Rock Engineering
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Lecture-38
Elastic Stress Distribution Around Circular Tunnels-02

Hello everyone. In the previous class, we started our discussion on elastic stress distribution around the circular tunnels and we were deriving those expression using the theory of elasticity. And we could do up to the application of the boundary condition at r tending to infinity, we are still left with few boundary conditions at the tunnel periphery that is at $r=A$. So, let us apply that and see how we can get the complete solution towards this problem?

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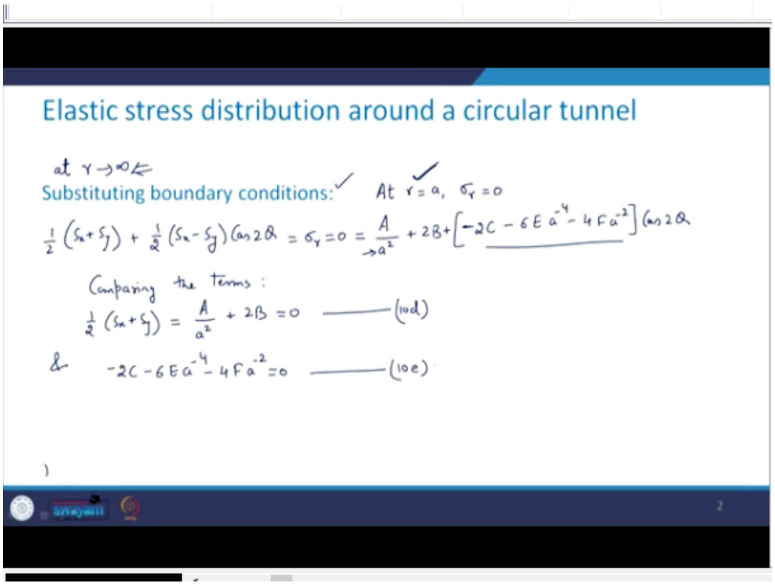

So, in case of this boundary condition in the previous class, we already substituted the boundary conditions, which were at r tending to infinity that we did earlier. So, now, we have the other boundary conditions which are like at $r = a$ you have $\sigma_r = 0$. Now, when this $\sigma_r = 0$, what is the expression that we are going to have is this that is

$$\frac{1}{2}(S_x + S_y) + \frac{1}{2}(S_x - S_y)\cos 2\theta = \sigma_r = 0 = \frac{A}{a^2} + 2B + (-2C - 6Ea^{-4} - 4Fa^{-2})\cos 2\theta$$

Now, as we did in the previous class, let us again compare the terms on both the sides of the equation.

What we will get is

$$\frac{1}{2}\left(S_x + S_y\right) = \frac{A}{a^2} + 2B = 0 \rightarrow (10d)$$

$$\left(-2C - 6Ea^{-4} - 4Fa^{-2}\right) = 0 \ \rightarrow (10e)$$

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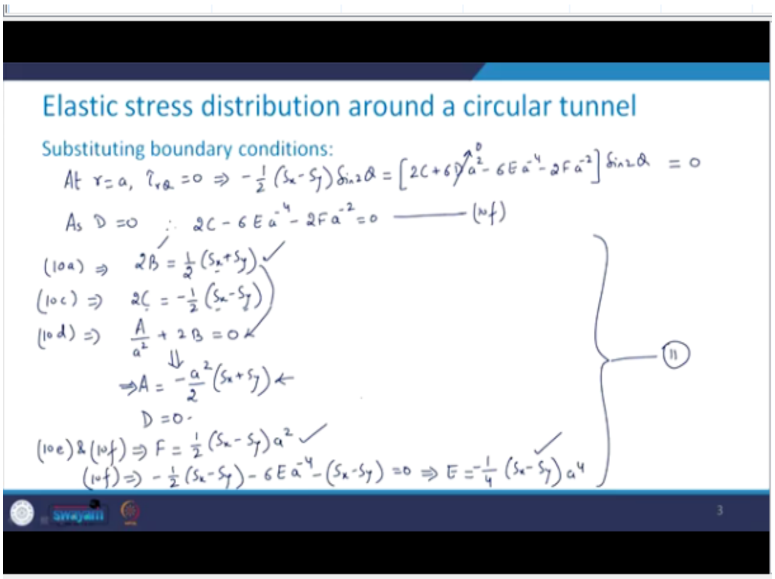

What will happen at $r = a$ to $\tau_{r\theta}$ that is going to be at $r = a$, your $\tau_{r\theta}$ is also equal to 0. And therefore, what we will get from here is

$$-\frac{1}{2}\left(S_x - S_y\right)\sin 2\theta = \left(2C + 6Da^2 - 6Ea^{-4} - 2Fa^{-2}\right)\sin 2\theta = 0$$

