Rock Engineering
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Lecture-39
Elastic Analysis of Circular Tunnels-Displacements

Hello everyone. In the previous class, we discussed about the elastic analysis of circular tunnels, we saw that how the stress distribution can be obtained all around the tunnel periphery or along any direction say θ equal to 0 and θ equal to 90°. We saw these things with respect to hydrostatic state of stress as well as for uniaxial state of stress. So, today, we will continue that discussion and now we will see that how the displacements can be obtained in this elastic analysis of circular tunnel.

Because, it is not only the stresses, but the displacements are also very important from the design point of view. So, today we will learn about the aspects related to the determination of displacements all around the tenor periphery. So, these expressions can be obtained by integrating the stress displacement equations for plane stress state.

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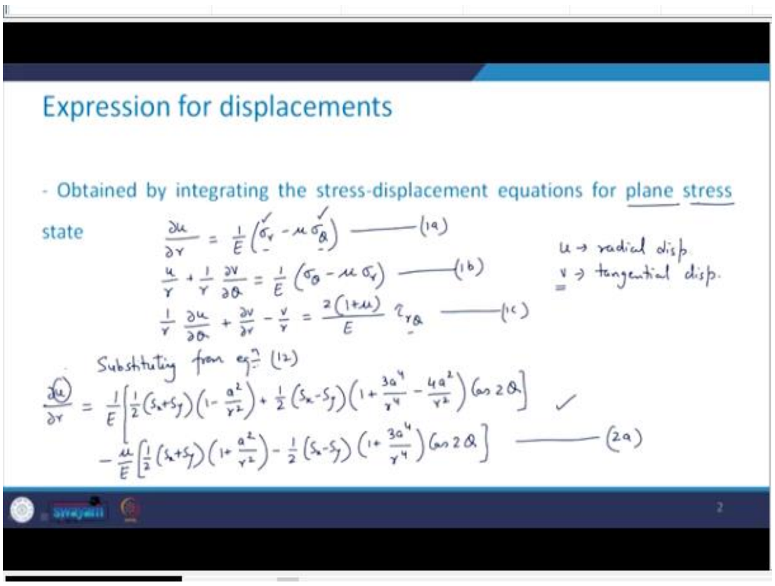

Let us see how? So, far the plane stress state we have these equations which are

$$\frac{\partial u}{\partial r} = \frac{1}{E}(\sigma_r - \mu\sigma_\theta) \rightarrow (1a)$$

$$\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{E}(\sigma_\theta - \mu\sigma_r) \rightarrow (1b)$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}=\frac{2(1+\mu)}{E}\tau_{r\theta}\rightarrow(1b)$$

We discussed that this u is the radial displacement and v is the tangential displacement which is in the direction perpendicular to the direction of u. Now, recall our discussion of the previous class, we had equation number 12 where the expression forσ_r, σ_θ etcetera was given. So, whatever the expression for these quantities which are here sσ_r, σ_θ and $\tau_{r\theta}$,let us substitute it in these 3 equations that is *1a*, *1b*, and *1c*.

So, we have here substituting from equation number 12. So, first let us see that what will happen to this equation *1a*. This is going to be

$$\frac{\partial u}{\partial r}=\frac{1}{E}\left[\frac{1}{2}(S_x+S_y)\left(1-\frac{a^2}{r^2}\right)+\frac{1}{2}(S_x-S_y)\left(1+\frac{3a^4}{r^4}-4\frac{a^2}{r^2}\right)\cos 2\theta\right]$$
$$-\frac{\mu}{E}\left[\frac{1}{2}(S_x+S_y)\left(1-\frac{a^2}{r^2}\right)-\frac{1}{2}(S_x-S_y)\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]\rightarrow(2a)$$

Make this equation as equation number *2a*. Now, if we integrate this equation *2a* we will be getting the expression for u, let us see how?

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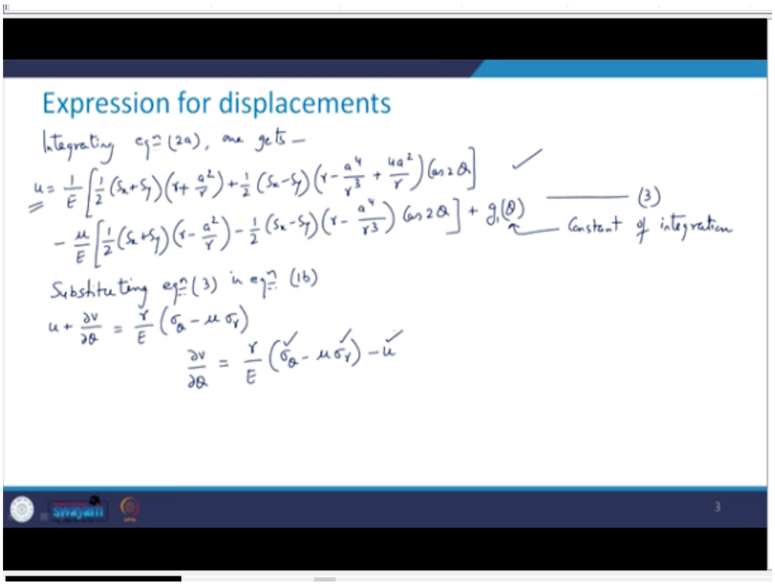

So, integrating this equation *2a* what one can get,

$$u = \frac{1}{E}\left[\frac{1}{2}(S_x + S_y)\left(r + \frac{a^2}{r}\right) + \frac{1}{2}(S_x - S_y)\left(r - \frac{a^4}{r^3} + 4\frac{a^2}{r}\right)\cos 2\theta\right]$$
$$- \frac{\mu}{E}\left[\frac{1}{2}(S_x + S_y)\left(r - \frac{a^2}{r}\right) - \frac{1}{2}(S_x - S_y)\left(r - \frac{a^4}{r^3}\right)\cos 2\theta\right] + g_1(\theta) \rightarrow (3)$$

