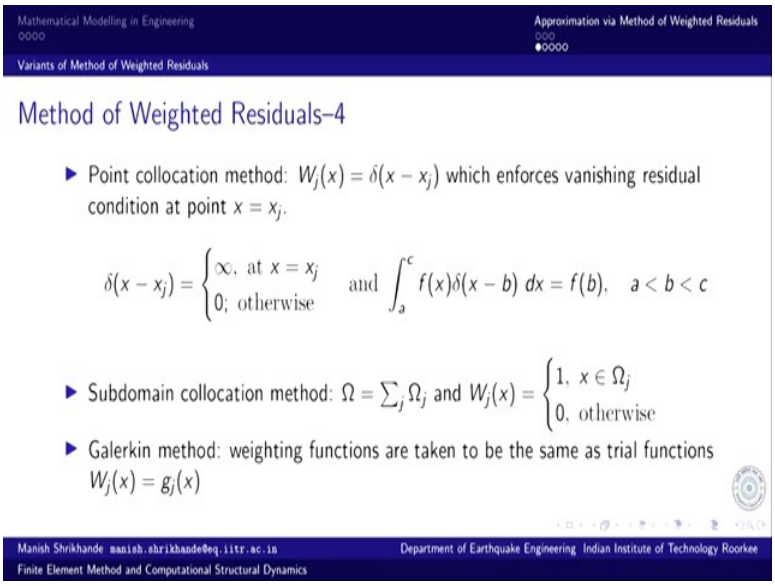
Finite Element Method and Computational Structural Dynamics
Prof. Manish Shrikhande
Department of Earthquake Engineering
Indian Institute of Technology, Roorkee

Lecture - 10
Mathematical Modelling and Approximate Solutions – III

Hello. So, in the last lecture, we discussed about the method of weighted residual and how we develop the approximate solution for any differential equation. Today we will look at the choice of weighting function on which the whole method hinges upon. So, how do we choose these weighting functions? There are different types of weighting functions and that is what gives rise to different variants of method of weighted residual.

(Refer Slide Time: 01:09)



First one is what we call as the **point collocation method**. In this method we enforce the residual to be zero at a particular point.

So, that is same as saying that we consider the weighting functions as Dirac delta functions. Dirac delta functions are special functions which shoots to infinity at a particular point and are zero otherwise with the additional condition that the integral of this function is equal to unity.

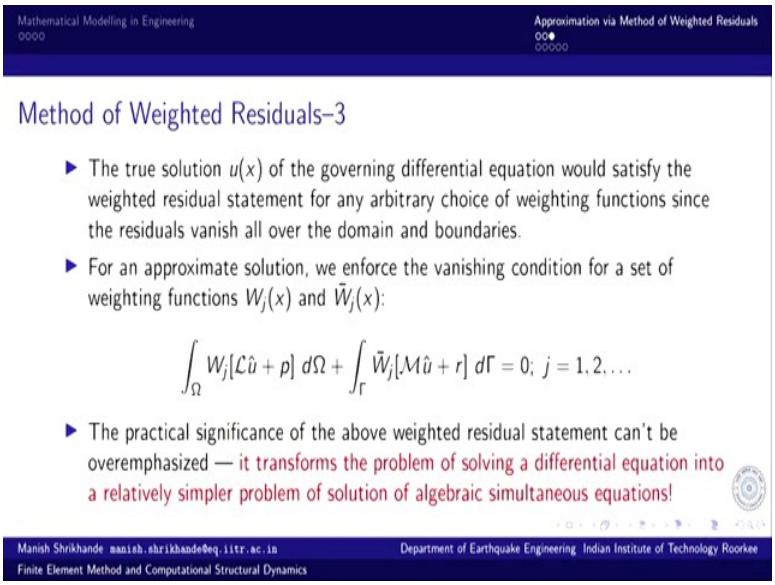
So, the whole idea of Dirac delta function is that it can be used to model discontinuities such as what we are trying to impose here that residual or the error of approximation

should be zero at a particular point. Iterating the property of Dirac delta function below as we will use it very often.

$$\int f(x)\delta(x-b)dx=f(b) \quad where \quad b \in [a,c]$$

In the above equation, the Dirac delta function $\delta(x-b)$ indicates that the function has zero value everywhere except at $x=b$. At $x=b$ the function value is infinite and integral of Dirac delta is unity.