Now, we have seen that the constant $D = 0$. So, therefore, if I substitute $D = 0$, so, here this term will become equal to 0 and what I am going to get is

$$2C - 6Ea^{-4} - 2Fa^{-2} = 0 \rightarrow (10f)$$

Now, let us see from the previous class I will rewrite these equations for your ready reference. So, we had equation

$$\frac{1}{2}\left(S_x + S_y\right) = 2B \rightarrow (10a)$$

$$-\frac{1}{2}\left(S_x - S_y\right) = 2C \rightarrow (10c)$$

$$\frac{A}{a^2} + 2B = 0 \rightarrow (10d)$$

$$A = -\frac{a^2}{2}(S_x + S_y)$$

We know that D = 0. Now, this *10e* and *10f* combined form they will give me

$$-\frac{1}{2}(S_x - S_y) - 6Ea^{-4} - (S_x - S_y) = 0$$

$$E = -\frac{1}{4}(S_x - S_y)a^4$$

So, I write all these equations in a combined manner as equation number 11.

So, all these constants which were arbitrary, we could obtain in terms of these apply the stresses at *r* tending to infinity which are S_x and S_y.

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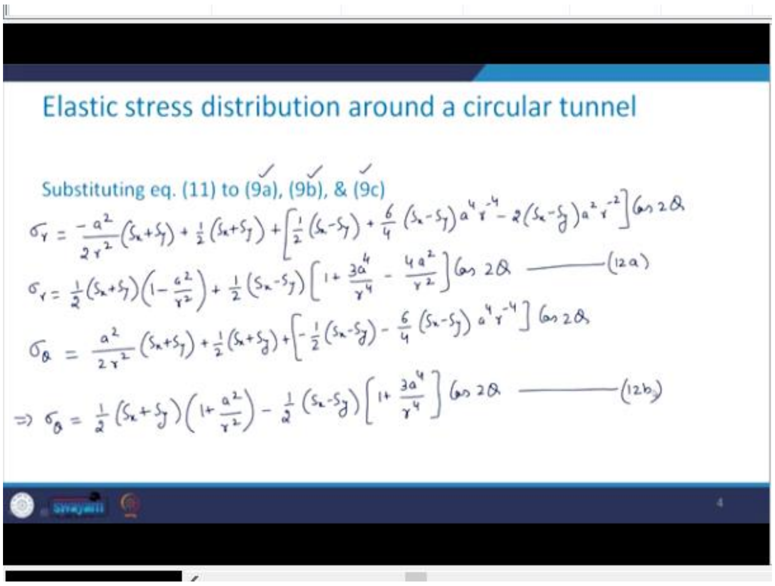

Now, we substitute this equation 11 to equations *9a*, *9b* and *9c* and see how it looks like, please do it very patiently, otherwise, you will make the mistake

$$\sigma_r = \frac{a^2}{2r^2}(S_x + S_y) + \frac{1}{2}(S_x + S_y) + \left[\frac{1}{2}(S_x - S_y) + \frac{6}{4}(S_x - S_y)a^4r^{-4} - 2(S_x - S_y)a^2r^{-2}\right]\cos 2\theta$$

$$\sigma_r = \frac{1}{2}(S_x + S_y)\left(1 - \frac{a^2}{r^2}\right) + \frac{1}{2}(S_x - S_y)\left[1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right]\cos 2\theta \rightarrow (12a)$$

Write it as equation number 12a. Now, what will happen to σ_θ? This is going to be

$$\sigma_\theta = \frac{a^2}{2r^2}(S_x + S_y) + \frac{1}{2}(S_x + S_y) + \left[-\frac{1}{2}(S_x - S_y) - \frac{6}{4}(S_x - S_y)a^4 r^{-4}\right]\cos 2\theta$$

$$\sigma_\theta = \frac{1}{2}(S_x + S_y)\left(1 + \frac{a^2}{r^2}\right) - \frac{1}{2}(S_x - S_y)\left[1 + \frac{3a^4}{r^4}\right]\cos 2\theta \rightarrow (12b)$$

This is going to be equation number 12b.