And there is going to be one constant of integration which will only be the function of θ. So, make this equation as equation number 3, where this $g_1(\theta)$ is the constant of integration. Now, you substitute this equation number 3 and expression for u in equation number *1b*. So, what we are going to get is this substituting equation *3* in equation *1b*. So, we have here that

$$u + \frac{\partial v}{\partial \theta} = \frac{r}{E}(\sigma_\theta - \mu\sigma_r)$$

$$\frac{\partial v}{\partial \theta} = \frac{r}{E}(\sigma_\theta - \mu\sigma_r) - u$$

Now, here you again substitute the expression for σ_θ and σ_r and also for u and then simplify it further let us see what we get.

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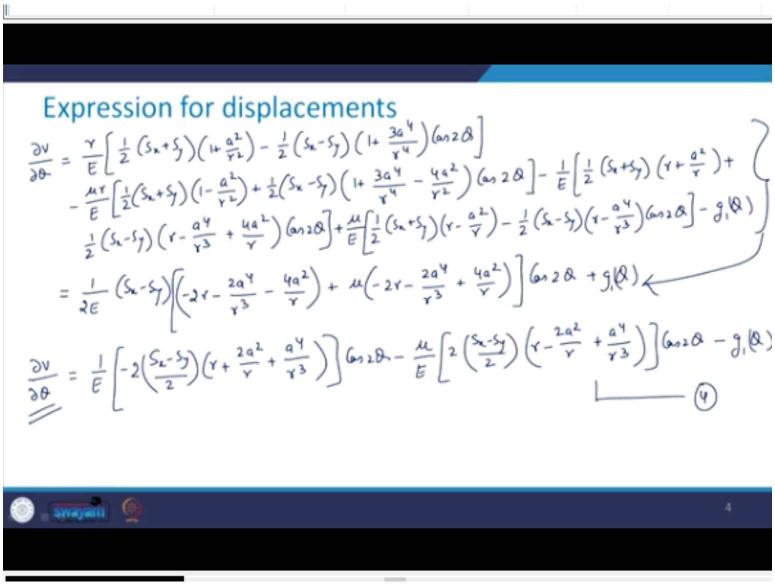

So, we have here

$$\frac{\partial v}{\partial \theta} = \frac{r}{E}\left[\frac{1}{2}(S_x + S_y)\left(1 - \frac{a^2}{r^2}\right) - \frac{1}{2}(S_x - S_y)\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta\right]$$
$$- \frac{\mu r}{E}\left[\frac{1}{2}(S_x + S_y)\left(1 - \frac{a^2}{r^2}\right) + \frac{1}{2}(S_x - S_y)\left(1 + \frac{3a^4}{r^4} - 4\frac{a^2}{r^2}\right)\cos 2\theta\right]$$
$$- \frac{1}{E}\left[\frac{1}{2}(S_x + S_y)\left(r + \frac{a^2}{r}\right) + \frac{1}{2}(S_x - S_y)\left(r - \frac{a^4}{r^3} + 4\frac{a^2}{r}\right)\cos 2\theta\right]$$
$$+ \frac{\mu}{E}\left[\frac{1}{2}(S_x + S_y)\left(r - \frac{a^2}{r}\right) - \frac{1}{2}(S_x - S_y)\left(r - \frac{a^4}{r^3}\right)\cos 2\theta\right] - g_1(\theta)$$

$$= \frac{1}{2E}(S_x - S_y)\left[\left(-2r - \frac{2a^4}{r^3} - \frac{4a^2}{r}\right) + \mu\left(-2r - \frac{2a^4}{r^3} - \frac{4a^2}{r}\right)\right]\cos 2\theta + g_1(\theta)$$

$$\frac{\partial v}{\partial \theta} = \frac{1}{E}\left[-2\left(\frac{S_x - S_y}{2}\right)\left(r + \frac{2a^2}{r} + \frac{a^4}{r^3}\right)\right]\cos 2\theta$$
$$- \frac{\mu}{E}\left[2\left(\frac{S_x - S_y}{2}\right)\left(r - \frac{2a^2}{r} + \frac{a^4}{r^3}\right)\right]\cos 2\theta - g_1(\theta) \rightarrow (4)$$

Now, if we integrate this equation, which is equation number 4 with respect to θ and I will be getting the expression for v.

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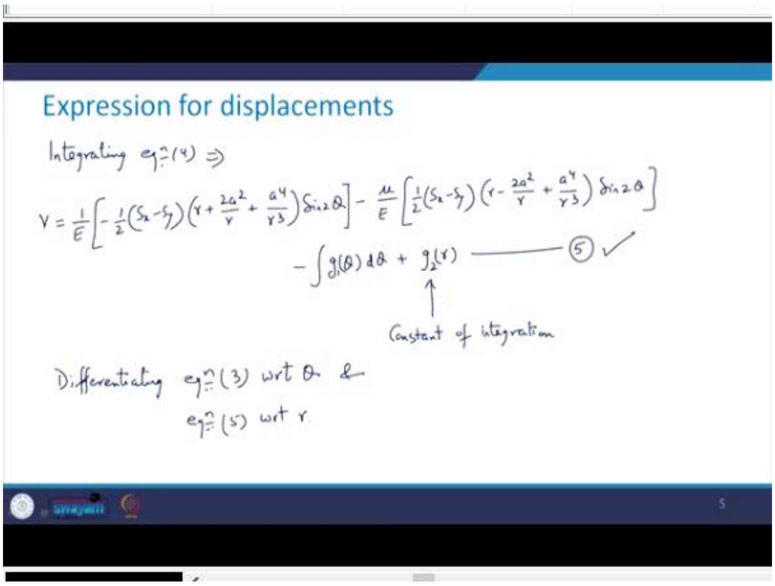

