Rock Engineering

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Lecture-40
Thick Wall Cylinder in Biaxial Stress Field

Hello everyone, in the previous class we discussed about the complete solution towards the elastic analysis of a circular tunnel, we have seen some special cases like hydrostatic state of stress and uniaxial state of stress and how the distribution for the stresses and displacement can be derived and obtained for any particular location r comma theta (r, θ). So, that we saw in detail. So, today we will learn about the analysis for the thick wall cylinder in biaxial stress field.

This basically simulates the condition of a concrete lining in case of the tunnel. So, again this is an elastic analysis, so we will have the detailed derivation with respect to the analysis for thick wall cylinder in biaxial stress field.

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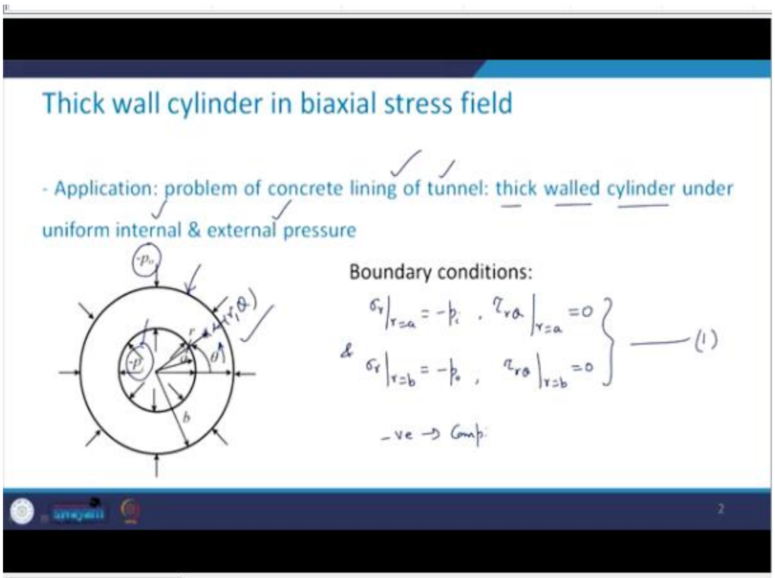

So, as I mentioned that the application is in the problem of concrete lining of the tunnel and to do this we consider a thick walled cylinder under uniform internal as well as external pressure, you can take a look at this figure where any point in the space can be represented by the coordinate r comma theta (r, θ) as we have been discussing for the few earlier classes. Again here theta is measured from the horizontal direction in the anticlockwise direction.

The radius of the tunnel is a and the thickness of this thick wall cylinder is b – a, that means this outer periphery it has a outer circle has a radius of b. Internal pressure is being denoted by p i there is a negative sign associated with it and the external pressure is denoted by - p 0. Let us take a look at the boundary conditions. So, we have for boundary conditions sigma r at r = a that is - p i and tau r theta at r = a will be = 0 and sigma r at r = b will be - p 0 and tau r theta at r = b will be = 0. So, these equations I mark them as 1. As we have taken earlier this negative sign will indicate to the compression.

$$\left.\sigma_r\right|_{r=a} = -p_i \qquad \left.\tau_{r\theta}\right|_{r=a} = 0$$
$$\left.\sigma_r\right|_{r=b} = -p_0 \;, \qquad \left.\tau_{r\theta}\right|_{r=b} = 0 \qquad (1)$$

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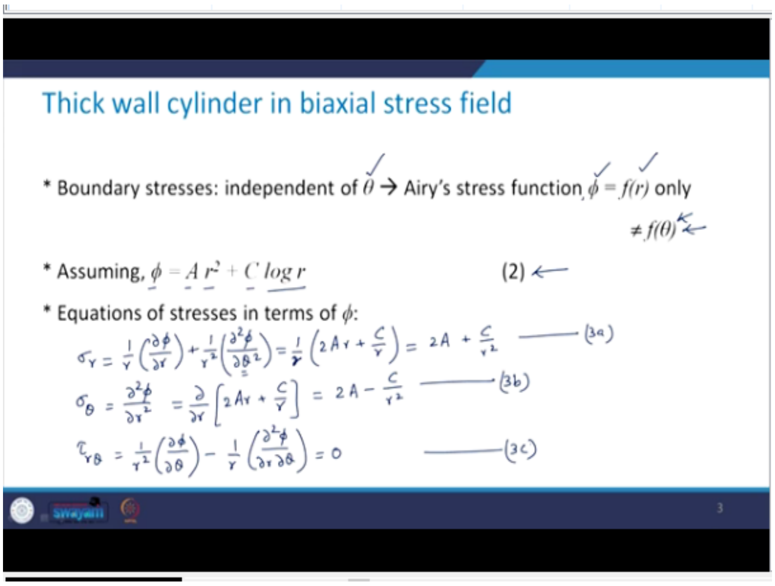

