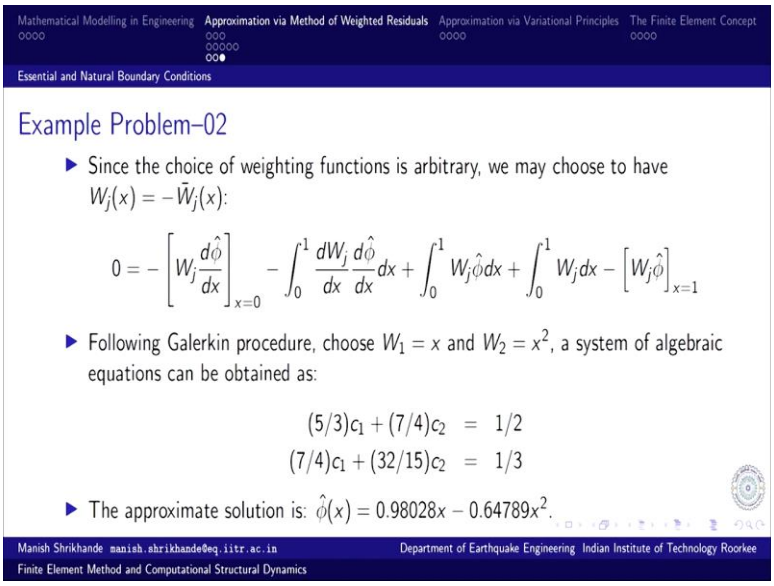

So, since the choice of weighting functions is arbitrary, we may choose to have W j is equal to minus of $-\bar{W}_j$ and once we do that; then two terms cancel out and then we are left with a simpler expression with fewer number of terms. If we do not choose we choose anything else no harm done, it will just be a little more complicated expression that is all.

So, after in this expression once we have this $\hat{\emptyset}$ approximation c_1 x + c_2 x^2 and then we can choose weighting function in sequence W_1 as x and W_2 as x^2 following Galerkin procedure. And, then all the terms are pretty well known it is a definite integral and all the integrals can be evaluated and all derivatives are taken care of and we end up with a set of algebraic simultaneous equations in terms of unknowns c_1 and c_2.

As you can see with because of use of Galerkin approach with weak form we end up with a symmetric system of equations and we can solve for this and c_1 can be evaluated as 0.98 and c_2 is evaluated at - 0.64 and0.65. And, the approximate solution is substitute c 1 and c 2 in the approximate solution and we have the desired approximation.

And, that is the one quality of approximation two-term approximation, the degree of approximation can of course, be improved if we add more number of terms. So, progressively the approximation quality can be improved and the error in approximation can be reduced as we add more and more number of terms I mean a cubic term, quadratic term and so on.

So, that more or less completes our discussion on method of weighted residual, Galerkin approach. We discussed why we use Galerkin approach, I mean it is much more the whole idea of using Galerkin approach weak form is to I mean added advantage is it leads to a symmetric system of equations which has a very desirable property. I mean it is very stable system of equations that very well conditioned system of equations and that it has very good numerical properties and very stable numerical solution result from it.

Now, we look at another method of approximate solution that has been in use for a long long time. So, these are the two parallel stands that had been going on method of weighted residual is approximations of via a method of weighted residual has been the preferred root of the mathematicians, the mathematical fraternity and the variational principles. They were more of applied mathematical fraternity or engineering and scientists I mean the they were more tuned towards the use of variational principles.

But, the purpose my intention here is to establish equivalence between the two approaches wherever they exist, wherever variational principles exist for a first issue is variational principles may not exist for all problems and wherever they exist the Galerkin weak formulation method of residual weak formulation of the method of weighted residual is synonymous. I mean it is exactly parallel with the variational principles approach.So, it is the system of equations that we end up with are exactly identical.

But what is a variational principle? So, variational principles they are based on something called a functional and what is a functional? It is a mathematical concept it can be said as a function of functions, not a function. It is a function of functions.