So, let us do that integrating equation number *4* what we will get is

$$v = \frac{1}{E}\left[-\frac{1}{2}(S_x - S_y)\left(r + \frac{2a^2}{r} + \frac{a^4}{r^3}\right)\sin 2\theta\right] - \frac{\mu}{E}\left[\frac{1}{2}(S_x - S_y)\left(r - \frac{2a^2}{r} + \frac{a^4}{r^3}\right)\sin 2\theta\right]$$
$$- \int g_1(\theta) d\theta + g_2(r) \rightarrow (5)$$

This is equation number *5* and this $g_2(r)$ is again the constant of integration. Now, what we can do is differentiate equation number *3* with respect to θ and equation number 5 with respect to r. So, this is what that I am going to get is differentiating equation 3 with respect to θ and equation number 5 which is this equation with respect to r.

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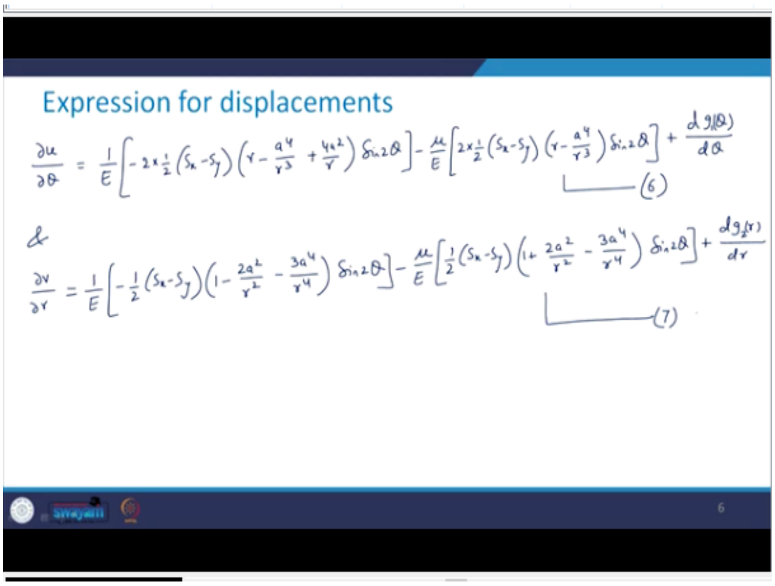

So, see this is what that we will get, say

$$\frac{\partial u}{\partial \theta} = \frac{1}{E}\left[-2 \times \frac{1}{2}(S_x - S_y)\left(r - \frac{a^4}{r^3} + \frac{4a^2}{r}\right)\sin 2\theta\right] - \frac{\mu}{E}\left[2 \times \frac{1}{2}(S_x - S_y)\left(r - \frac{a^4}{r^3}\right)\sin 2\theta\right] + \frac{dg_1(\theta)}{d\theta} \rightarrow (6)$$

$$\frac{\partial v}{\partial r} = \frac{1}{E}\left[-\frac{1}{2}(S_x - S_y)\left(1 - \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\sin 2\theta\right] - \frac{\mu}{E}\left[\frac{1}{2}(S_x - S_y)\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\sin 2\theta\right] + \frac{dg_2(r)}{dr} \rightarrow (7)$$

So, you will not have any difficulty. Now, what we will do is, we will substitute these equations 5, 6, 7 in equation number *1c*.

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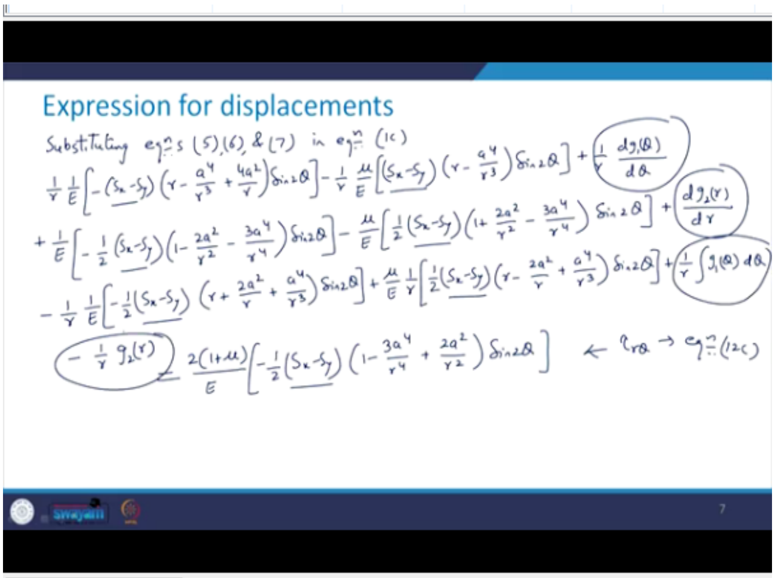