Now these boundary stresses they are independent of theta, so airy's stress function which was phi can be taken as the function of r only and not the function of theta. So, in this case it is assumed that the phi can be taken by this equation 2 that is phi = A r square + C log of r. These A and r they will be determined from appropriate boundary conditions. So, what are going to be the equations of stresses in terms of phi.

$$\phi = Ar^2 + C\log r \qquad (2)$$

Let us see these are sigma r will be = 1 upon r del phi del r + 1 upon r square del 2 phi del theta 2, just substitute this expression for phi here, so what you will have here is 1 upon r and 2 A r + C upon r and this is the differentiation with respect to theta and since phi is independent of theta. So, this term will become = 0. So, this is going to be 2 a + C upon r square, equation 3a.

$$\sigma_r = \frac{1}{r}\left(\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 \phi}{\partial r^2}\right) = \frac{1}{r}\left(2Ar + \frac{C}{r}\right) = 2A + \frac{C}{r^2} \tag{3a}$$

$$\sigma_\theta = \left(\frac{\partial^2 \phi}{\partial r^2}\right) = \frac{\partial}{\partial r}\left(2Ar + \frac{C}{r}\right) = 2A - \frac{C}{r^2} \tag{3b}$$

$$\tau_{r\theta} = \frac{1}{r^2}\left(\frac{\partial \phi}{\partial \theta}\right) - \frac{1}{r}\left(\frac{\partial^2 \phi}{\partial r \partial \theta}\right) = 0 \tag{3c}$$

Then sigma theta = del 2 phi del r 2 which is = say del del r and I differentiate phi once. So, that is 2 A r + C upon r then I have to differentiate it again. So, this will be 2 a - C upon r square. That is equation 3b. Then tau r theta will be = 1 upon r square del phi del theta - 1 upon r del 2 phi del r del theta and what will be this? This will simply be = 0 because phi is not a function of theta. So, differentiation with respect to theta will be = 0. So, that is how we can obtain these equations of stresses in terms of phi.

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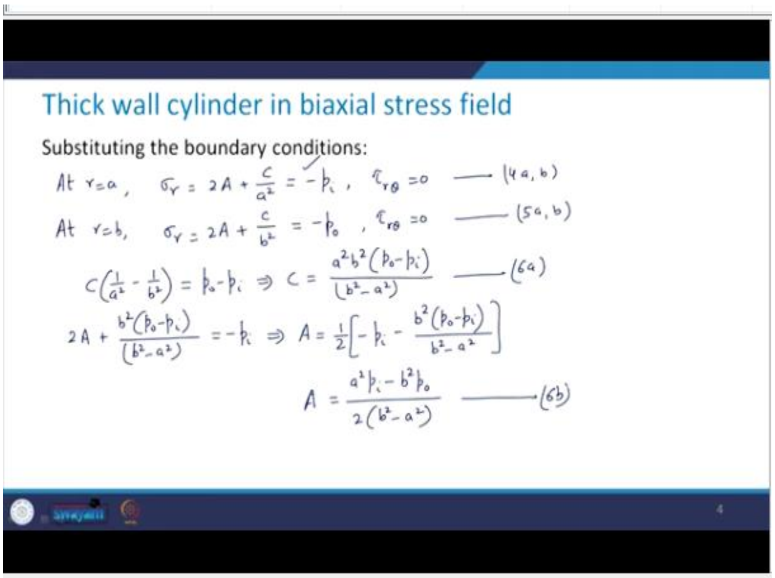

