Rock Engineering
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Lecture - 41
Elasto-Plastic Stress Distribution Around Circular Tunnel

Hello everyone. In the previous class, we discussed about the elastic analysis of the circular tunnel. We saw the stress distribution around the circular tunnel. And also, we saw that how the deformations or the displacements are going to occur? That was followed by our discussion on the analysis of the concrete lining. And that was also the elastic analysis. So, today, we will start with the elasto-plastic stress distribution around the circular tunnel.

And here, we will take the help of Tresca yield criterion in order to define the plastic region of the rock. Let us see how we proceed in this direction.

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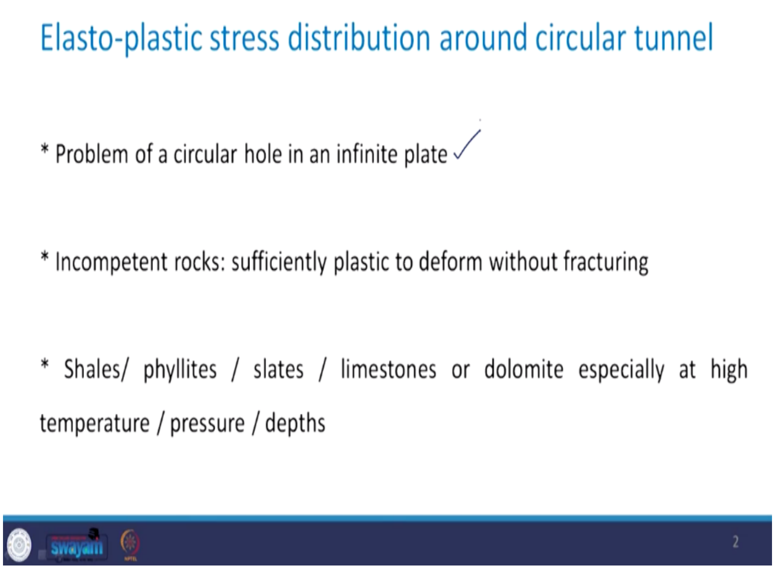

This resembles to a problem of circular hole in an infinite plate. In case of the incompetent rocks, which are sufficiently plastic to deform without fracturing, such elasto-plastic situation will be relevant. Rocks such as shales, phyllites, slates, limestones or dolomite they exhibit elasto-plastic behavior specially at high temperature, pressure and depths.

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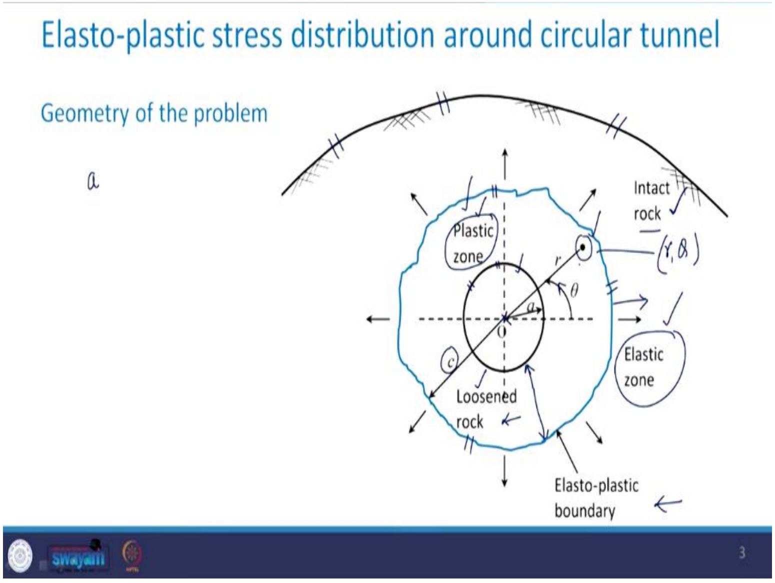

So, this figure shows the geometry of the problem. You see that this is what is the ground condition. Here, this is the intact rock. And in this rock, you have made the excavation in the form of this circular opening which has a radius of *a*. Then, because this rock is in such a way that there is the occurrence of a plastic zone all around this cavity, which is the circular tunnel and that has been represented by this blue colored curve.

It has been assumed to be more or less circular. Why is it so? We will discuss that in detail just in subsequent slides. And the radius of this zone from the center, that has been taken as *c*. So, the rock which is beyond this boundary, this is in elastic zone. And the rock in between the boundary and the cavity, this is in the plastic zone. So, this is what we call as the intact rock. And here in this plastic zone that is from here to here, you will have the loosened rock.

So, this boundary which is drawn with the help of the blue color is the boundary between the elastic and the plastic zone. And therefore, we call this as elasto-plastic boundary. As we discussed in case of the elastic analysis in this case also any point in the rock is denoted by r,θ, where r is its radial distance from this center point O and θ is an angle which is measured from the horizontal in the anti-clockwise direction as shown in this figure.

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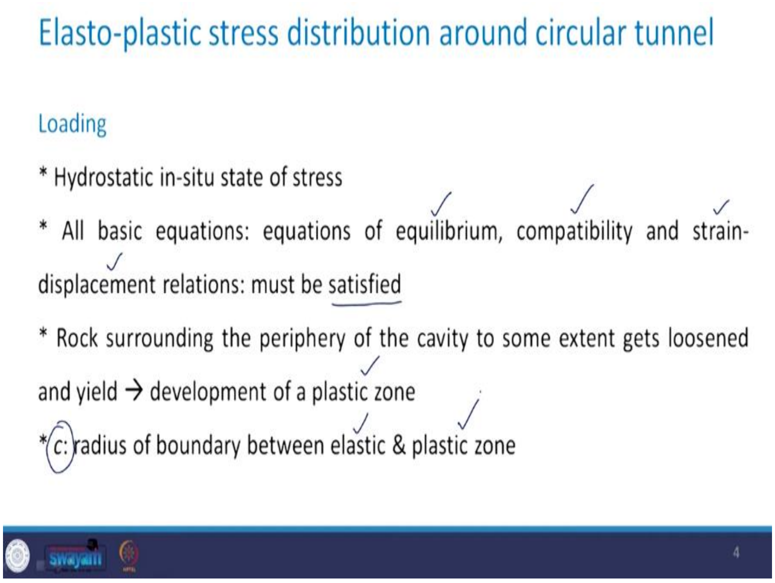

Coming to the loading condition, it is assumed that there occurs the hydrostatic in situ state of stress. That means that the vertical as well as the horizontal in situ stresses, they are same. All the basic equations, such as equations of equilibrium, compatibility and strain displacement relationship, all these must be satisfied. Rock surrounding the periphery of the cavity to some extent gets loosened and yield.

And this results into the development of a plastic zone, as I just explained you, when we were discussing about the geometry of the problem. And c is defined as the radius of boundary between elastic and the plastic zone.