So, what is a function? Function maps a set of variables onto a scalar. So, essentially one function can have several independent variables. But, for a set of independent variables the function maps those variables on to a single number.

So, it is a mapping. If we extend this concept by replacing independent variables, by a set of different functions then we are looking at a something called something that we can look at as a function of functions and that function of functions is mapped on to a scalar and that scalar is called a functional.

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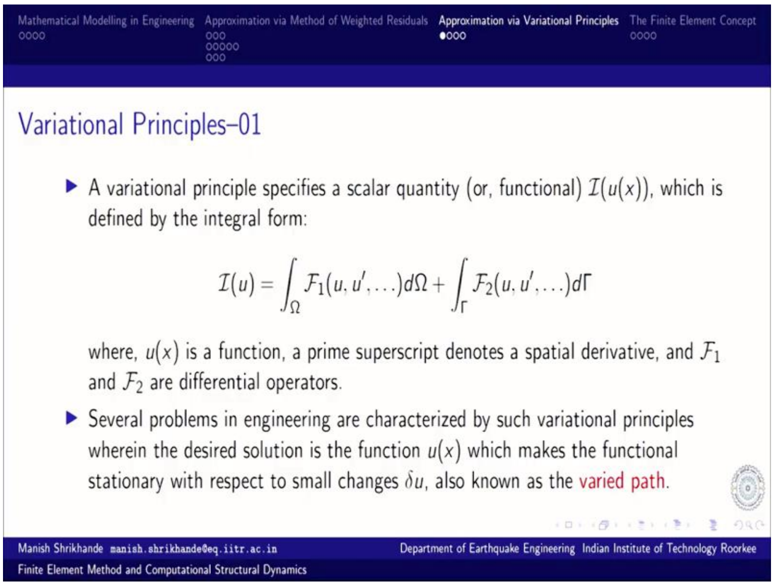

So, that is what it is denoted here in a operator form, integral form. So, u is a function or class of functions is given by a domain integral over some domain. F_1 is some differential operator; it is some operator and it comprises of it operates on several functions u, u prime and so on. So, this that means, u is a function, u prime is the first derivative of u and u double prime can be second order derivative of u and so on. So, it is a quantity which depends on function u and its derivatives and evaluated over the domain. And, then there is also a boundary evaluation, boundary term another operator and again depends on several functions and its derivatives.

So, several problems in engineering are characterized by such variational principles wherein the desired solution to the problem u is the unknown basic unknown of the problem that we need to find, but the desired solution the true solution u of the problem is such that the functional that we are looking at has an extreme value; extreme value means it is either maxima or minima and as we all know for anything to be maximum or minimum it has to have the stationarity condition.

At the point of maxima it becomes stationary; stationary means it becomes invariant to small changes in the whatever influencing parameter it has. So, when the functional becomes stationary with respect to small changes delta u, delta u is referred to as the varied path. So, these are the small changes from the true solution. If I am in the neighborhood of the true solution then little bit here and there is not going to significantly influence the functional value. So, functional value is going to be more or less stationary in that neighborhood. So, that is what distinguishes, that is what identifies that the first criteria first requirement for extreme value, any quantity reaching its maxima or minima is that it has to become stationary with respect to its parameters. So, in this case it is parameter is the varied path.