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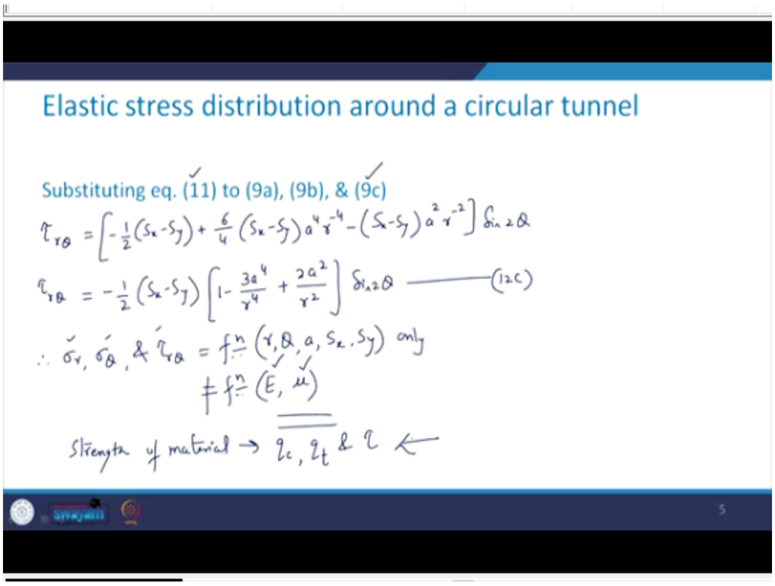

Similarly, we can write the expression for $\tau_{r\theta}$ when we substitute this equation 11 to equation number 9*c*. So, this is what that we are going to get is

$$-\frac{1}{2}(S_x - S_y)\sin 2\theta = (2C + 6Da^2 - 6Ea^{-4} - 2Fa^{-2})\sin 2\theta = 0$$

$$\tau_{r\theta} = \left[-\frac{1}{2}(S_x - S_y) + \frac{6}{4}(S_x - S_y)a^4 r^{-4} - (S_x - S_y)a^2 r^{-2}\right]\sin 2\theta$$

$$\tau_{r\theta} = -\frac{1}{2}(S_x - S_y)\left[1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right]\sin 2\theta \rightarrow (12c)$$

And this equation will become equation 12c.

Now, notice here that σ_r, σ_θ and $\tau_{r\theta}$ these are the function of r, θ, a, S_x and S_y only. These are not the function of the elastic properties of the rock. So, what does that mean? That elastic properties of the rock are not going to influence these stresses physically this statement does not make any sense. So, we know that strength of the material is given in terms of its UCS, its tensile strength and its shear strength.

So, indirectly these elastic constants they play a role through these strength characteristics. We will discuss about this in more detail little later in this class.

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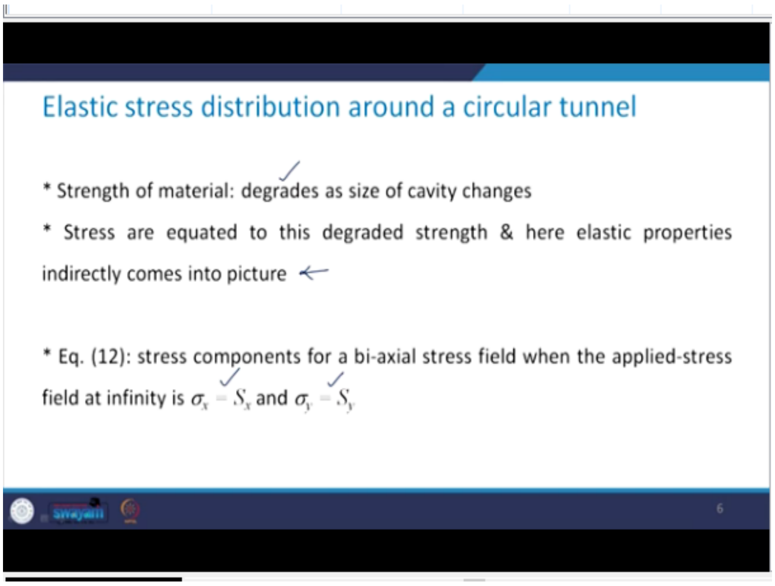

Now, this strength of the material, it reduces as the size of the cavity changes, stresses are equated to this degraded strength and here the elastic properties indirectly come into picture. So, we need to be careful. Equation number 12 which we derived just now, these gave the stress components for a biaxial stress field when the applied a stress field at the infinity is $\sigma_x = S_x$ and $\sigma_y = S_y$.