So, substituting these equations *5*, *6* and *7* in equation number *1c*. So, what we are going to get

$$\frac{1}{r}\frac{1}{E}\left[-(S_x-S_y)\left(r-\frac{a^4}{r^3}+\frac{4a^2}{r}\right)\sin 2\theta\right]-\frac{1}{r}\frac{\mu}{E}\left[(S_x-S_y)\left(r-\frac{a^4}{r^3}\right)\sin 2\theta\right]+\frac{1}{r}\frac{dg_1(\theta)}{d\theta}$$
$$+\frac{1}{E}\left[-\frac{1}{2}(S_x-S_y)\left(1-\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta\right]$$
$$-\frac{\mu}{E}\left[-\frac{1}{2}(S_x-S_y)\left(1+\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta\right]+\frac{dg_2(r)}{dr}$$
$$-\frac{1}{r}\frac{1}{E}\left[-\frac{1}{2}(S_x-S_y)\left(r+\frac{2a^2}{r}+\frac{a^4}{r^3}\right)\sin 2\theta\right]$$
$$+\frac{1}{r}\frac{\mu}{E}\left[\frac{1}{2}(S_x-S_y)\left(r-\frac{2a^2}{r}+\frac{a^4}{r^3}\right)\sin 2\theta\right]+\frac{1}{r}\int g_1(\theta)d\theta-\frac{1}{r}g_2(r)$$

$$\tau_{r\theta}=\frac{2(1+\mu)}{E}\left[-\frac{1}{2}(S_x-S_y)\left(1+\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta\right]\rightarrow(12c)$$

Now, what we need to do is as we did in the case of the stress distribution, we compare the terms on either side of these equation. So, this is what that we are going to get. See this all these terms they have S_x - S_y in with their terms. So, these terms that is this, this, this one and this one, these have to be equal to 0 in a combined fashion.

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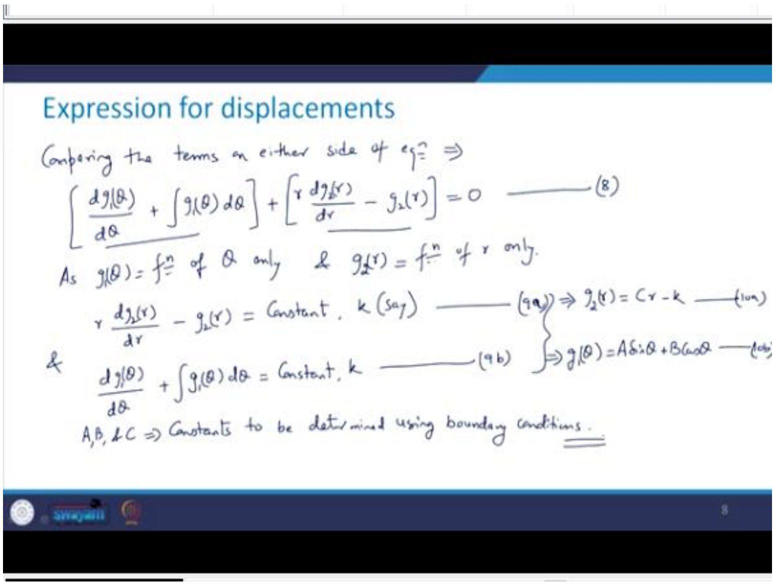

So, comparing the terms on either side of this equation, this is going to give us,

$$\left[\frac{dg_1(\theta)}{d\theta} + \int g_1(\theta)d\theta\right] + \left[r\frac{dg_2(r)}{dr} - g_2(r)\right] = 0 \rightarrow (8)$$

Now as this $g_1(\theta)$ is a function of θ only and your $g_2(r)$ is the function of r only.

So, separately this and this they both will be equal to a constant. So, that is what that we are going to write that is

$$r\frac{dg_2(r)}{dr} - g_2(r) = constant, k \ \rightarrow (9a) \Rightarrow g_2(r) = Cr - K \rightarrow (10a)$$

$$\frac{dg_1(\theta)}{d\theta} + \int g_1(\theta)d\theta = constant, k \ \rightarrow (9b) \Rightarrow g_1(\theta) = A\sin\theta + B\cos\theta \rightarrow (10b)$$

These A, B and C, they are the constants to be determined from the boundary conditions using the boundary conditions. So, let us see further what we can do. So, this *10b* I will substitute in equation number *3*.

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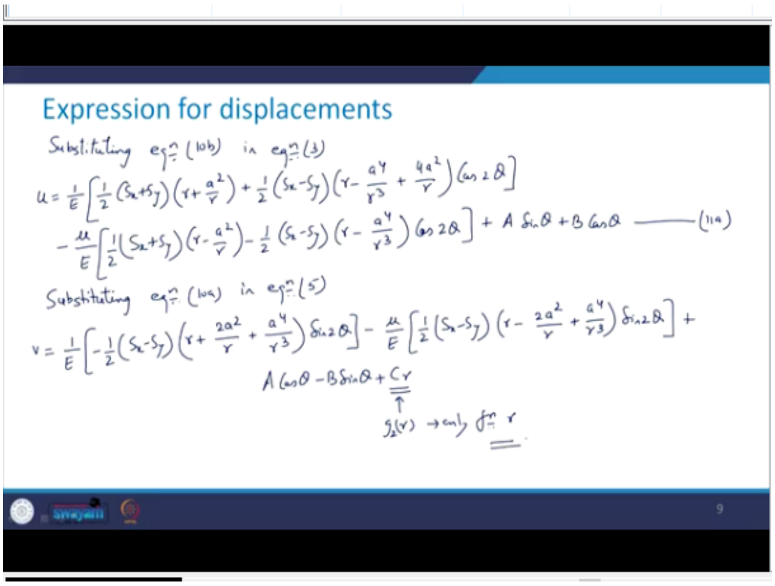