(Refer Slide Time: 03:16)



So, in the point collocation method; we take the domain residual and Dirac delta function at one point and we make this residual vanish at that particular point. We can generate as many points as required in the domain according to the number of unknown coefficients Each point giving an algebraic equation and solving the set of algebraic equations to find out unknowns.

So, essentially transforming differential equation into a set of simultaneous algebraic equations. So, we generate as many equations as we need by enforcing the residuals to vanish at different points in the domain and solving the resulting simultaneous equations. The next method, or the next variant of method of weighted residual is **subdomain collocation**. In this method, instead of forcing the error to vanish at a particular point, we insist on vanishing the error over part of the domain.

We divide the domain of the problem into sum of non-overlapping subdomain; so total

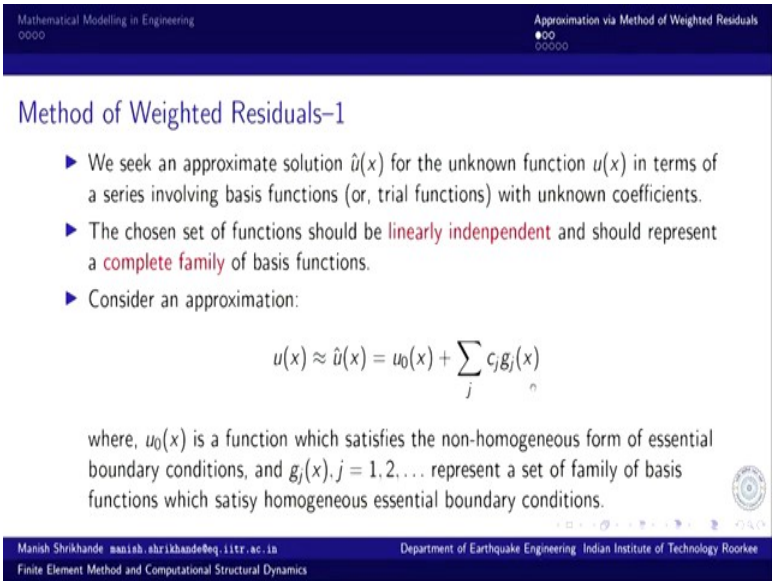
domain of the problem Ω can be represented as a sum of several subdomains Ω_j, which are non-overlapping in nature. And then we integrate the error over each subdomain and put it to 0.

So, on an average the error vanishes over the individual subdomains and we can use as many subdomains as required according to the number of unknowns in the problem, which will fetch as many equations as the number of unknowns.

Essentially this translates to saying that, weighting function is equal to unity over individual subdomain and it is equal to 0 over other subdomain. So, for sub domain Ω_j, the weighting function is equal to 1 and it is 0 everywhere else. And then it is just evaluating the definite integral and that will fetch a set of algebraic simultaneous equations in terms of the unknown coefficients which can be solved for evaluating those coefficients.

The third and the most popular method of all in methods of weighted residuals is called Galerkin method. In Galerkin method, the weighting functions are taken to be the same as that of the trial functions in the series solution that we had earlier.

(Refer Slide Time: 06:20)



So, the approximate solution is given as linear combination of these trial functions; c_j are the unknown coefficients and g_j are the trial functions and we choose these trial functions as the weighting functions. This gives us Galerkin method for method of

weighted residuals.

And we can go ahead and evaluate the integrals and that will fetch us a set of simultaneous equations and we can use those for solution of simultaneous equations. Galerkin method is very popular and there are reasons for that; first thing is because we are choosing the weighting functions as the trial functions. So, the whole problem of uncertainty over what weighting functions to chose vanishes.