Now substitute the boundary conditions and see what we get that is at r = a, sigma r is going to. So, these equations will become 4a and b. Now at r = b we have sigma r as 2A + C upon b square which is = - p 0 and tau r theta will be = 0. This is 5a comma 5b.

$$\text{At } r = a,\ \sigma_r = 2A + \frac{C}{a^2} = -p_i,\ \tau_{r\theta} = 0 \tag{4a,b}$$

$$\text{At } r = b,\ \sigma_r = 2A + \frac{C}{b^2} = -p_0,\ \tau_{r\theta} = 0 \tag{5a,b}$$

$$C\left(\frac{1}{a^2}\frac{1}{b^2}\right)=p_0-p_i \Rightarrow C=\frac{a^2b^2\left(p_0-p_i\right)}{\left(b^2-a^2\right)} \tag{6a}$$

$$2A+\frac{b^2\left(p_0-p_i\right)}{\left(b^2-a^2\right)}=-p_i \Rightarrow A=\frac{a^2p_i-b^2p_0}{2\left(b^2-a^2\right)} \tag{6b}$$

And if we just solve this, this is what that we are going to get half - p i - b square p 0 - p i divided by b square - a square or if you further simplify this you will get A = a square p i - b square p 0 divided by 2 times b square - a square, make it 6b. Now you substitute these expressions 6a and 6b in equations 3a 3b and 3c.

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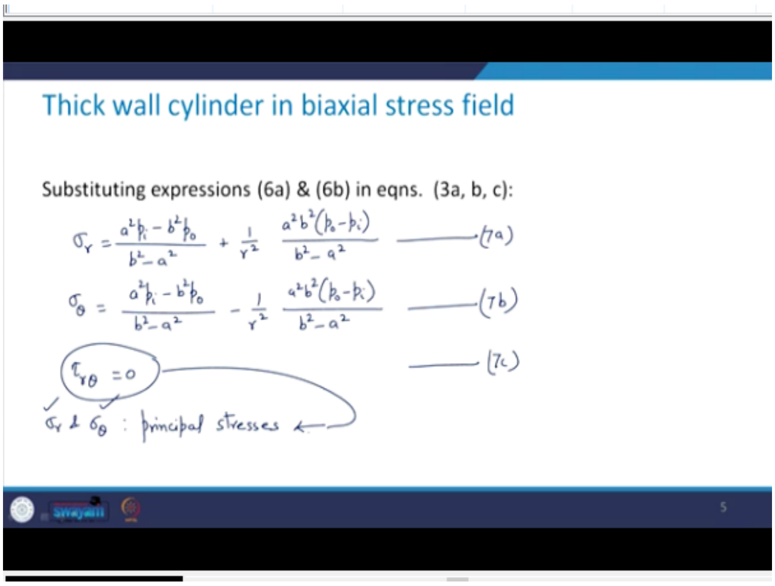

This is what that you are going to get that is sigma r will be = a square p i - b square p 0 divided by b square - a square + 1 upon r square a square b square p 0 - p i divided by b square - a square 7a. Sigma theta will be a square p i - b square p 0 divided by b square - a square - 1 upon r square, a square b square p 0 - p i divided by b square - a square. This is 7b and we have seen that tau r theta will be = 0.

$$\sigma_r=\frac{a^2p_i-b^2p_0}{\left(b^2-a^2\right)}+\frac{1}{r^2}\frac{a^2b^2\left(p_0-p_i\right)}{\left(b^2-a^2\right)} \tag{7a}$$

$$\sigma_\theta=\frac{a^2p_i-b^2p_0}{\left(b^2-a^2\right)}-\frac{1}{r^2}\frac{a^2b^2\left(p_0-p_i\right)}{\left(b^2-a^2\right)} \tag{7b}$$

$$\tau_{r\theta}=0 \tag{7c}$$

So, this is equation number 7c. This sigma r and sigma theta, these are the principal stresses, these are radial and tangential as well as these are the principal stresses why, because the shear stress is 0. So, these will become principal stresses as well.