So, substituting equation *10b* in equation number *3*. So, what we will get is

$$u = \frac{1}{E}\left[\frac{1}{2}(S_x + S_y)\left(r + \frac{a^2}{r}\right) + \frac{1}{2}(S_x - S_y)\left(r - \frac{a^4}{r^3} + 4\frac{a^2}{r}\right)\cos 2\theta\right]$$
$$- \frac{\mu}{E}\left[\frac{1}{2}(S_x + S_y)\left(r - \frac{a^2}{r}\right) - \frac{1}{2}(S_x - S_y)\left(r - \frac{a^4}{r^3}\right)\cos 2\theta\right] + A\sin\theta$$
$$+ B\cos\theta \rightarrow (11a)$$

This equation I will mark as *11a*. So, similarly, I will substitute equation *10a* in equation number *5* which is the expression for *v*.

$$v = \frac{1}{E}\left[-\frac{1}{2}(S_x - S_y)\left(r + \frac{2a^2}{r} + \frac{a^4}{r^3}\right)\sin 2\theta\right] - \frac{\mu}{E}\left[\frac{1}{2}(S_x - S_y)\left(r - \frac{2a^2}{r} + \frac{a^4}{r^3}\right)\sin 2\theta\right]$$
$$+ A\cos\theta - B\cos\theta + Cr$$

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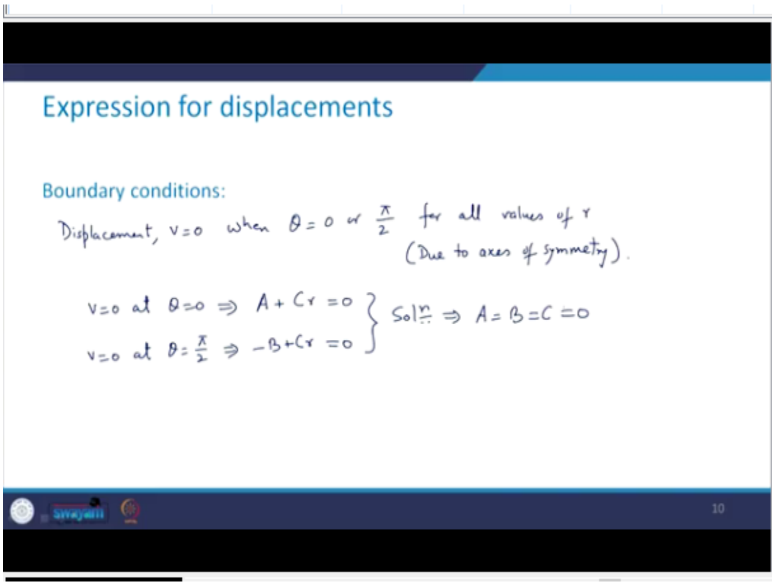

Now, after getting this is the time to apply the boundary conditions. So, we need to apply here the boundary conditions with respect to the displacement. So, what do we have here is that displacement *v* will be equal to 0 then *θ* = 0 or *Π/2* for all values of *r*. Now, what is the reason behind this? It is due to the axis of symmetry. So, this is due to axis of symmetry. Now, this

$$v = 0 \; at \; \theta = 0 \; \Rightarrow A + Cr = 0$$

$$v = 0 \; at \; \theta = \frac{\pi}{2} \Rightarrow -B + Cr = 0$$

And if you try to solve both of these equations, so, the solution will be A = B = C = 0. Now, like we did in case of the stresses in this case also we will take the 2 conditions, one for the biaxial and the second one is for the uniaxial.

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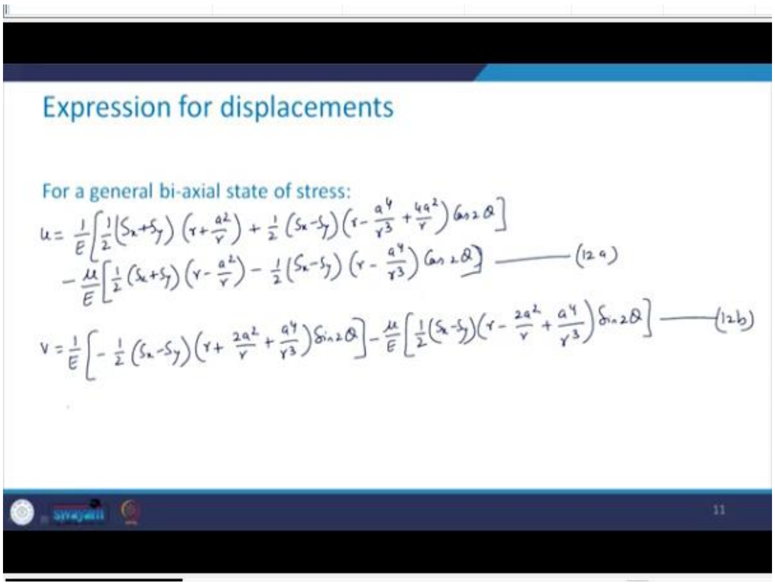

So, for the general biaxial state of stress what we will get

$$u=\frac{1}{E}\left[\frac{1}{2}(S_x+S_y)\left(r+\frac{a^2}{r}\right)+\frac{1}{2}(S_x-S_y)\left(r-\frac{a^4}{r^3}+4\frac{a^2}{r}\right)\cos 2\theta\right]$$

$$-\frac{\mu}{E}\left[\frac{1}{2}(S_x+S_y)\left(r-\frac{a^2}{r}\right)-\frac{1}{2}(S_x-S_y)\left(r-\frac{a^4}{r^3}\right)\cos 2\theta\right]\rightarrow(12a)$$

$$v=\frac{1}{E}\left[-\frac{1}{2}(S_x-S_y)\left(r+\frac{2a^2}{r}+\frac{a^4}{r^3}\right)\sin 2\theta\right]-\frac{\mu}{E}\left[\frac{1}{2}(S_x-S_y)\left(r-\frac{2a^2}{r}+\frac{a^4}{r^3}\right)\sin 2\theta\right]$$

$$\rightarrow(12b)$$

That is 12b. Now, what will happen at the tunnel boundary?