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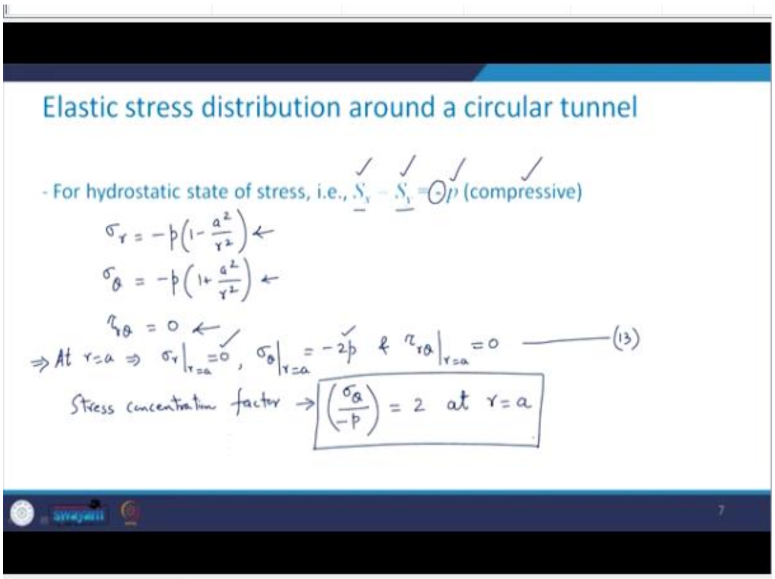

Now, we take some of the special cases and see how the form of the equation 12 becomes for these special cases. So, when we have the hydrostatic state of stress that means, that $S_x = S_y = -p$ and since it is in compressive in nature, so, I am introducing a negative sign with this state of stress which is p. So, what we are going to get if we just substitute this equal to S_y in equation number 12 what we are going to get is

$$\sigma_r = -p\left(1-\frac{a^2}{r^2}\right)$$

$$\sigma_\theta = -p\left(1+\frac{a^2}{r^2}\right)$$

$$\tau_{r\theta} = 0$$

And that will happen to the boundary condition that is at

$$\sigma_r|_{r=a} = 0, \sigma_\theta|_{r=a} = -2p, \tau_{r\theta}|_{r=a} = 0 \rightarrow (13)$$

So, this equation I will write as equation number 13. Now, I define a factor called stress concentration factor this I define as

$$\left(\frac{\sigma_\theta}{-p}\right) = 2\ at\ r = a$$

Please remember this we are going to use this information while we plot the stress distribution around a circular tunnel.

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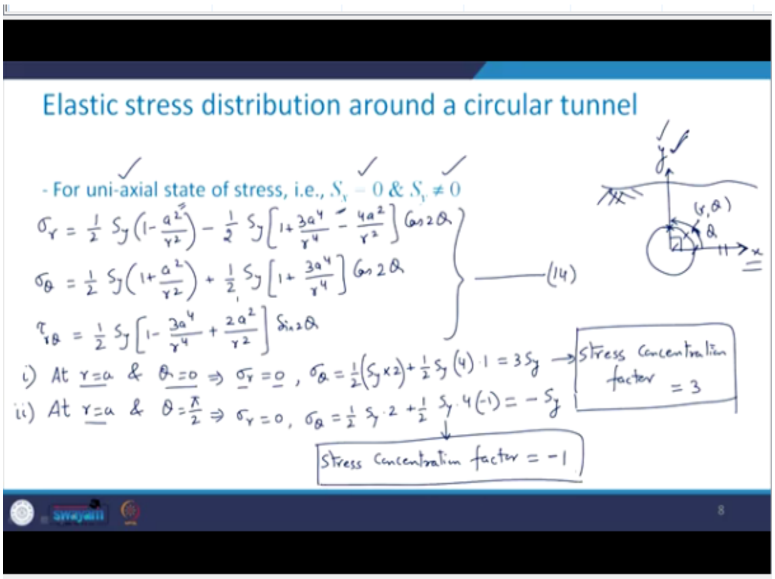

Now, let us take another state of stress which is the uniaxial state of stress in which case I will have a $S_x = 0$ that means, the applied stresses in the x direction they are 0 and in y direction they have the finite value. So, in such a situation what will happen to σ_r?

$$S_x = 0 \; and \; S_y \neq 0$$

$$\sigma_r = \frac{1}{2} S_y \left(1 - \frac{a^2}{r^2}\right) - \frac{1}{2} S_y \left[1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right] \cos 2\theta$$

$$\sigma_\theta = \frac{1}{2} S_y \left(1 + \frac{a^2}{r^2}\right) + \frac{1}{2} S_y \left[1 + \frac{3a^4}{r^4}\right] \cos 2\theta$$

$$\tau_{r\theta} = \frac{1}{2} S_y \left[1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right] \sin 2\theta$$

$$i) \; At \; r = a \; and \; \theta = 0 \; \rightarrow \sigma_r = 0, \sigma_\theta = \frac{1}{2}(S_y \times 2) + \frac{1}{2}(S_y \times 4) = 3S_y$$

$$ii) \; At \; r = a \; and \; \theta = \frac{\pi}{2} \; \rightarrow \sigma_r = 0, \sigma_\theta = \frac{1}{2}(S_y \times 2) + \frac{1}{2}(S_y \times 4 \times -1) = -S_y$$

I made these equations as equation number 14. Now, let us take few conditions. So, the first condition is at $r = a$ and $\theta = 0$.