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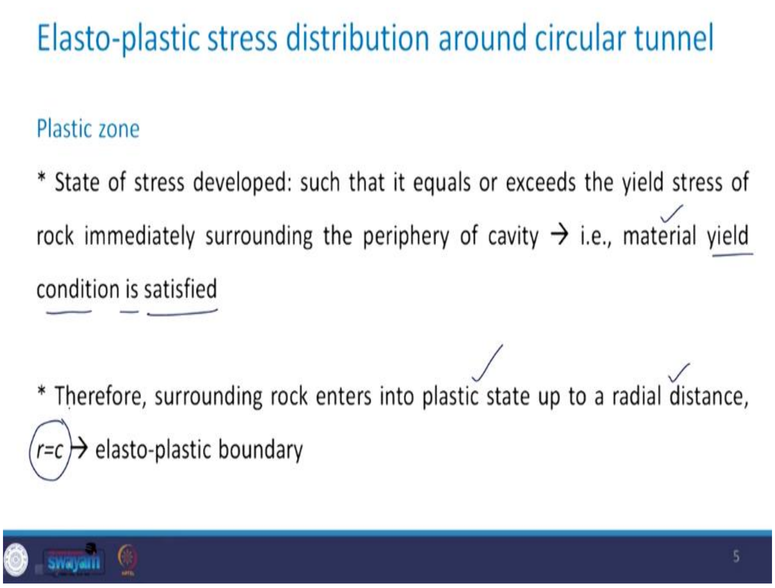

Coming to some discussion on the plastic zone, the state of a stress which is developed is such that it equals or exceeds the yield stress of rock immediately surrounding the periphery of the cavity. What does that mean? That is the material yield condition is satisfied. And therefore, the surrounding rock enters into the plastic state up to a radial distance of c, which we are calling as elasto-plastic boundary.

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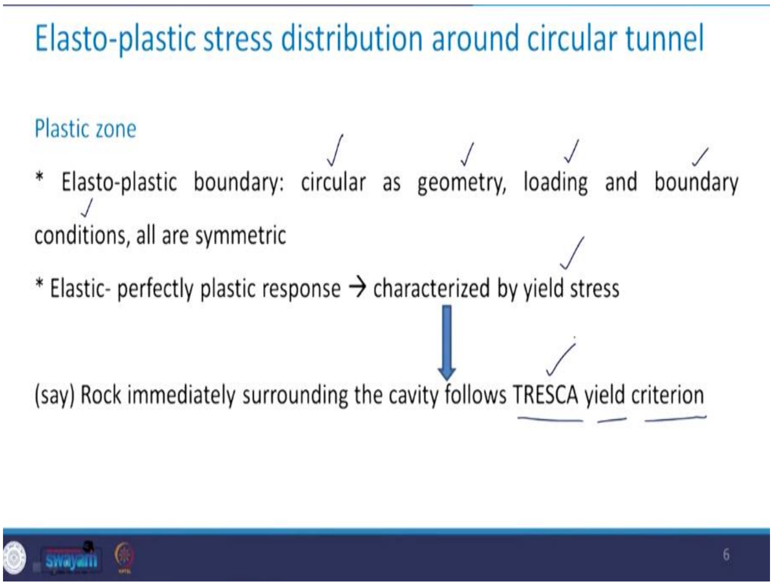

This elasto-plastic boundary is circular. Why? Because the geometry, loading and the boundary condition of the problem they all are symmetric. So, for all those conditions like condition of equilibrium, compatibility condition everything's should be satisfied. So, this elasto-plastic boundary has to be circular. We are assuming this elastic perfectly plastic response of the rock which is characterized by a yield stress.

And it is assumed that rock which is immediately surrounding the cavity. It follows the TRESCA yield criterion. It is not necessary that it follows the TRESCA yield criterion, it can follow Mohr Coulomb failure criterion as well. But here as far as this lecture and the scope of this course is concerned, we will be restricting only to the TRESCA yield criterion. However, the procedure that I am going to explain for the elasto-plastic analysis is general enough.

And you can apply any failure criterion to represent the rock immediately surrounding the cavity.

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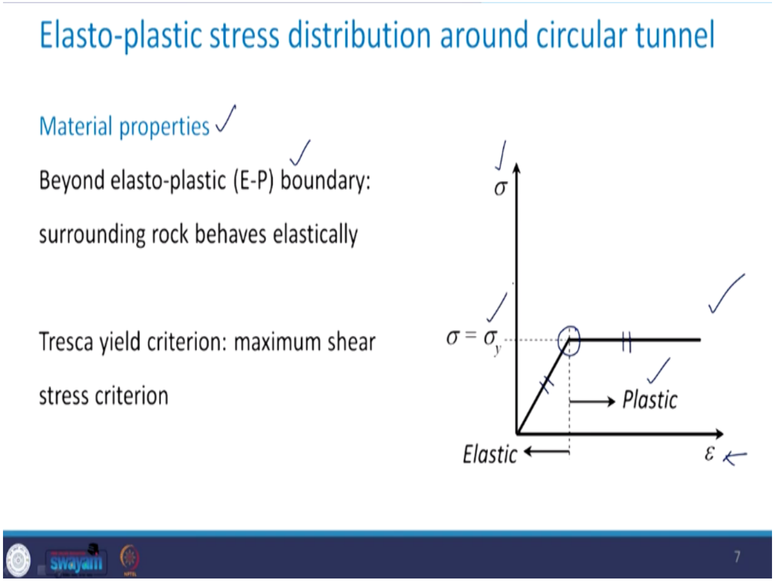

Coming to the material properties, beyond the elasto-plastic boundary, the surrounding rock will behave in an elastic manner. As we have assumed that the rock is following TRESCA yield criterion, which is the maximum shear stress criterion. Take a look at this figure where σ versus ε has been plotted up to the elastic limit it goes in a linear fashion in this manner as shown here.

And once it enters or once the material yields then it enters in the plastic domain. And this is how the stress strain relationship will be represented. So, here you get the maximum stress which is the yield stress. And it is represented by $\sigma = \sigma_y$. So, what is the state of stress in this case?

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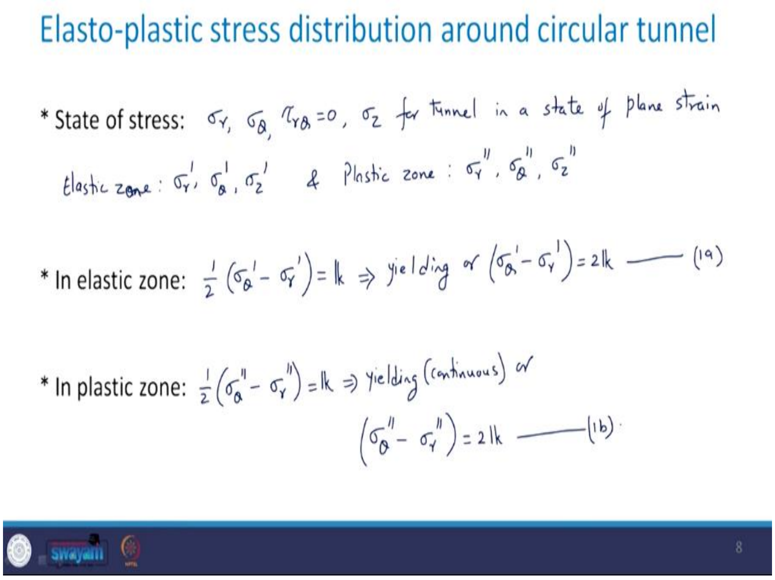

Let us see that. That is going to be σ_r, σ_θ and $\tau_{r\theta}$ which will be equal to 0. And you will have σ_z for tunnel in a state of plane strain. Then in the elastic zone, we have the stresses as σ_r', σ_θ' and σ_z'. And in the plastic zone, these are represented as σ_r'', $\sigma_\theta''$and σ_z''.