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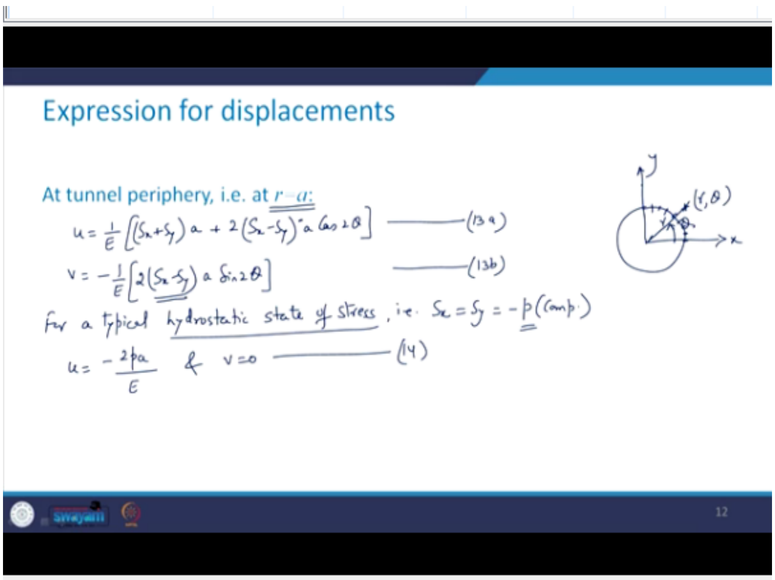

So, at the tunnel boundary that is, at $r = a$ just substitute $r = a$ in the earlier expressions. So, you will be getting

$$u = \frac{1}{E}\left[\left(S_x + S_y\right)a + 2\left(S_x - S_y\right)a\cos 2\theta\right] \rightarrow (13a)$$

$$v = \frac{1}{E}\left[2\left(S_x - S_y\right)a\sin 2\theta\right] \rightarrow (13b)$$

Now, you know that any point on the periphery will be defined by θ, if you just recall this figure that we had.

So, for any point we had this r, θ, this was x direction and this was y direction r was this distance and θ was here. So, along the periphery any point will be defined by the value of θ because r is going to be equal to the radius of the tunnel which is equal to A. So, for a typical hydrostatic state of stress what will happen? That is $S_x = S_y = -p$. Why I am writing - p here because tension we have taken as positive in this case, so, this is compressive.

So, with a negative sign, substitute this whole thing here and what you will get,

$$u = \frac{-2pa}{E} \; and \; v = 0 \; \rightarrow (14)$$

So, your v will be equal to 0, make it equation number *14*. Now, this is typically for the hydrostatic state of stress.

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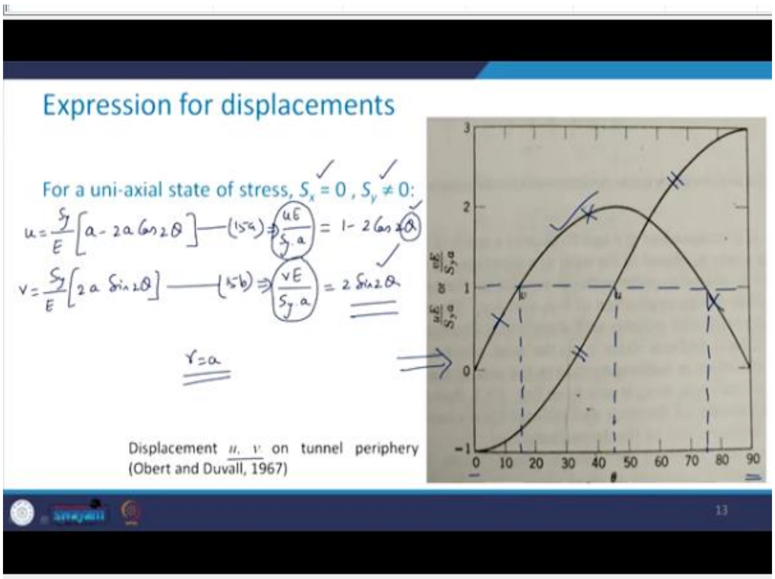

Now, what will happen in case if you have the uniaxial state of stress where you have $S_x = 0$ and S_y is nonzero. So, you will get here as

$$u = \frac{S_y}{E}[a - 2a\cos 2\theta] \rightarrow (15a) \Rightarrow \frac{uE}{S_y a} = 1 - 2\cos 2\theta$$

$$v = \frac{S_y}{E}[2a\sin 2\theta] \rightarrow (15b) \Rightarrow \frac{vE}{S_y a} = 2\sin 2\theta$$

Now, the variation of this quantity and this quantity with respect to θ has been shown in this figure on the tunnel periphery that is, when you have $r = a$. That is what is the case that we are considering? So, accordingly this plot is giving you the variation of u and this one is giving you the variation of v.

So, just keep on substituting the value of θ from 0 to 90° and you will be able to generate this kind of smooth curve, some typical values which you can take let us say that when you have here $\theta = 15°$, just substitute $\theta = 15°$ and see what you get the value of v it will work out to be 1. Similarly, for $\theta = 75°$ also this v will work out to be 1.

And this for 45° your u will work out to be 1. So, this is how you keep on substituting the value of θ in these expressions and you will be able to get this variation of the displacement u and v on the tunnel periphery. Now let us see that what happens in case of the plane strain. So, this is what that we discuss till now was the plane stress state.