So, if this was the tunnel, I had the directions of these axis in this manner equal to that is the horizontal axis was representing x direction and vertical was y and if this is what was the rock mass. So, any point was there, which was having they coordinate as r ,θ and this θ, I took in the anti clockwise direction from the x axis. So, when I see r equal to a and θ equal to 0 this means what?

R equal to a means, it is here at the tunnel periphery and θ equal to 0 means, it is in the x direction. So, what I am going to get here is $\sigma_r = 0$, just substitute this $r = a$. So, you see this term will become equal to 0 and this term will also become equal to 0 resulting into $\sigma_r = 0$, then what will happen to σ_θ. Let us see, substitute $r = a$ here.

So, what will happen at this point to all these stress components, see here σ_r will become equal to 0 again because $r = a$ and this term will become equal to 0.

So, in this case your stress concentration factor is going to be equal to 3 while in this case the stress concentration factor will be equal to - 1. So, here it is 3 in the first case and in the second case it equal to - 1.

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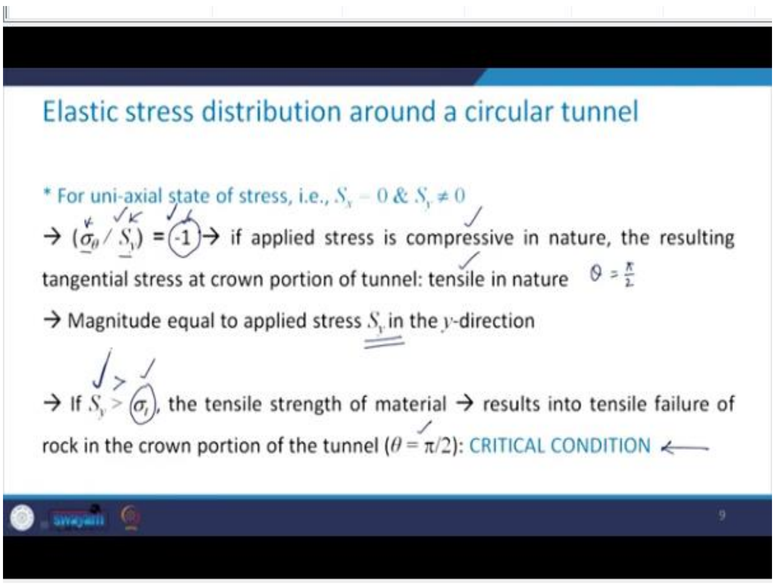

So, here I have summarized what we discussed for the uniaxial state of stress, that is when there is no applied stress in the x direction and only in the y direction these stresses are present. In that case (σ_θ / S_y) we have seen that it is - 1. If the applied stress is compressive in nature, the resulting tangential stress and the crown portion of the tunnel becomes tensile in nature, because you see that the ratio of these 2 is - 1.

So, in case if I take this S_y to be equal to compressive obviously with that logic this σ_θ will become tensile in nature and we are talking about the tangentially stress at the crown portion because this was the value at θ equal to 90 degree or Π /2. Its magnitude at this location would be equal to the applied stress S_y which is in the y direction. Now, let us say that this applied stress in the y direction becomes more than the tensile strength of the material, which is represented by σ_t.

What do we mean that is S_y is more than σ_t, then what will happen? Stress is more than the strength in the same mode which is in tension. So, this would result into the tensile failure of rock in the crown portion of the tunnel which is θ equal to Π/2 and this is one of the most critical conditions as far as the analysis of the tunnels is concern. So, one needs to be very, very careful about the stresses at the crown portion of the tunnel.