Now, what will happen to the yielding condition in the elastic zone is

$$\frac{1}{2}(\sigma_\theta' - \sigma_r') = \kappa$$

And when this condition is satisfied, we would say that the yielding is taking place. And then we can write this equation in this form that

$$\sigma_\theta' - \sigma_r' = 2\kappa \rightarrow (1a)$$

This is equation number 1a. What happens in plastic zone?

We will have

$$\frac{1}{2}(\sigma_\theta'' - \sigma_r'') = \kappa$$

Here it is the yielding which is continuous. Or, we can write it in this particular form this equation as

$$\sigma_\theta'' - \sigma_r'' = 2\kappa \rightarrow (1b)$$

This is equation number 1b.

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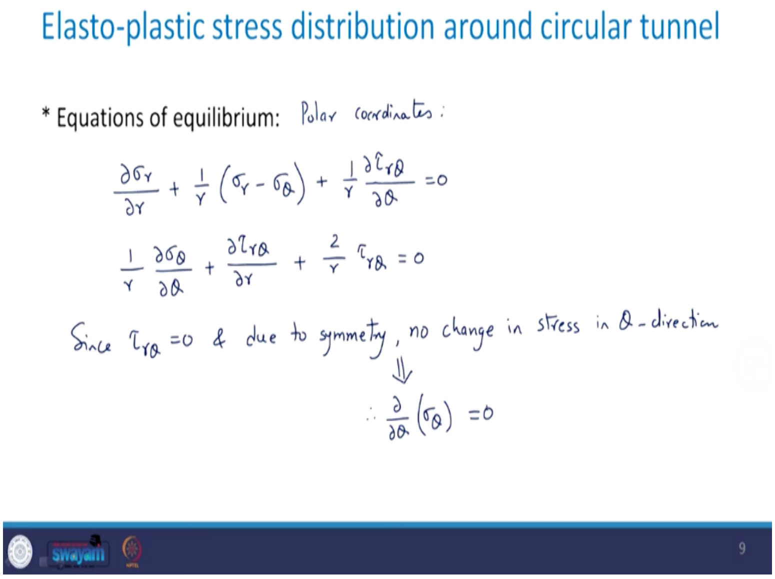

Now, what are going to be the equations for the equilibrium in the polar coordinate systems? So, here we have

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r}\tau_{r\theta} = 0$$

Now, we have seen that since $\tau_{r\theta} = 0$ and due to symmetry, no change in the stress which is in θ direction. So, this is going to give me

$$\frac{\partial(\sigma_\theta)}{\partial \theta} = 0$$

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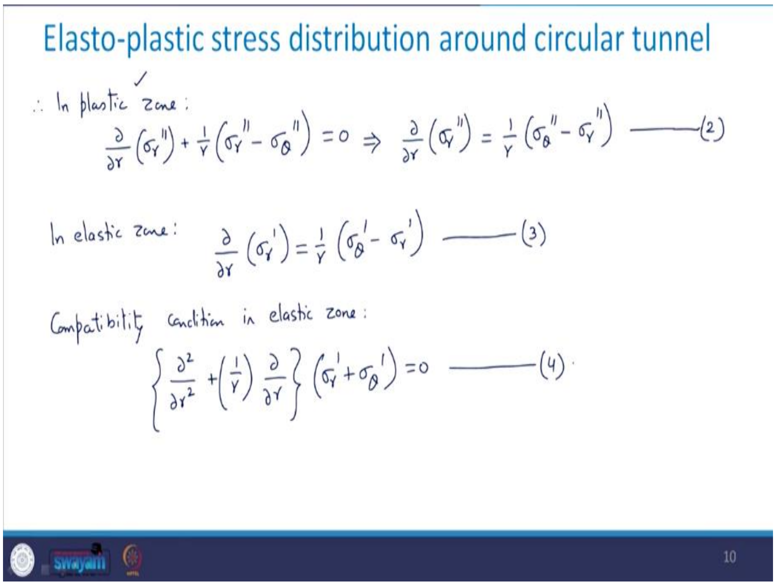

So, in the plastic zone, what will happen?

$$\frac{\partial}{\partial r}(\sigma_r'') + \frac{1}{r}(\sigma_r'' - \sigma_\theta'') = 0 \Rightarrow \frac{\partial}{\partial r}(\sigma_r'') = \frac{1}{r}(\sigma_\theta'' - \sigma_r'') \rightarrow (2)$$

This equation I am marking as equation number 2. What will happen in case of the elastic zone? Similarly, I can apply that equation of the equilibrium. So, we are going to get

$$\frac{\partial}{\partial r}(\sigma_r'') = \frac{1}{r}(\sigma_\theta' - \sigma_r') \rightarrow (3)$$

That is equation number 3.

Now, what will be the compatibility condition in the elastic zone? So, that will be given by

$$\left\{\frac{\partial^2}{\partial r^2} + \left(\frac{1}{r}\right)\frac{\partial}{\partial r}(\sigma_\theta' + \sigma_r{}')\right\} = 0 \rightarrow (4)$$

This equation is going to be equation number 4. Now, these equations of equilibrium and the compatibility equation are to be satisfied subjected to boundary condition of the problem.

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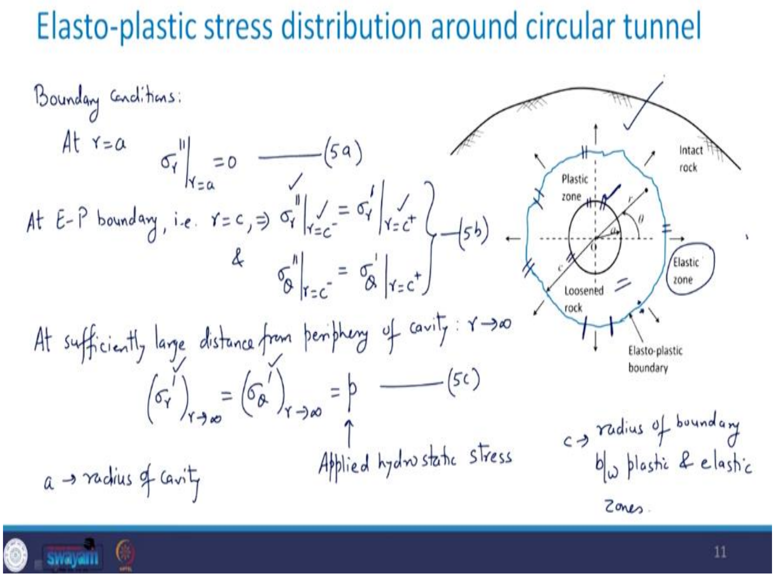

So, we have here boundary conditions. So, take a look at this figure. This boundary that is at r=a. It is the free boundary because here excavation has taken place. So, this is the boundary of the cavity. So, therefore, what we will have here is that

$$\sigma_r{}''|_{r=a} = 0 \rightarrow (5a)$$

σ r double*'* will be equal to 0. Now, at the elasto-plastic boundary, what is *r* at elasto-plastic boundary which is this elasto-plastic boundary? What is *r* here? That is at r = c.

There is going to be the continuity of the stresses across this boundary. So, because of that what you are going to have is

$$\sigma_r{}''|_{r=c^-} = \sigma_r{}''|_{r=c^+} \longrightarrow (5b)$$

$$\sigma_\theta{}''|_{r=c^-} = \sigma_\theta{}''|_{r=c^+} \longrightarrow (5b)$$

Please understand that here this is the elasto-plastic boundary.