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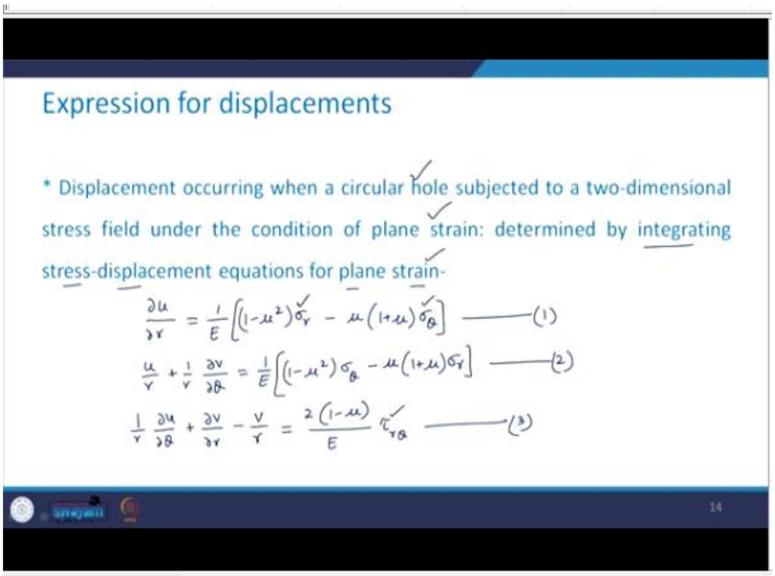

Now what will happen in case of the plane strain, displacement which is occurring when the circular hole is subjected to a two-dimensional stress field under the condition of the plane strain? So, in this case we have to integrate these stress displacement equations which are there for the plane strain. In the previous case we integrated the equations for plane stress. Now from the theory of elasticity what are the equations for the plane strain?

See these look like this

$$\frac{\partial u}{\partial r} = \frac{1}{E}[(1-\mu^2)\sigma_r - \mu(1+\mu)\sigma_\theta] \rightarrow (1)$$

$$\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{E}[(1-\mu^2)\sigma_\theta - \mu(1+\mu)\sigma_r] \rightarrow (2)$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \frac{2(1-\mu)}{E}\tau_{r\theta} \rightarrow (3)$$

So, these are the equations for stress displacement relationship for the plane strain situation. Now we can substitute the expression for σ_r σ_θ and $\tau_{r\theta}$ again as we did in the previous case.

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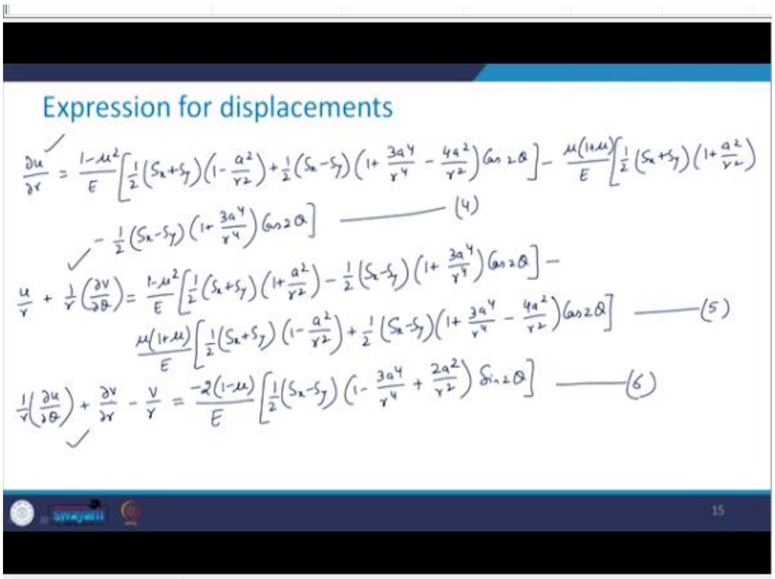

So, let us see what we get is

$$\frac{\partial u}{\partial r} = \frac{1-\mu^2}{E}\left[\frac{1}{2}(S_x+S_y)\left(1-\frac{a^2}{r^2}\right) + \frac{1}{2}(S_x-S_y)\left(1+\frac{3a^4}{r^4} - 4\frac{a^2}{r^2}\right)\cos 2\theta\right]$$
$$-\frac{\mu(1+\mu)}{E}\left[\frac{1}{2}(S_x+S_y)\left(1+\frac{a^2}{r^2}\right) - \frac{1}{2}(S_x-S_y)\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right] \rightarrow (4)$$

Similarly, you will have

$$\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}=\frac{1-\mu^2}{E}\left[\frac{1}{2}(S_x+S_y)\left(1+\frac{a^2}{r^2}\right)-\frac{1}{2}(S_x-S_y)\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]$$
$$-\frac{\mu(1+\mu)}{E}\left[\frac{1}{2}(S_x+S_y)\left(1-\frac{a^2}{r^2}\right)+\frac{1}{2}(S_x-S_y)\left(1+\frac{3a^4}{r^4}-4\frac{a^2}{r^2}\right)\cos 2\theta\right]$$
$$\rightarrow (5)$$

And the last equation which is

$$\frac{1}{r}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r}=\frac{-2(1-\mu)}{E}\left[\frac{1}{2}(S_x-S_y)\left(1-\frac{3a^4}{r^4}+2\frac{a^2}{r^2}\right)\sin 2\theta\right]\rightarrow (6)$$

So, now if you integrate these equations this, this.