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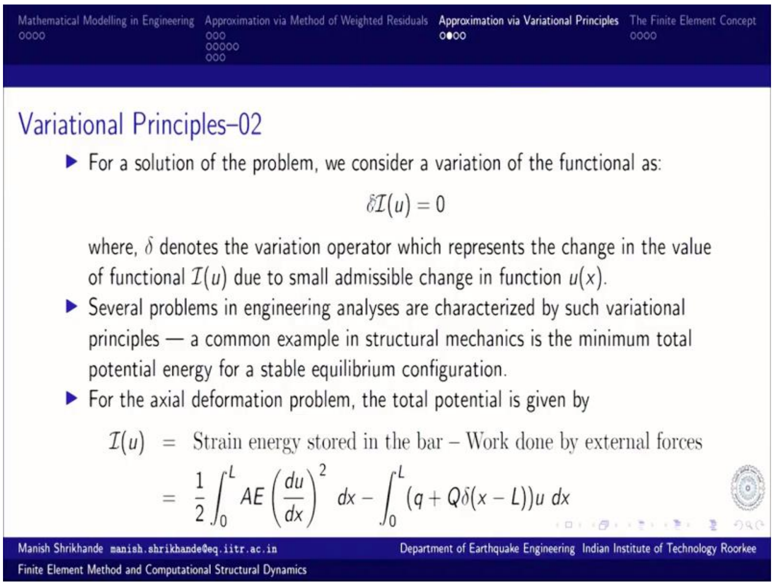

So, for a value solution of the problem we consider that the variation of the functional whatever the associated functional of the problem is we say it is variation should be 0. So, that corresponds to the extremal condition. So, delta here represents the variation operator.

It is a linear operator, behaves similarly to the differential operator, but it also commutes with differential operator and the integral operator integration operator. So, essentially this represents change in the value of the functional I (u) due to small admissible changes in the function u. So, admissible is the key word here. When we change the functional, when we change the function or explore the changes in the function u the boundary condition whatever the function u satisfied the boundary conditions essential boundary condition continued to be satisfied by the varied path or small variations that we are looking at. So, the boundary conditions are not violated, essential boundary conditions. So, that is what the admissible change refers to.

So, several problems in engineering analysis are characterized by a variational principles suitable corresponding variational principles. And, a common example in the structural mechanics problems is the minimum total potential energy. The stable equilibrium configuration of any body, deforming under the influence of external loads is characterized by minimum total potential energy.

So, everybody strives to achieve the state of minimum total potential energy and that corresponds to the stable equilibrium configuration. So, if we refer go back to our original problem that we started discussion of this course the axial deformation problem the total potential is given by the strain energy stored in the bar and, minus the work done by external force, , skin friction, the surface traction, or the end load and so on.

So, the strain energy stored in the bar is simply half of integral from 0 to L, the domain of the problem $\mathrm{AE}(\frac{du}{dx})^2 dx$. So, that is the strain energy simple volume integral of so, product of stress and strain. And, then minus the work done by the external forces; so, q is the distributed traction surface traction and that works along the as the axial deformation u moves at every point. And then this capital Q is the end force and that is what is indicated by this Dirac delta function. So, x minus L, so, that implies that this force Q acts only at point L. So, that is the way we model discontinuities in by using Dirac-delta function.

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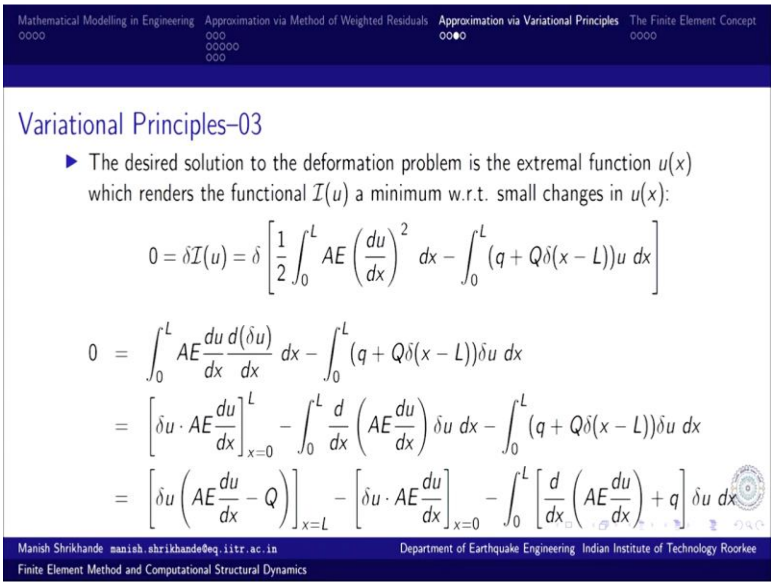

So, now the stable equilibrium configuration would correspond to the variation first variation being 0. So, we consider the variation of this problem. So, desired solution to the deformation problem is an extremal function u $_x$. So, u $_x$ should render the functional I (u) to a minimum. I mean minimum because we know it has to be a minimum potential energy, but first variation only it talks about the stationarity and that is the prerequisite of minima.