And for this system to be in equilibrium there has to be the continuity of stresses across this boundary. Now, this when I say r =c^- that means, we are somewhere in the plastic zone, but very near to this elasto-plastic boundary. When I say r =c+ we are in the elastic zone, but again we are near to the elasto-plastic boundary. That is what is the meaning of these r =c- and r =c+

That is, we are at the boundary, but just inside and just outside the boundary what exactly is going to happen. So, this equation I am calling as 5b. Why we are writing for $\sigma_r''|_{r=c^-}$? Because when r =c- , we are in the plastic zone. And in the plastic zone, we are representing the state of stresses with double *'*. Now, the next boundary condition is that at sufficiently large distance from the periphery of the cavity, what will happen?

What does this mean? That I say that r is tending to infinity. We have seen in case of the elastic analysis that at r tending to infinity the stresses are going to be equal to the in-situ stresses. So, that we can write as that

$$\sigma_r'|_{r\to\infty} = \sigma_\theta'|_{r\to\infty} = p \longrightarrow (5c)$$

Make this equation at 5c. When I say r tending to infinity essentially this is going to be in the elastic zone.

And therefore, the state of stresses is being represented by single *'*. This *p* is the applied hydrostatic stress. What is *a*? This is the radius of the cavity. And *c* is the radius of boundary between plastic and elastic zones as it has been shown in this figure.

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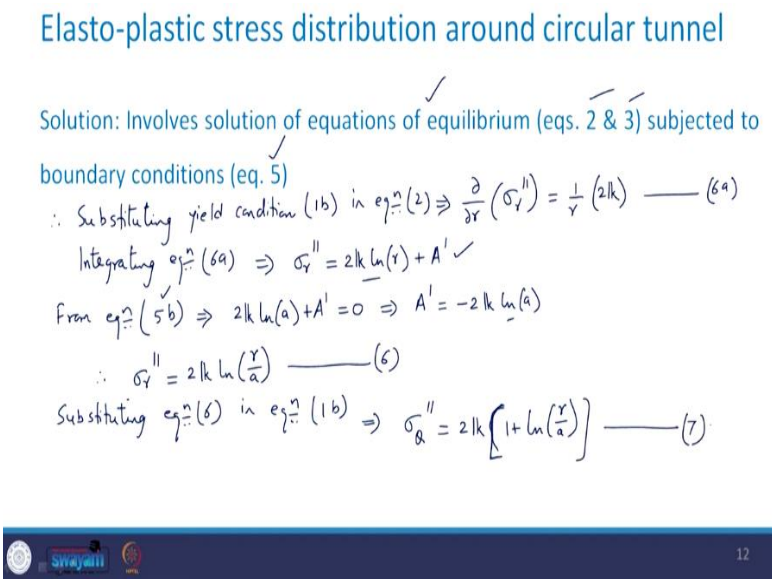

Now, coming to the solution of these, so, these solutions will involve the solution of equations of equilibrium which were given by equations 2 and 3 and subjected to the boundary conditions which we just derived which are equation number 5. So, what we will do is we just substitute these yield condition given by 1b in equation number 2. What we get is

$$\frac{\partial}{\partial r}(\sigma_r'') = \frac{1}{r}(2\kappa) \longrightarrow (6a)$$

This is 6a. Now, you integrate this equation.

So, integrating equation 6a, we are going to get

$$\sigma_r'' = 2\kappa \ln(r) + A'$$

Now, from equation number 5b,

$$2\kappa \ln(a) + A' = 0 \;\Rightarrow A' = -2\kappa \ln(a)$$

That is what is your equation number 5b. This will be equal to 0. Now substitute it here in this expression.

$$\sigma_r'' = 2\kappa \ln(\frac{r}{a}) \longrightarrow (6)$$

So, this equation I am making it as 6. Now, I substitute this equation number 6 in equation 1b.

$$\sigma_\theta'' = 2\kappa \left[1 + ln(\frac{r}{a})\right] \longrightarrow (7)$$

This is equation number 7.

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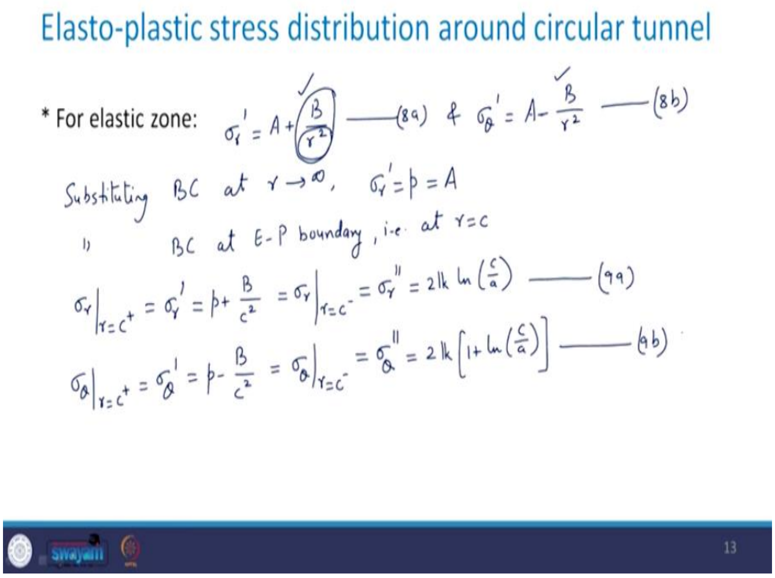

Now, what will happen in case if you have the elastic zone? You will have these stresses in this form that is

$$\sigma_r' = A + \frac{B}{r^2} \longrightarrow (8a)$$

$$\sigma_\theta' = A - \frac{B}{r^2} \longrightarrow (8b)$$

Substituting BC at $r \longrightarrow \propto \quad \sigma_r' = p = A$

Substituting BC at E-P boundary, i.e r=c

$$\sigma_r|_{r=c^+} = \sigma_r' = p + \frac{B}{c^2} = \sigma_r|_{r=c^-} = \sigma_r'' = 2\kappa \ln(\frac{c}{a}) \longrightarrow (9a)$$

$$\sigma_\theta|_{r=c^+} = \sigma_\theta' = p + \frac{B}{c^2} = \sigma_\theta|_{r=c^-} = \sigma_\theta'' = 2\kappa \ln(1 + \frac{c}{a}) \longrightarrow (9b)$$

This is 8a. Please refer to our discussion related to the elastic analysis. There we derived these expressions for σ_r and σ_θ. And if you just correlate, they were in this particular form. So, I am directly taking it from there.

Now, we substitute the boundary condition at r tending to infinity. So, substituting the boundary condition at r tending to infinity. So, what will happen? What is that condition? So, when this r is tending to infinity, this term will become very large tending to because r is tending to infinity. And this whole term will tend to 0. So, therefore, this will become equal to A.

Now, substituting this boundary condition at the elasto-plastic boundary which is at r=c. So, what was that?

We just obtained this. So, make this equation as 9a. Similarly, we have it for θ. This is 9b. So, this is how we obtain the expression for σ_r'' , σ_θ'' and the respective stresses in the elastic zone as well.