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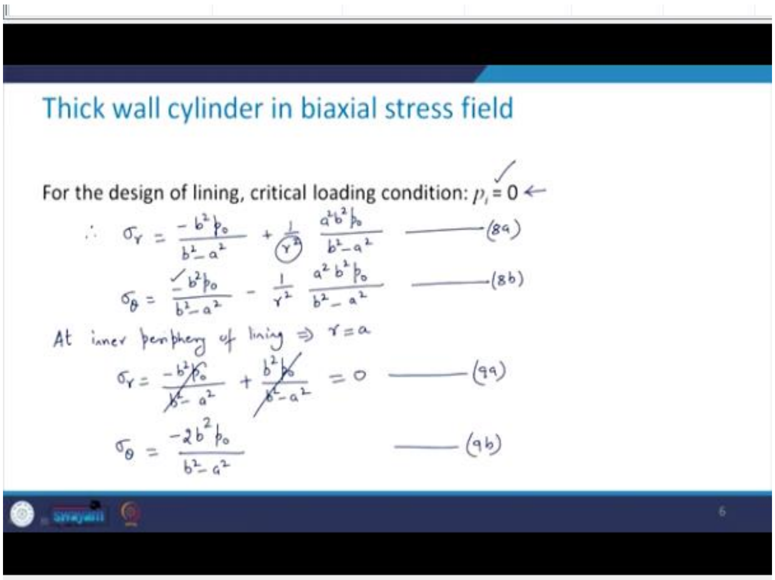

Now for the design of the lining the critical loading condition will be when your p i that is the internal pressure becomes = 0. So, when this becomes = 0 what will happen to sigma r? That will become - b square p 0 divided by b square - a square + 1 upon r square a square b square p 0 divided by b square - a square. That is equation 8a and your sigma theta will be = - b square p 0 divided by b square - a square - 1 upon r square a square b square p 0 divided by b square - a square. This will become 8b.

$$\sigma_r = \frac{-b^2 p_0}{\left(b^2 - a^2\right)} + \frac{1}{r^2}\frac{a^2 b^2 \left(p_0\right)}{\left(b^2 - a^2\right)} \tag{8a}$$

$$\sigma_\theta = \frac{-b^2 p_0}{\left(b^2 - a^2\right)} - \frac{1}{r^2}\frac{a^2 b^2 \left(p_0\right)}{\left(b^2 - a^2\right)} \tag{8b}$$

At inner periphery of lining, i.e; $r = a$

$$\sigma_r = 0 \tag{9a}$$

$$\sigma_\theta = \frac{-2b^2 p_0}{\left(b^2 - a^2\right)} \tag{9b}$$

Now what will happen at the inner periphery of the lining and what will this represent that is when r = a, your sigma r will become = - b square p 0 upon b square - a square and see here this a square here if you just substitute a square in place of r square this will get cancel out and what you will get is b square p 0 upon b square - a square.

These will get cancel out and this will become = 0 9a and your sigma theta will be see this is - b square p 0 upon b square - a square and the same thing will be here. So, I can write down simply - 2b square p 0 upon b square - a square. That is 9b.

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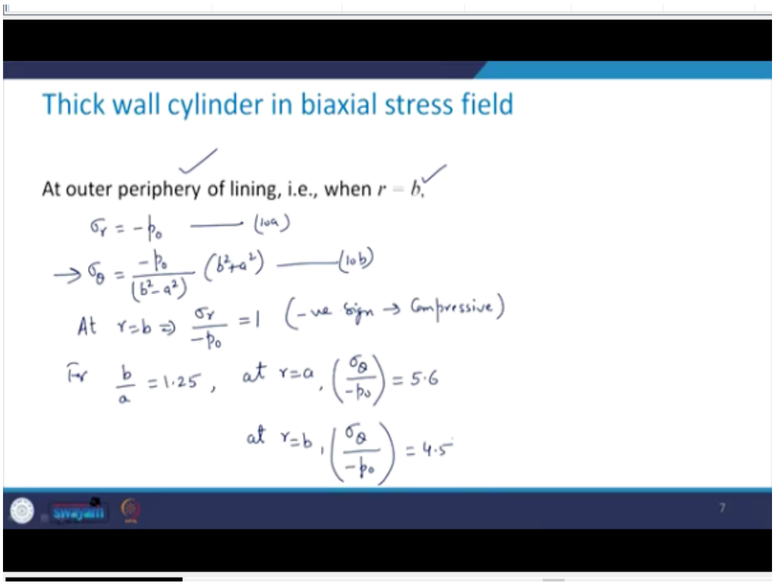

Now what will happen at the outer periphery of the lining which means that when r becomes = b, this sigma r is going to be = - p 0, that is 10a and sigma theta will become - p 0 divided by b square - a square into b square + a square, this is 10b. Now at r = b, that means at outer periphery of the lining your sigma r upon - p 0 = 1. Negative sign once again what is it? This is compressive in nature.