And because the trial functions satisfy the homogeneous boundary conditions; the error in the domain are weighted more heavily than at the boundaries.

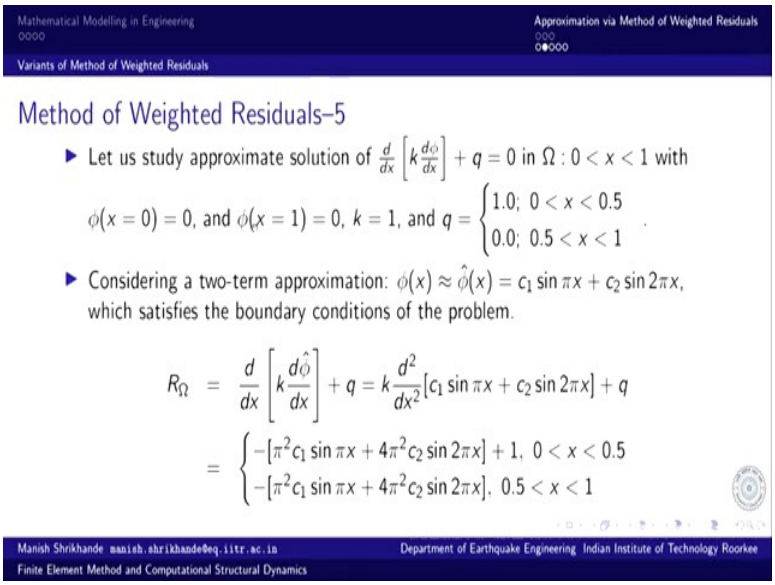
Let us try to solve a second order differential equation similar to the actual deformation problem or it could also be a heat conduction problem. Interpretation of coefficients and terms may vary from application to application; but essentially it is a second order differential equation with constant coefficients.

$$\frac{d}{dx}\left(k\frac{d\phi}{dx}\right)+q=0 \quad \Omega:\{x \mid 0<x<1\}$$

with

$$\phi(x=0)=0;\ \ \phi(x=1)=0;\ \ k=1;\ \ q=\begin{Bmatrix}1.0; & 0<x<0.5\\ 0.0; & 0.5<x<1\end{Bmatrix}$$

(Refer Slide Time: 09:08)



This is a second order differential equation and the domain is 0 to 1 and the function q in this particular case, is defined to be unity over half of the domain and it is equal to 0 over other half of the domain. It is a second order differential equation, so we need to

have two boundary conditions. And in this particular case, both boundary conditions are specified on the basic unknown of the problem.

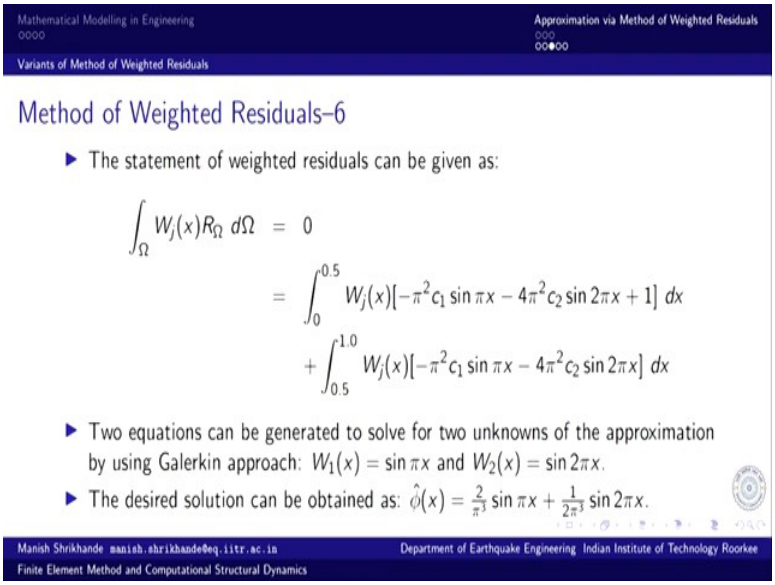
So, the unknown ϕ is equal to 0 at $x=0$ and it also vanishes at the other end of the boundary. So we have homogeneous boundary conditions at both ends and for homogeneous boundary conditions a very simple approximation can be sine function, $\sin(\pi x), \sin(2\pi x), \sin(3\pi x)$ as they all will vanish at $x=0$ and $x=1$.