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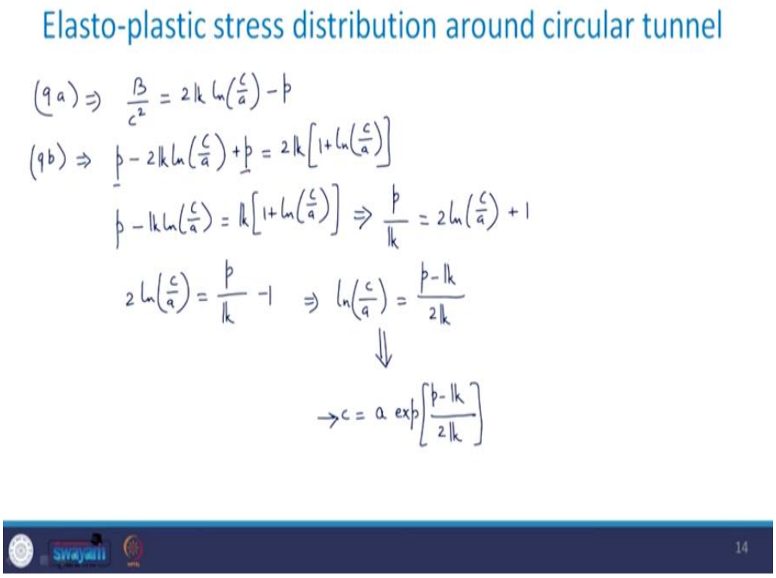

Now, if we look at equation number 9a, this will give me

$$\frac{B}{c^2} = 2\kappa \ln\left(\frac{c}{a}\right) - p \longrightarrow (9a)$$

$$p - 2\kappa \ln\left(1 + \frac{c}{a}\right) + p = 2\kappa \ln\left(1 + \frac{c}{a}\right) \longrightarrow (9b)$$

$$p - \kappa \ln\left(\frac{c}{a}\right) = \kappa \ln\left(1 + \frac{c}{a}\right) \Rightarrow \frac{p}{\kappa} = 2 \ln\left(\frac{c}{a}\right) + 1$$

$$\frac{p}{\kappa} = 2 \ln\left(\frac{c}{a}\right) + 1$$

$$2 \ln\left(\frac{c}{a}\right) = \frac{p}{\kappa} - 1 \Rightarrow \ln\left(\frac{c}{a}\right) = \frac{p - \kappa}{2\kappa} \Rightarrow c = a\, exp\left[\frac{p-\kappa}{2\kappa}\right]$$

So, this is how the radius of this elasto-plastic boundary is obtained in terms of the other quantities.

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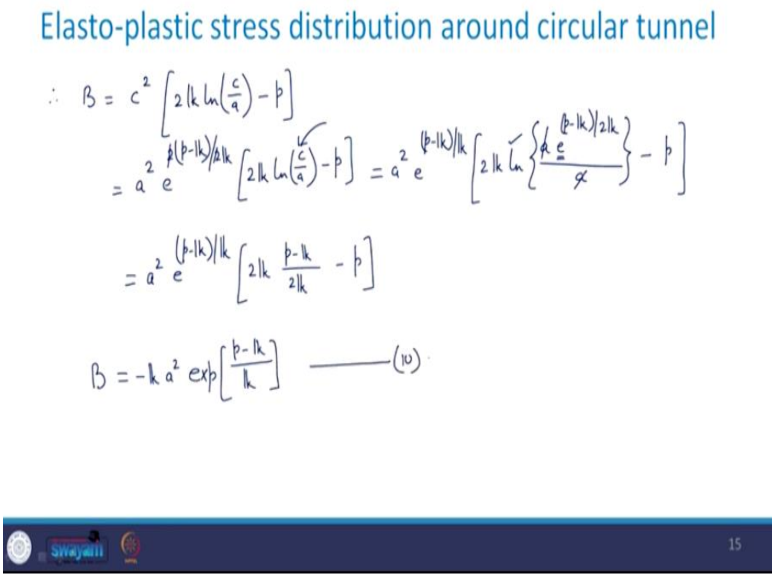

And therefore, what will happen to this B? That is going to be

$$B = c^2 \left[2\kappa \ln\left(\frac{c}{a}\right) - p\right]$$

$$= a^2 e^{\left(\frac{2(p-\kappa)}{2\kappa}\right)} \left[2\kappa \ln\left(\frac{c}{a}\right) - p\right] = a^2 e^{\left(\frac{(p-\kappa)}{\kappa}\right)} \left[2\kappa \ln\left\{\frac{a e^{\left(\frac{(p-\kappa)}{\kappa}\right)}}{a}\right\} - p\right]$$

$$= a^2 e^{\left(\frac{(p-\kappa)}{\kappa}\right)} \left[2\kappa \ln\left(\frac{p-\kappa}{\kappa}\right) - p\right]$$

$$B = -\kappa a^2 exp\left[\frac{p-\kappa}{\kappa}\right] \longrightarrow (10)$$

This equation I will write as equation number 10.

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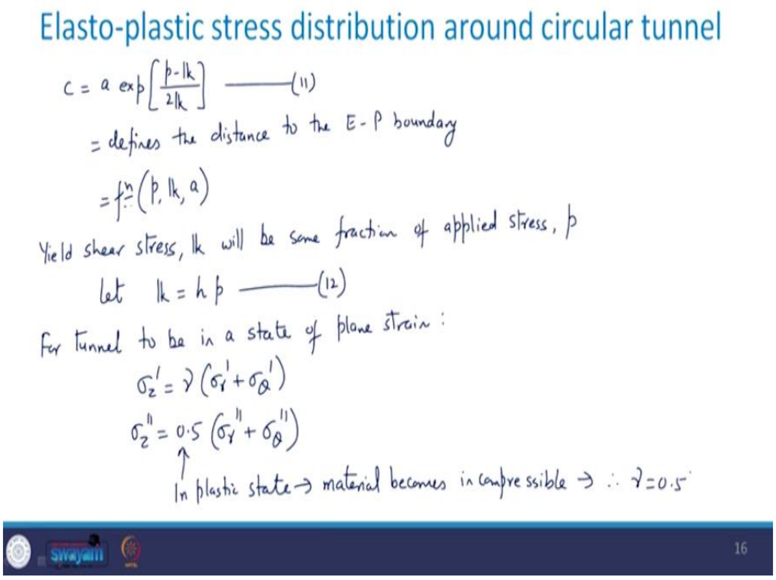

$$c = a\, exp\left[\frac{p-\kappa}{2\kappa}\right] \longrightarrow (11)$$

So, this basically defines the distance to the elasto-plastic boundary. And this is a function of p, κ and a. So, this yield shear stress which I am representing by this κ will be some fraction of this applied stress which is p. So, I assume that this κ is equal to some fraction h of this applied stress p.

$$Let\ \kappa = hp \ \longrightarrow (12)$$

Make it equation number 12.

So, for tunnel to be in a state of plane strain, remember we are doing this derivation for this situation.

$$\sigma_z' = \nu(\sigma_r' + \sigma_\theta')$$

$$\sigma_z'' = 0.5(\sigma_r'' + \sigma_\theta'')$$

Now, why this 0.5? That in the plastic state your material becomes incompressible. And you know that for an incompressible material ν can be taken as 0.5.