So, minimum with respect to extreme small changes in u x. So, first variation force to 0 and once we carry out this variation, then we end up with as I said the variational operator operates in similar fashion as the derivative operator. And, after we get this we perform the integration by parts on this domain integral and we end up with a boundary term and similar familiar terms and after that we can collect the terms. And, resolve this integral as this boundary term that will come here and we end up with one boundary term here one boundary term here and a domain integral here. And, since the admissible variations are arbitrary delta u can be anything as long as the only restriction is it has to satisfy the essential boundary conditions of the problem. So, the only way this identity can hold is when individually all these three terms vanish and when that happens.

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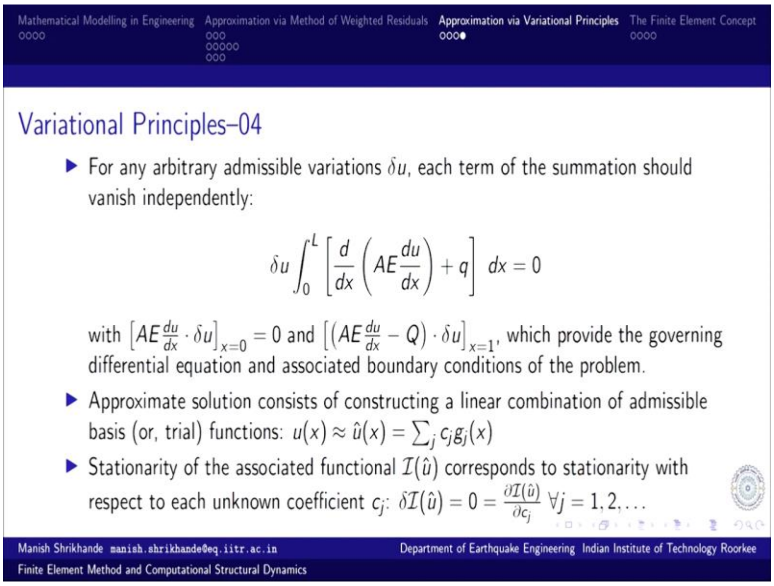

So, for any arbitrary variable a variation δu each term of the summation should vanish and when that we impose that we get the familiar equations of governing differential equation I mean it can be δu times this integral. So, for any arbitrary δu it is obviously not 0. So, then this particular integrand has to vanish and that is the governing differential equation of the problem. And, then the other two terms they give us the governing boundary condition of the problem.

Now, the we can work out the approximate solution by constructing a linear combination of admissible boundary basis function or trial functions. So, u (x) can be constructed as $\hat{u}$ given by linear combination of c_j g_j and where g_j are linearly independent basis functions which satisfy the essential boundary condition of the problem.

And, once we have that then the only unknowns of the problem are c_j and the first variation of the functional corresponds to the partial derivatives with respect to each of these unknowns of the problem. So, first variation of functional being 0 is same as saying partial derivative of the functional with respect to each of these unknown coefficient c_1, c_2, c_3 being put to 0. And, that is what yields the governing the algebraic system of equations that we derived as in the case of weak form of method of weighted residual using Galerkin approach. And, if you do this if you solve the same problem using this variational approach you will find that you will get exactly the same set of equations and same set of solutions.

So, I will stop here for the this lecture and now, in the next lecture we will discuss about how we extend this simple approximate solution to the finite element method. How we develop the concept of finite element method from these basic classical methods of approximation, that have been followed for very long time in mathematical community.

Thank you.