We consider two term approximation of $\phi(x)$ as $(x)=c_1\sin(\pi x)+c_2\sin(2\pi x)\phi$. This obviously, satisfies the specified boundary conditions of the problem. Now, when we substitute this in the governing differential equation, that gives us the domain residual. There is only domain residual in this case, there are no boundary residuals as boundary conditions are exactly satisfied.

So, once we substitute this approximate solution in the governing differential equation, we get the domain residual as :

$$\begin{aligned}\int_\Omega W_j(x) R_\Omega \, d\Omega &= 0 \\ &= \int_0^{0.5} W_j(x)\left[-\pi^2 c_1 \sin\pi x - 4\pi^2 c_2 \sin 2\pi x + 1\right] dx + \\ &\quad + \int_{0.5}^{1.0} W_j(x)\left[-\pi^2 c_1 \sin\pi x - 4\pi^2 c_2 \sin 2\pi x\right] dx\end{aligned}$$

(Refer Slide Time: 11:45)



So, the statement of weighted residual, I mean the whole basis of approximate solution is; we can enforce weighted sum of the domain residual to vanish. In the above problem, the entire domain Ω is split into two parts, because the domain residual is discontinuous over two parts. First half is 0 to 0.5, second half is 0.5 to 1. Now, at this point we can choose any of the three methods that we discussed; point collocation, subdomain collocation, and Galerkin method.

Let us use Galerkin method, though other methods could as well be used. So, we choose two weighting functions. First we choose $W_1 = \sin \pi x$, substitute and evaluate the integral that will fetch us one equation in terms of unknown c_1 and c_2 . In this particular case, since $\sin \pi x$ is orthogonal to $\sin 2\pi x$; the integral of $\sin \pi x$ and $\sin 2\pi x$ will vanish, so eventually we will have only one term involving c_1 .

Similarly we now repeat this operation with the choice of $\sin 2\pi x$ as the weighting function. And again after carrying out the integral, we will have another set of equations and once we solve these two set of equations simultaneously, we get the unknown coefficient c_1 and c_2 as :

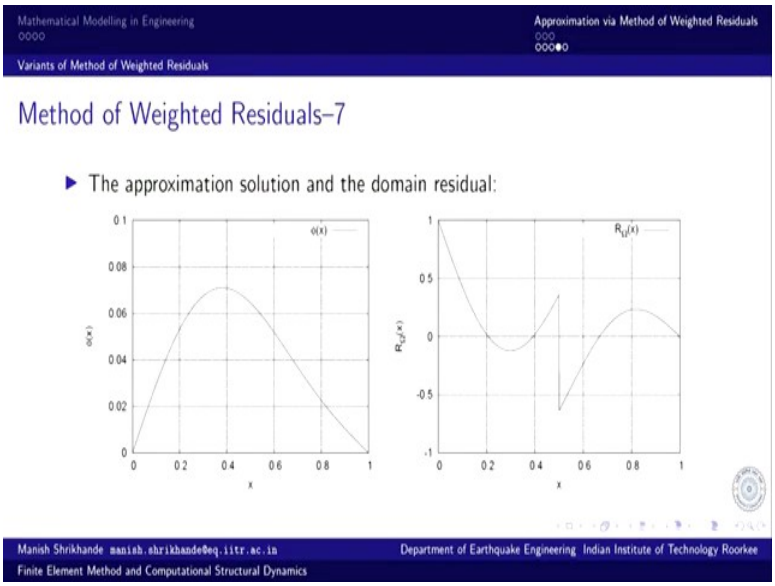
$$c_1 = \frac{2}{\pi^3}; \quad c_2 = \frac{1}{2\pi^3}$$

Substituting it back in the approximation

$$(x) = c_1 \sin \pi x + c_2 \sin 2\pi x \phi$$

that is the desired approximate solution of the differential equation. Now, it is instructive to look at how the solution looks like and what happens to the residuals, what is the quality of solution and quality of approximation.