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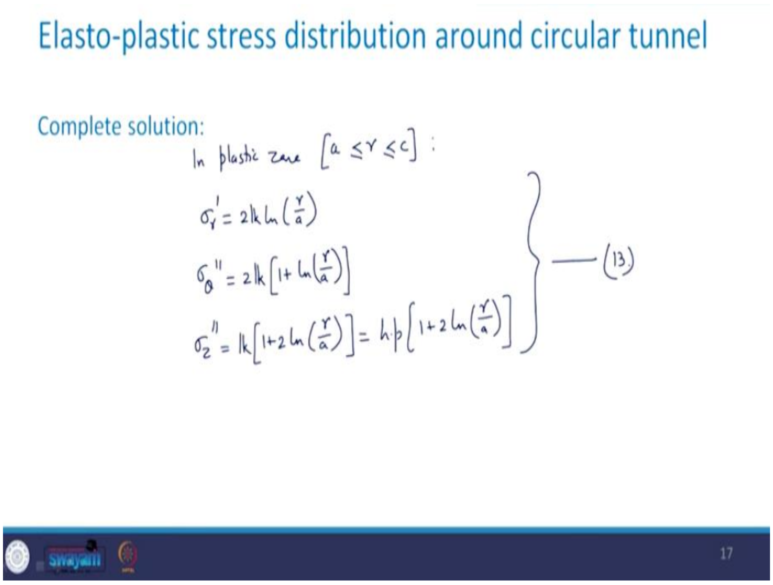

So, let us try to write the complete solution here. First, I write in the plastic zone. That is $a \leq r \leq c$.

$$\sigma_r' = 2\kappa \ln\frac{r}{a};\ \ \sigma_\theta'' = 2\kappa\left[1+\ln\frac{r}{a}\right];\ \sigma_z'' = \ \kappa\left[1+\ln\frac{r}{a}\right] = hp\left[1+2\ln\frac{r}{a}\right] \Longrightarrow (13)$$

These equations I am writing as equation number 13. Now, what will happen in the elastic zone?

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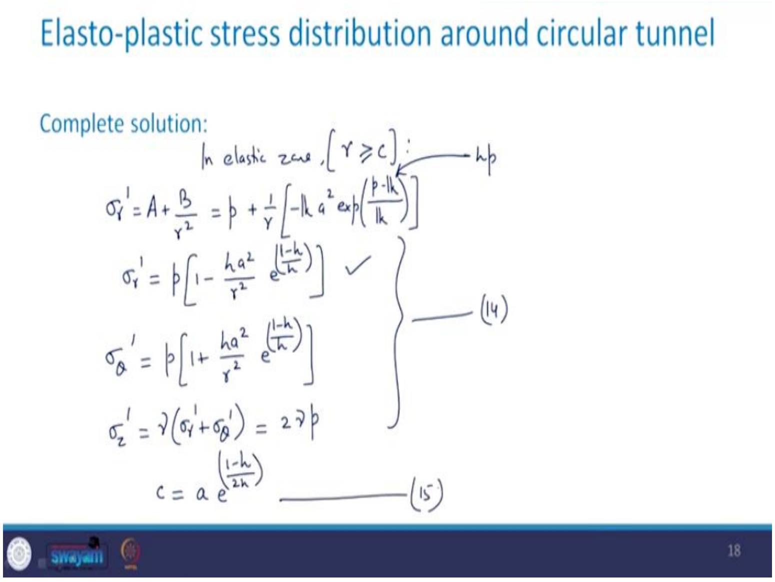

So, in the elastic zone, your r ≥ c. We are going to have is

$$\sigma_r' = A + \frac{B}{r^2} = p + \frac{1}{r}\left[-\kappa a^2 exp\left[\frac{p-\kappa}{\kappa}\right]\right]$$

$$\sigma_r' = p\left[1 - \frac{ha^2}{r^2}e^{\left(\frac{1-h}{h}\right)}\right]$$

$$\sigma_\theta' = p\left[1 + \frac{ha^2}{r^2}e^{\left(\frac{1-h}{h}\right)}\right]$$

$$\sigma_z' = \nu(\sigma_r' + \sigma_\theta') = 2\nu p$$

So, first we will have here as equation number 14, these 3 equations.

And we can write this radius of the elasto-plastic boundary again in the form of

$$c = ae^{\left(\frac{1-h}{2h}\right)} \rightarrow (15)$$

This equation I will write as equation number 15.

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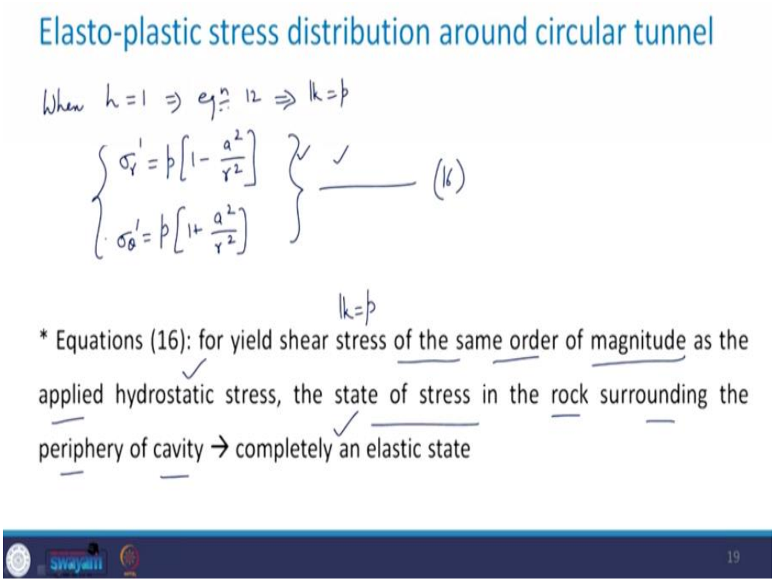

Now, what will happen when this h becomes equal to 1? Your equation 12 will become what κ is equal to p. And in that situation, what will happen to σ_r'? That will be

$$\sigma_r' = p\left[1 - \frac{a^2}{r^2}\right]$$

$$\sigma_\theta' = p\left[1 + \frac{a^2}{r^2}\right]$$

This you write as equation number 16. Now, this equation 16 which we have derived here for the yield shear stress of the same order of the magnitude as the applied hydrostatic stress that means when κ becomes equal to p.

The state of stress in the rock surrounding the periphery of the cavity will be completely in an elastic state. You can have a look here on the equation number 16. And if you just compare the result with the elastic analysis, you will be able to have that idea that right now it is the completely and elastic state.

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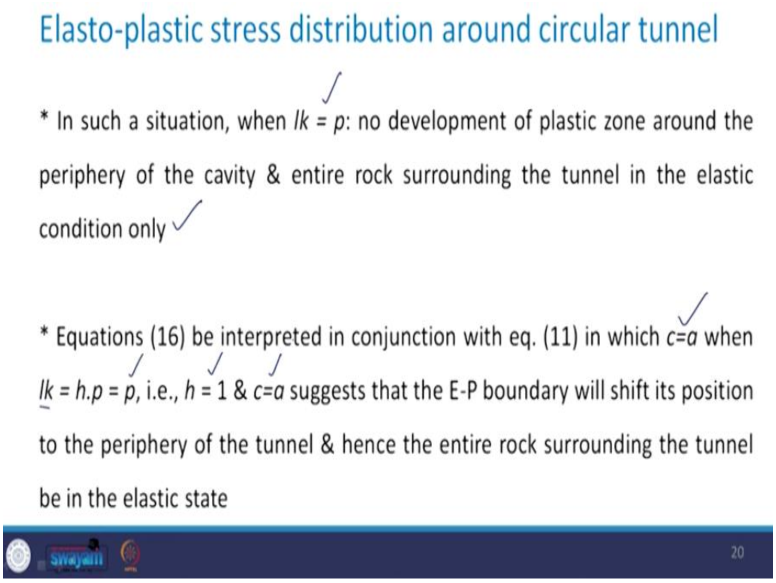

So, in such a situation, when this κ becomes equal to p, there will be no development of plastic zone around the periphery of the cavity. And the entire rock surrounding the tunnel will be in the elastic condition only. This equation 16 can be interpreted in conjunction with the equation 11 in which c will be equal to a when this κ becomes equal to p. That is for h =1 and c =a.