At outer periphery of lining, i.e; r = b

$$\sigma_r = -p_0 \tag{10a}$$

$$\sigma_\theta = \frac{-p_0}{\left(b^2 - a^2\right)}\left(b^2 + a^2\right) \tag{10b}$$

At $r = b$; $\dfrac{\sigma_r}{-p_0} = 1$ (negative sign $\rightarrow$ Compressive)

For b by a = 1.25 and I take at r = a, your sigma theta upon - p 0, that means this quantity. From this equation, this is going to be = 5.6 and at r = b your sigma theta upon - p 0 will be

4.5. So, let us try to draw the variation of this stress concentration factor which is sigma upon p 0 with respect to the lining thickness.

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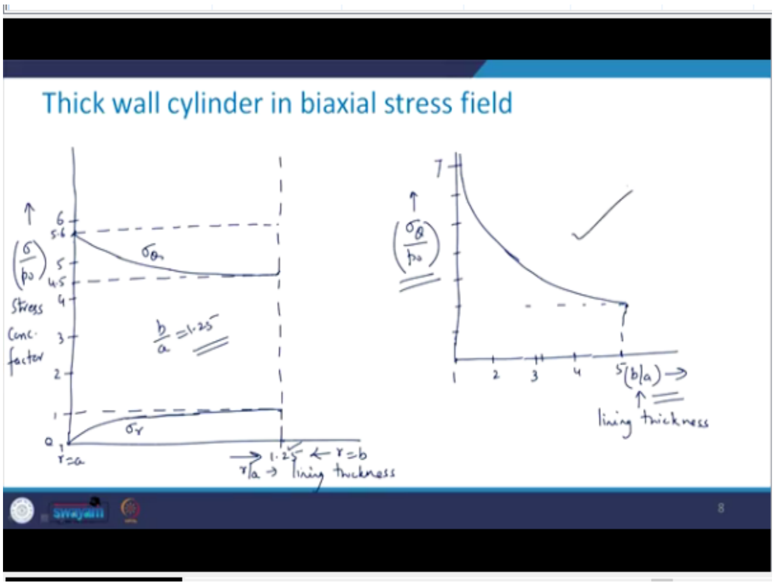

Let us see how it looks like. So, we have one axis vertical axis here and horizontal axis. In the horizontal axis I am drawing r upon a which is the lining thickness and here I have a value I draw a vertical line corresponding to r by a = 1.25, that means here it is r = b because we just took b by a = 1.25 fine. So, here it is r = a that means here it is 1 and here it is 1.25. Now on the y axis say this is = 0 and I divide it into some parts 2 3 4 5 and then 6.

So, this is here 1 2 3 4 5 and 6. So, here what we have is sigma upon p 0 which is increasing in this direction and what do we call this as stress concentration factor. Now I just draw a horizontal line from this 1 and we have seen that this at r = b your sigma r becomes = 1 and at r = a, this is 0. So, this is how its variation is going to look like. Here it will become = 1. So, it will be this is going to be the variation for sigma r.

And in case of the sigma theta we have seen that at r = a it is 5.6. So, let us say this for 5.6 so I draw a line and r = b we have 4.5. So, another line let us say here this is corresponding to say 4.5. This is your 4.5. So, this is like r = a it is 5.6 and then it goes to 4.5 when r = b, that is your b by a I have considered as 1.25. So, its variation will look like this. So, this is the variation for sigma theta.

Now let us try to draw the variation of this stress concentration factor with respect to b by a, because in this case we took b by a = 1.25. Now it is not necessary that always it will be 1.25.

So, here I have b upon a again this is the representation of the lining thickness and increasing in this direction and here I have sigma theta upon p 0. So, I have here 1, 2 maybe this is say 3, this is 4 and here I have 5, and on this axis I have 1 2 3 4 5 6 and 7, let us see.