(Refer Slide Time: 14:41)



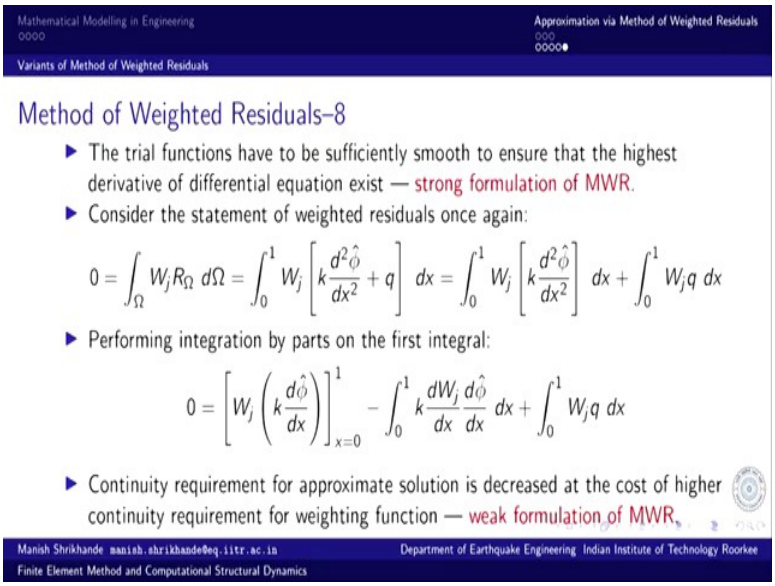
So, this is what the approximate solution looks like. And obviously, we have chosen the approximation function, such that boundary conditions are satisfied. So, we have the exact satisfaction of the boundary condition, i.e. at $x=0$, the function value is 0 and at $x=1$, the function value is 0 again, in between it rises and then it comes back.

Now, the domain residual as you can see oscillates around 0 throughout the domain. In other words, the entire error oscillates about this 0 line and it will keep on decreasing as we increase the number of terms in the series. So, the error will shrink closer to the 0 line and the approximation will become progressively better as we add more number of terms in the series.

So, instead of just two term approximation, if I use three term approximation, four term approximation, six term approximation; I will get better and better quality of results with smaller absolute error in the domain. This is similar to what we always know; we have always known this the quality of function approximation by using Fourier series. If we use the trigonometric function as the trial function; then effectively we are using Fourier series approximation.

And we all know about the convergence properties of the Fourier series. Any periodic function can be represented as a sum of trigonometric functions.

(Refer Slide Time: 17:08)



So, now one thing that we need to look at is, while we are looking at the governing differential equation - second order differential equation in this case. The approximation has to be consistent with the highest order of derivative that exist in the problem. For a solution of second order differential equation, I cannot choose an approximation which is just a constant value.

If I choose $\phi=\phi_0$ where ϕ_0 is a constant, then the entire thing breaks down. There is no working approximation to find as derivative of ϕ_0 will be 0 and I will be left with very absurd situation like q is equal to 0.

So, obviously, that does not work and that is a care that we need to take while constructing the approximate solution. The approximation that we develop has to be sufficiently smooth, it has to be sufficiently differentiable as dictated by the highest order of derivative in the problem.

And that is what we call as the strong formulation of method of weighted residuals. In the strong formulation, the approximate solution satisfies the continuity requirement to the highest degree as dictated by the governing differential equation.

So, in this case the continuity requirement is that the approximate function has to be at least twice differentiable; we have to choose a form of the approximate solution which will be differentiable at least two times. That is the strongest form of continuity being

enforced, that it has to be differentiable at least two times.