This suggests that elasto-plastic boundary will shift its position to the periphery of the tunnel. And hence, the entire rock which is surrounding the tunnel will be in the elastic state.

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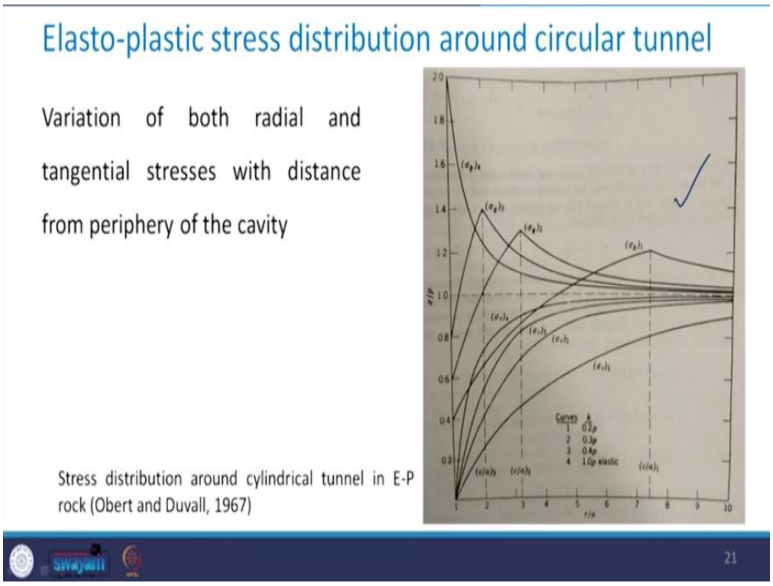

Now, the variation of both radial and the tangential stresses with the distance from the periphery of the cavity has been shown in this figure. Let us try to see some of the typical feature with respect to this stress distribution one by one.

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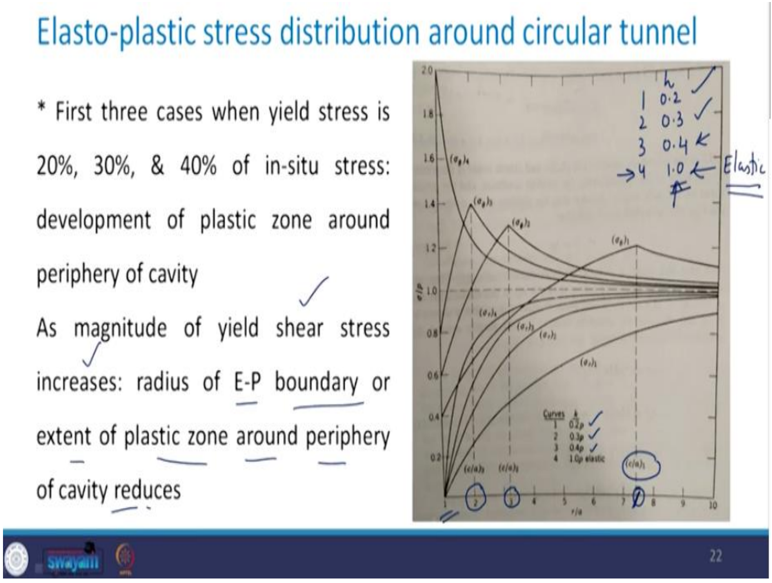

So, here we are taking 4 cases 1, 2, 3 and 4. The first case corresponds to when *h* is equal to 0.2, second case is for 0.3, third is for 0.4 and the fourth case *h* will become equal to 1. You can see here that it is written. So, when this is the fourth case as we have seen this is going to be fully elastic situation. So, as the magnitude of the yield shear stress increases, that means for these 3 values you can see as this is increasing from 0.2 times p to p.

The radius of elasto-plastic boundary or the extent of the plastic zone around the periphery of the cavity reduces. Because when it is equal to 1, we have seen that it is going to be completely elastic. And that is also shown here that *c*/ *a*. In the first case, you can see that it is quite large which is let us say about 7.3 something here. Then in case of *c* by in case of the second case that is when it *h* is equal to 0.3, it is further reducing.

And when this is increased 2.4 that is *h* is increased 2.4, this is further reduced to somewhere close to 2. And when you have the fourth case, obviously this is going to be equal to 1.

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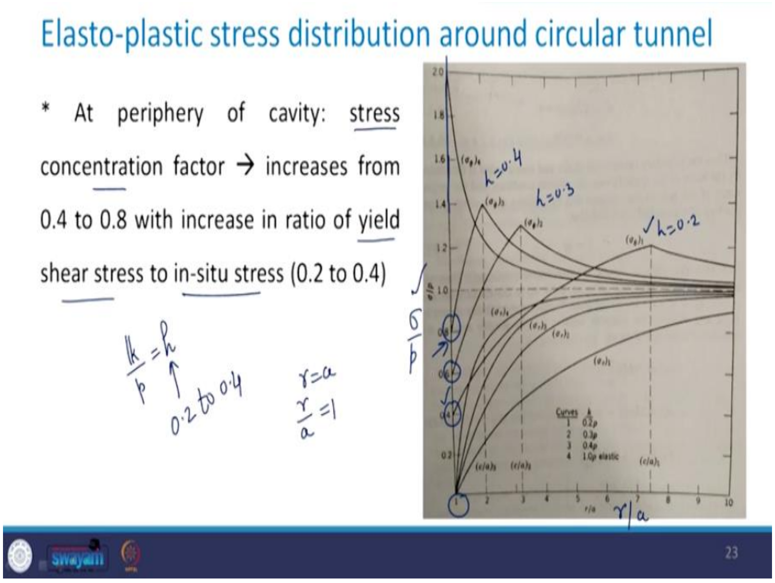

So, at the periphery of the cavity, what does that mean? That is, $r=a$. So, this ratio r /a equal to 1. And then you can see that here this axis we are plotting r /a. And here it is σ/ p. So, this is what is r/a = p. So, we move along this line if you want to see what is the situation at the periphery of the cavity? So, take a look here that this stress concentration factor which has been plotted on this axis, it increases from 0.4 to 0.8.

As the ratio of yield stress to the in-situ stress that is κ upon p which is equal to h is being increased from 0.2 to 0.4. So, you see that for the first case here h is equal to 0.2. So, this at the tunnel periphery the stress concentration factor is 0.4. And for this case when h is equal to 0.3, it increases to 0.6 and further increases, when you further increase this h to 0.4. And it increases to 0.8. So, some of these salient features you must keep in mind.