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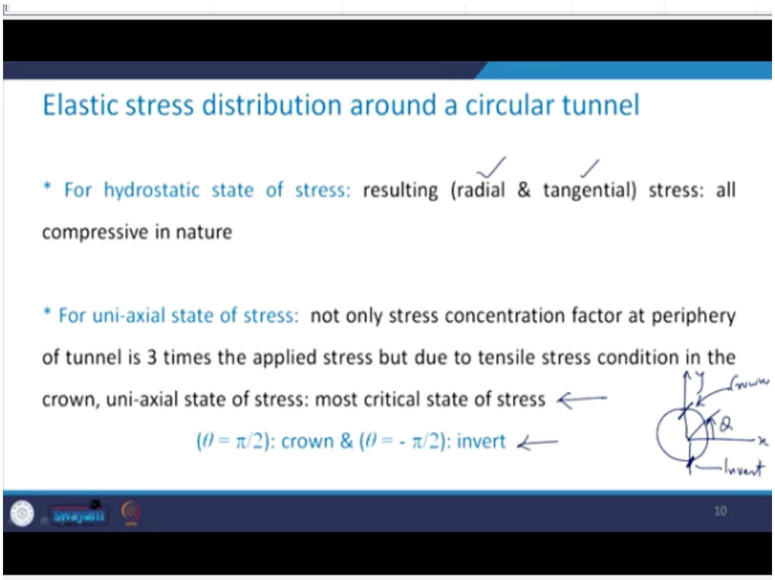

So, we have seen that for hydrostatic state of stress both the radius as well as the tangential stresses. They are all compressive in nature, we saw that the stress concentration factor was all positive in that case. But in case of the uniaxial state of stress, it is not only that the stress concentration factor at the periphery of the tunnel is 3 times the applied stress. But also due to the tensile stress condition in the crown portion of the tunnel.

This unit actually state of stress becomes most critical state of stress. So, once again here I have given for your ready reference that what do we mean by the crown of a tunnel and invert of the tunnel? So, if this is the tunnel and you have these axes like x and y. So, here this θ we are measuring from this direction. So, when θ is equal to 90 degree then this is what is the crown and when θ equal to - Π /2 which is this point, this is going to be the invert of the tunnel.

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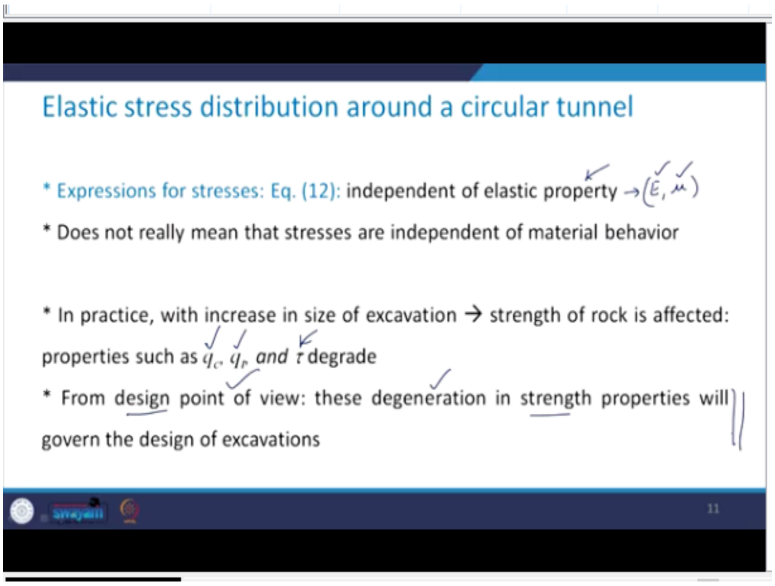

Now, the expressions which were given for these stresses using equation number 12 we have seen that these are independent of the elastic properties, what were those like E and μ. So, those expression for σ_r, σ_θ and $\tau_{r\theta}$, they were not the function of these E and μ. So, this does not really mean that stresses are independent of the material behavior. What happens in practice is that with increase in the size of the excavation, the strength of the rock is influenced and the strength properties such as UCS, tensile strength and the sheer strength they degrade.

So, from a design point of view this degeneration in the strength properties will govern the design of excavation. So, when those expression for these stresses when they come out to be independent of the elastic property, that does not mean that stresses are independent of the material characteristic or its behavior they come in this particular fashion as far as the design of those structures are concerned.

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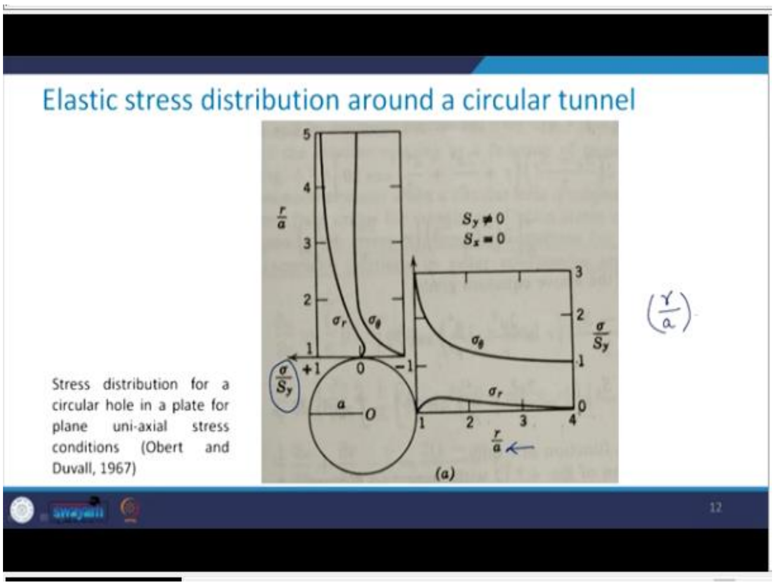