So, this is your 7 and at b by a = 5 it will be the variation is going to look like this something like this. So, this is what is the variation of this stress concentration factor with respect to b by a, if somebody asks you that how the variation of radial and the tangential stresses will be in case of the thick wall cylinder in biaxial stress field? This is how they will look like.

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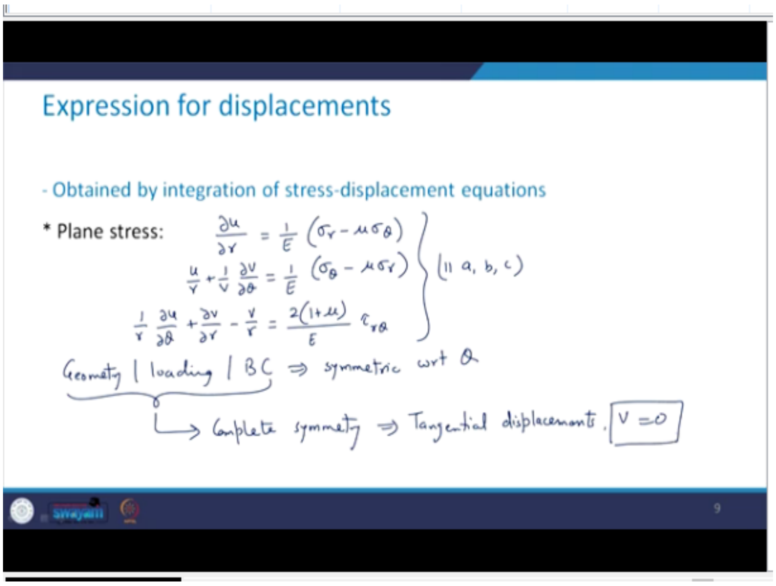

Now how to obtain the displacements? So, again as we have seen in the previous case of the elastic analysis of the circular tunnel, in this case also this will be obtained by the integration of stress displacement equation. So, we take the situation of a plane stress and you have seen that we have these set of equations which is sigma r - mu sigma theta. That is the first one, then we have u upon r + 1 upon v del v del theta is = 1 upon e sigma theta - mu times sigma r.

$$\frac{\partial u}{\partial r}=\frac{1}{E}\left(\sigma_r-\mu\sigma_\theta\right) \tag{11a}$$

$$\frac{u}{r}+\frac{1}{v}\frac{\partial v}{\partial \theta}=\frac{1}{E}\left(\sigma_\theta-\mu\sigma_r\right) \tag{11b}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}=\frac{2\left(1+\mu\right)}{E}\tau_{r\theta} \tag{11c}$$

So, we take these as equation 11 a, b, and c. Since the geometry loading and boundary conditions, they are all symmetric with respect to theta. So, when we have the symmetry with respect to all the 3 situation, this is what we call as the complete symmetry. And therefore the tangential displacements which are represented by v, this is going to be = 0 because of this complete symmetry.

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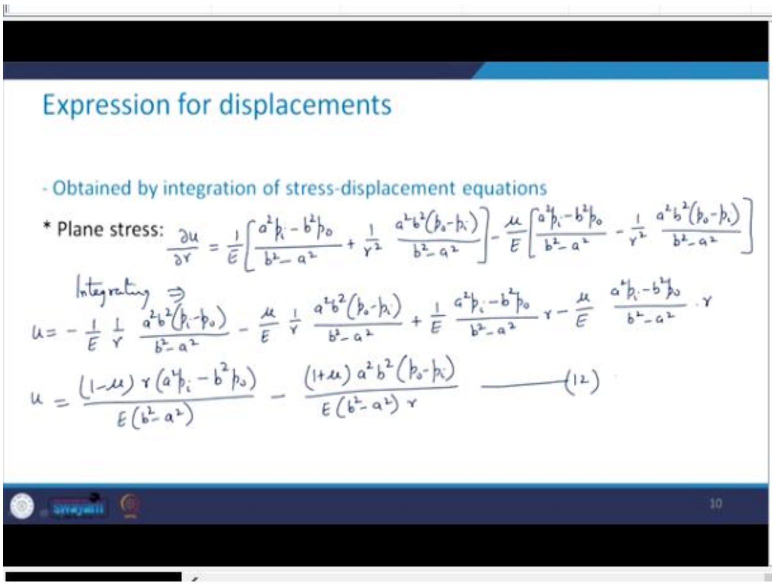