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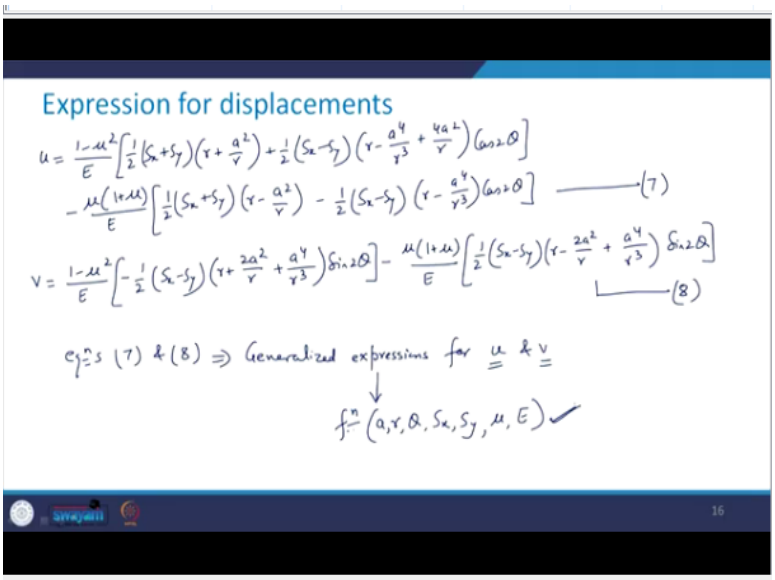

And this you will be able to get the expression for u and v how? Let us see,

$$u=\frac{1-\mu^2}{E}\left[\frac{1}{2}(S_x+S_y)\left(r+\frac{a^2}{r}\right)+\frac{1}{2}(S_x-S_y)\left(r-\frac{a^4}{r^3}+4\frac{a^2}{r}\right)\cos 2\theta\right]$$
$$-\frac{\mu(1+\mu)}{E}\left[\frac{1}{2}(S_x+S_y)\left(r-\frac{a^2}{r}\right)-\frac{1}{2}(S_x-S_y)\left(r-\frac{a^4}{r^3}\right)\cos 2\theta\right]\rightarrow (7)$$

This equation will be your equation number 7.

And what will be the expressions for v let us see that,

$$= \frac{1-\mu^2}{E}\left[-\frac{1}{2}(S_x - S_y)\left(r + \frac{2a^2}{r} + \frac{a^4}{r^3}\right)\sin 2\theta\right]$$

$$- \frac{\mu(1+\mu)}{E}\left[\frac{1}{2}(S_x - S_y)\left(r - \frac{2a^2}{r} + \frac{a^4}{r^3}\right)\sin 2\theta\right] \rightarrow (8)$$

Now these equations 7 and 8, these are the generalized expressions for u and v.

u is the radial displacement, v is the tangential displacement and you can see that these are the function of $a, r, \theta, S_x, S_y, \mu$ and E. If you recall in case of these stresses those expressions where not the function of the elastic properties, but in case of the displacement E and μ also come into the picture.

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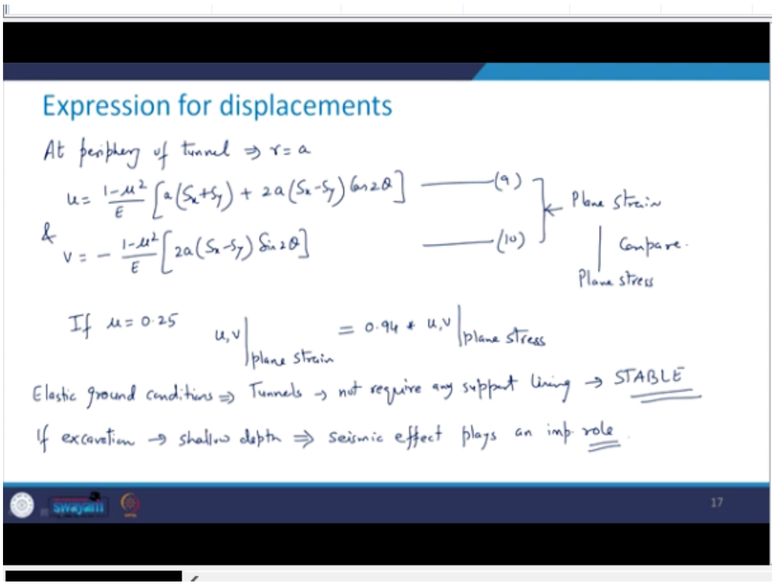

Now what happens at the periphery of the tunnel? That means at $r = a$. So, at periphery of the tunnel what does that mean? That is $r = a$,

$$u = \frac{1-\mu^2}{E}\left[(S_x + S_y)a + 2(S_x - S_y)a\cos 2\theta\right] \rightarrow (9)$$

$$v = \frac{1-\mu^2}{E}\left[2(S_x - S_y)a\sin 2\theta\right] \rightarrow (10)$$

This is equation number 10. Now if you compare these 2 equations of the plane strain with those of the plane stress equations, so compare.

And if I just substitute $\mu = 0.25$ then we will see that u and v for the plane strain situation they are approximately equal to 0.94 times the u and v for the plane stress case. So, in case if you have the elastic ground conditions, so tunnels do not require any support lining, they all stable. If the excavation is at shallow depth, then in this situation seismic effects place an important role.

So, as for as the expressions for displacement are concerned, we saw these for the plane stress as well as for the plane strain condition for a circular tunnel and we see that when we compared these 2, they are more or less of the same order for same $\mu = 0.25$ and this is how the analysis of the circular tunnel can be carried out. So, we discussed about not only the stresses but the distribution of the displacement or the variation of the displacement all along the tunnel periphery which is circular in shape.

So, this is what that I wanted to discuss with you as far as the elastic analysis of circular tunnel is concern, in the next class we will discuss about the elastic analysis for the lining of the tunnel. Thank you very much.