Now, here whatever are the expressions that we obtained in the form of equation number 12, that is the expression for σ_r, σ_θ and $\tau_{r\theta}$. So, the variations of those stresses have been shown here in this figure. So, these stresses they have been obtained as a part of the stress concentration factor and the radial distance has been normalized with the help of the radius of the tunnel, that is this all the data has been presented in the form of *r/a*.

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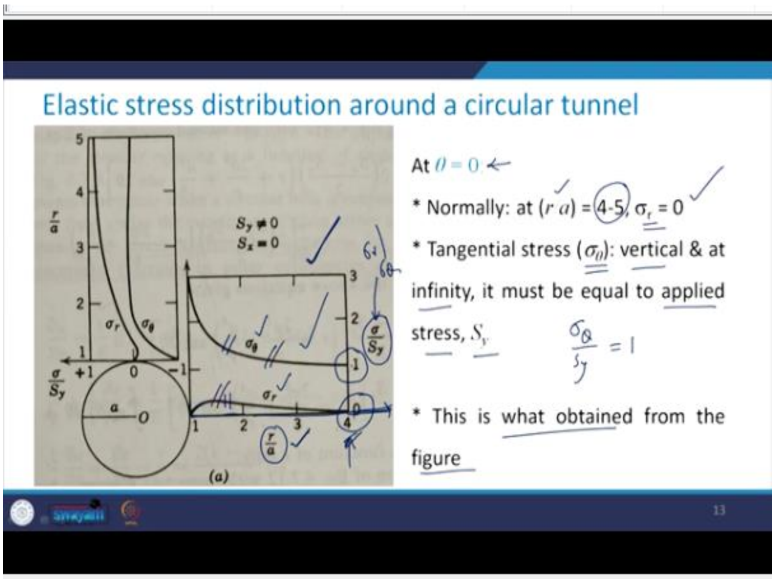

Let us have the detailed discussion about this. First you focus on this particular part of the figure, that is at $\theta = 0$. So, $\theta = 0$ means, that here it is this axis, here $\theta = 0$. So, I have on this axis *r /a* quantity and on the vertical axis we are plotting σ / S_y. Now, this σ can be σ_r or it can be σ_θ. So, this plot is showing the variation of σ_r with *r /a* and this plot is showing the variation of σ_θ with *r /a*.

So, if you just take a look at those expression of equation number 12 normally at *r /a* equal to 4 to 5, this σ_r would work out to be equal to 0 and that is what we can see here in this figure as well, that is when this *r /a* is to the tune of 4 or 5, this σ_r will become equal to 0. What happens to the tangential stress? So, in this case it is the vertical and at infinity it must be equal to the applied stress S_y and therefore, this σ_θ /S_y should be equal to 1 when this *r* tends to infinity.

So, in this case you see that here *r* tends to infinity means that I am taking that the radial distance is about 4 to 5 times the radius of the tunnel and that we can consider for all practical purposes to represent the infinite boundary in case of the elastic situation. So, you see that at *r /a* tending to 4 to 5, this σ_θ becomes equal to 1. So, this is what that we get from those equation and has been represented in this figure. Please remember these are the typical variation of these stresses at θ equal to 0.