So, we get this.

$$\frac{\partial u}{\partial \theta}=\frac{1}{E}\left(\frac{a^2p_i-b^2p_0}{\left(b^2-a^2\right)}+\frac{1}{r^2}\frac{a^2b^2\left(p_0-p_i\right)}{\left(b^2-a^2\right)}\right)-\frac{\mu}{E}\left(\frac{a^2p_i-b^2p_0}{\left(b^2-a^2\right)}-\frac{1}{r^2}\frac{a^2b^2\left(p_0-p_i\right)}{\left(b^2-a^2\right)}\right)$$

So, if we integrate it what we can get is the expression for you So, see how we will derive that, it will look

On Integrating;

$$u=\frac{(1-\mu)r\left(a^2p_i-b^2p_0\right)}{E\left(b^2-a^2\right)}-\frac{(1+\mu)a^2b^2\left(p_0-p_i\right)}{E\left(b^2-a^2\right)r} \qquad (12)$$

Just solve this and you will get this expression in this form 1 - mu r a square p i - b square p 0 whole divided by E times b square - a square - 1 + mu a square b square p 0 - p i divided by E times b square - a square into r. This is going to be equation number 12.

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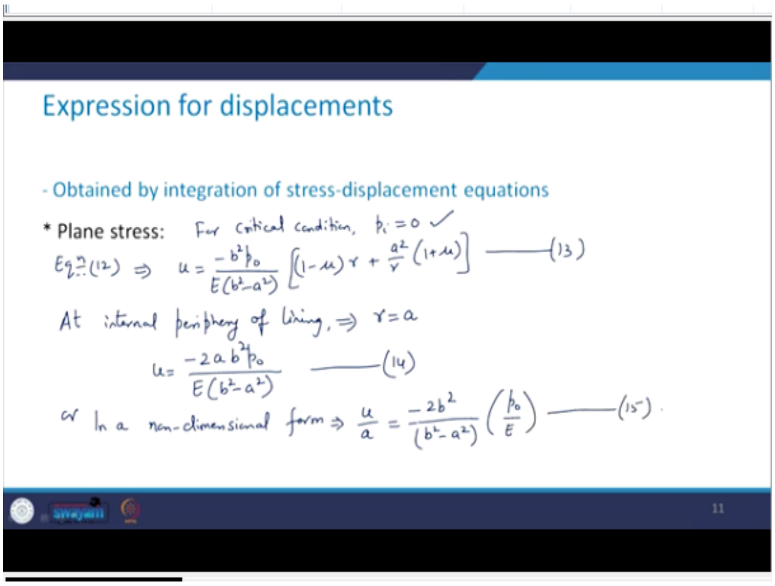

Now what is the situation when we have the critical condition? This will be equation number 13 and at the internal periphery of the lining what does this mean, what will be the value of r? This will be = a.

For critical condition: $p_i = 0$. Thus Eqn. 12 becomes:

$$u = \frac{-b^2 p_0}{E\left(b^2 - a^2\right)}\left((1-\mu)r + \frac{a^2}{r}(1+\mu)\right) \tag{13}$$

At internal periphery of lining; $r = a$

$$u = \frac{-2ab^2 p_0}{E\left(b^2 - a^2\right)} \tag{14}$$

Or in a non-dimensional form:

$$\frac{u}{a} = \frac{-2b^2}{\left(b^2 - a^2\right)}\left(\frac{p_0}{E}\right) \tag{15}$$

So, this is how we can obtain the displacement in case if we want to analyze the concrete lining in case of the circular tunnel. So, this is what is for the plane stress condition which will be the most relevant condition in this case.

So, in detail we discussed about the derivation of the stresses and displacement for the elastic analysis of the concrete lining in case of the circular tunnel. So, this is what that I wanted to discuss with you as far as the elastic analysis of the circular tunnel is concerned. In the next class we will learn about the elastoplastic analysis of the circular tunnel, thank you very much.