Now, let us look at it in a slightly different way. We can evaluate the integral of product of two functions W and the derivative of approximation. So, I can integrate it by parts. So, if I perform integration by parts on strong form, this is the result I get :

$$0=\left[W_j\left(k\frac{d\hat{\phi}}{dx}\right)\right]_0^1-\int_0^1 k\frac{dW_j}{dx}\frac{d\hat{\phi}}{dx}+\int_0^1 W_j q\,dx$$

Now, interestingly we have not violated any laws of mathematics here; but the statement of weighted residual now is different than before.

What is the difference? The first noticeable difference is, while the highest order of derivative in the strong form of weighted residual statement was 2; in this particular case the highest order of derivative is only 1. So, the approximate solution for this weighted residual statement to exist approximate solution only needs to be differentiable by 1 degree.

So, I do not need to construct an approximation, which will be differentiable at least twice; the approximation is good enough as long as I can get a finite derivative, as long as the first derivative exists. Of course, that comes at the cost of increased degree of continuity requirement on the weighting function. Earlier, there was no such requirement, the weighting functions were completely arbitrary, I could choose anything and that is how point collocation or subdomain collocations methods could exist, because W could be anything.

But now if I take the liberty of reducing the continuity requirement or the differentiability requirement on the approximate solution at the cost of increasing the continuity requirement of the weighting function; then I am looking at a situation in which the approximation can be constructed of lower degree polynomials which may not be sufficiently differentiable as dictated by the governing differential equation.

That is a very profound statement. The continuity requirement for the approximate solution is now decreased at the cost of higher continuity requirement for weighting functions and this is called the weak formulation of method of weighted residuals.

We can see that in the domain integral if I choose weighting functions same as the trial functions this integral essentially leads to a symmetric system of simultaneous equations, a much desired quality for numerical computations. A symmetric system of equations is numerically very stable and very efficient and robust solution techniques can be deployed for solution of simultaneous equations.

So, it is by using this weak form of method of weighted residual and using the Galerkin approach; it is guaranteed that the simultaneous equations that we will be getting for solution of unknown coefficients are going to be a symmetric system of equations. That is a very desirable property for numerical solution.

It may seem that now we have additional difficulty of choosing weighting function, which now has a higher continuity requirement. But that is not a problem because, we are using weighting function same as the trial functions. And as long as trial functions are sufficiently smooth according to the order of derivative required as dictated by this method a statement of weighted residual, we have just as many trial functions as we need for the generation of requisite number of equations for solving unknown coefficients.

So, it is a kind of made for each other kind of situation and leads to a very stable system of equations for solution. Now, another thing that we need to notice here is the boundary term; it is always a product of two terms weighting function W , and a derivative function. Interpretation of this boundary term varies from application to application.

In case of structural mechanics applications; this derivative term will always correspond to a force like quantities and, the weighting function term always corresponds to displacement like quantities. Another interpretation of weighting function can be as a virtual displacement.

So, this boundary term can be interpreted as the work done by the boundary forces in moving through the virtual displacements at the boundaries. And looking at this boundary term also gives us important idea about the identification of what are the essential boundary conditions.

Given the differential equation and the associated boundary conditions; how do we recognize which of the boundary conditions are essential boundary conditions and which are natural boundary conditions? or in mathematical parlance, they are referred to as

Dirichlet boundary conditions and Neumann boundary conditions, respectively.

It is important to recognize this distinction; because we need to construct our approximate solution in such a way that, the essential or Dirichlet boundary conditions are always satisfied. Natural boundary conditions or the Neumann boundary conditions would be satisfied through the process during the process of approximation. We do not really need to bother about imposing the Neumann boundary conditions during the process of approximation itself.

More on this in our next lecture and how we go about and how to interpret and how this weak formulation is similar to another formulations of approximate solutions and how we can establish a parity between this weak formulation of method of weighted residual with the variational methods of approximate solution or the variational calculus. We will discuss that in our next lecture.

Thank you.