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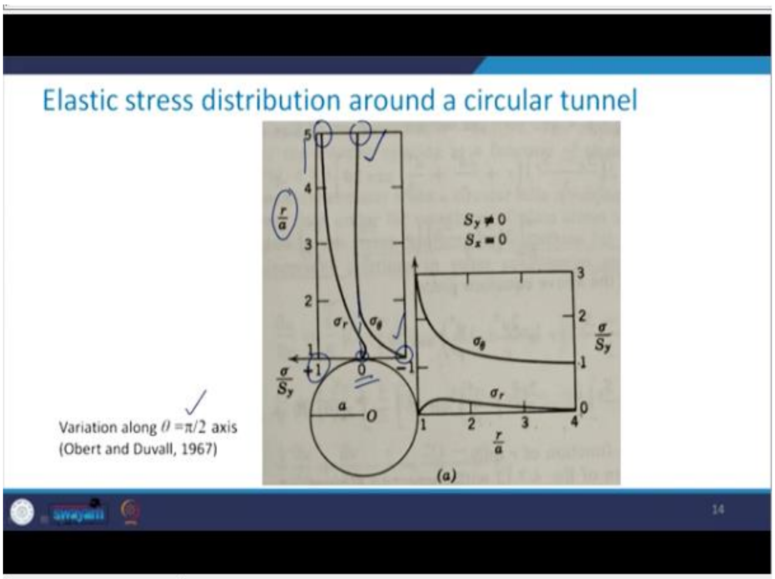

Then focus on this portion of the figure. See here this is at the crown which is at θ equal to Π/2 axis. So, when you have *r /a* equal to 1 that means, at r =a that is an eternal periphery, you have σ_θ /S_y to be equal to – 1 and we saw that it is like in that condition that a stress concentration factor was coming out to be – 1 and that is what has been represented in this figure.

And when it goes to infinity that is when *r /a* becomes equal to 4 to 5, this becomes equal to 0, that is σ_θ is approximately equal to 0. In case of the σ_r at *r* tending to infinity or *r /a* goes to 4 to 5 this σ_r /S_y it tends to + 1 you see here. So, this is how the variation along the crown it looks like.

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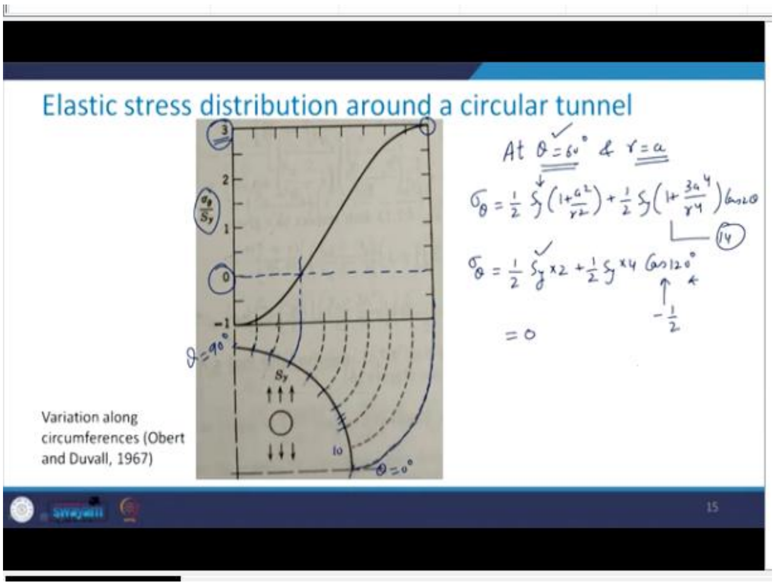

Now, this is the variation along the circumference. So, along the circumference from θ equal to 0 degree to this is θ equal to 90 degree. So, this is 0 degree, then you have here 10, 20, 30, 40, 50, 60, 70, 80 and this is point is 90 degree. Now, take a look here that what happens at θ equal to 60 degree and obviously, because we are discussing about this variation along the circumference.

So, r is going to be equal to a. So, the σ_θ will be what?

$$\sigma_\theta = \frac{1}{2} S_y \left(1 + \frac{a^2}{r^2}\right) + \frac{1}{2} S_y \left[1 + \frac{3a^4}{r^4}\right] \cos 2\theta$$

$$\sigma_\theta = \frac{1}{2} S_y \times 2 + \frac{1}{2} S_y \times 4 \cos 120 = 0$$

So, this is going to give me σ_θ to be equal to 0, you see that this is S_y and this will become - S_y. So, they will get cancelled out. Now, take a look here 10, 20, 30, 40, 50 and 60. So, we just have its projection like this here.

And then take it like this and here you have the stress concentration factor. So, you see that corresponding to θ equal to 60 degree, you have this σ_θ to be equal to 0. That is how your curve is also giving you. Now, at θ equal to 0, you have seen that σ_θ upon S_y works out to be 3. So, that is what has been shown here in this figure θ equal to 0 have the projection and you go along this line and see this point corresponding to this point your σ_θ upon S_y works out to be 3.

So, this is how one can show the variation of these stresses all along the circumference, along any axis, say we have seen that along x axis or y axis. So, this is what that I wanted to discuss with you as far as the elastic stress distribution around a circular tunnel is concerned. We will see other aspects related to this topic in the next class. Thank you very much.