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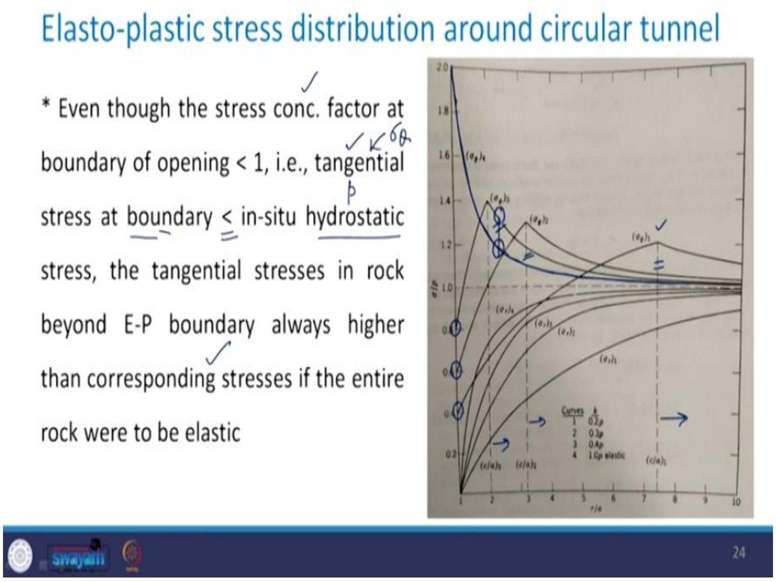

Now, even though the stress concentration factor at the boundary of the opening is less than 1. We have seen that it is 0.8, 0.6 or 0.4. The tangential stress at the boundary, when I say tangential stress means I am talking about σ_θ. This σ_θ is less than the in situ hydrostatic stress which is p. The tangential stresses in the rock beyond the elasto-plastic boundary, they are always higher than the corresponding stresses if the entire rock mass were to be elastic.

Try to understand this, if the entire rock mass were to be elastic, this is what is the variation of σ_θ with r/a. See this is the fourth case. Now, take a look at these respective curves beyond these respective dotted lines. In all the cases, the state of stress beyond this is more than what it was in case of the elastic. So, you see that you drop a line here. State of stress here is more than the state of stress corresponding to the elastic state.

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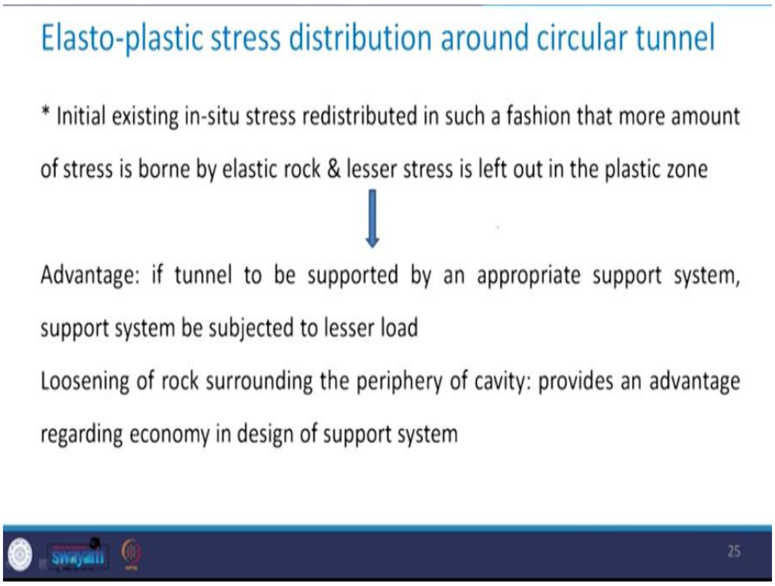

So, the initial existing in situ stress gets redistributed in such a fashion that more amount of stress is borne by the elastic rock and lesser stress is left out in the plastic zone. The advantage of this is that if the tunnel is to be supported by an appropriate support system, that support system would be subjected to the lesser load. What happens? All around the periphery of the cavity in the surrounding rock, there occurs the loosening of the rock.

And this provides an advantage as far as the economy in the design of support system is concerned. Now, as the rock immediately surrounding the cavity, it enters into the plastic state.

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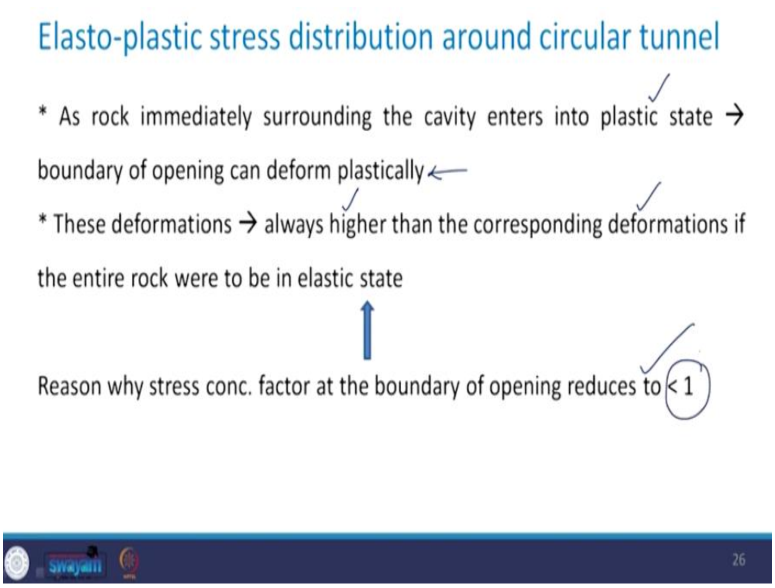

Boundary of the opening can deform plastically. These deformations are always going to be higher than the corresponding deformations if the entire rock were to be in the elastic state. That is the reason why stress concentration factor at the boundary of the opening becomes less than 1.

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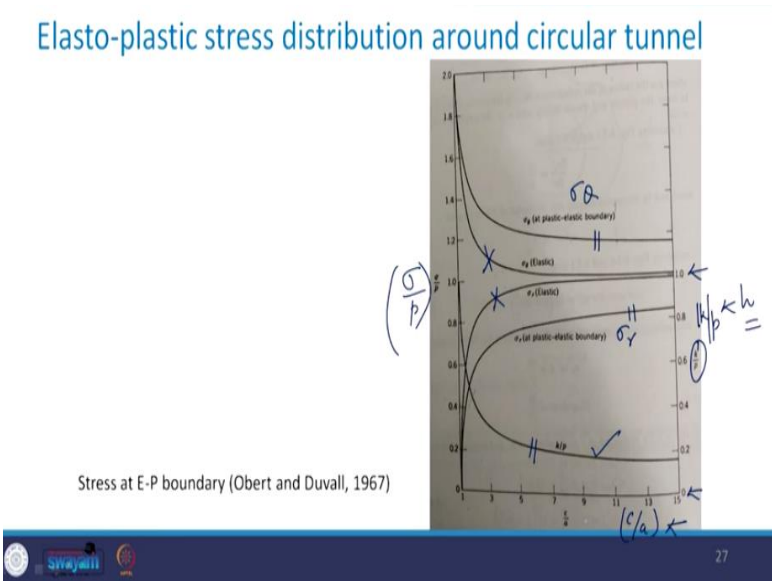

Now, this is the typical variation of σ_r and σ_θ at the elasto-plastic boundary. See this one is for the variation of σ_r at the elasto-plastic boundary. And this is for the σ_θ. While in case of the elastic state, this is the variation for σ_r. And this is the variation for σ_θ. Here on the x-axis you have c/a. And on y axis you have the stress concentration factor as σ /p.

And then one curve has also been shown for this the variation of κ/p, you can see here that it is the κ /p. That is this is *h*. So, that has been shown and it will vary from 0 to 1. And the variation of this has been shown with respect to this c/a by this plot. So, this is what that I wanted to discuss with you as far as the elasto-plastic stress distribution around circular tunnel is concerned.

We could see some of the typical features from this analysis which was based upon the TRESCA yield criterion. So, in the next class, we will learn about some of the failure modes and mechanism in case of the underground excavation. Thank you very much.