Rock Engineering
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Lecture-37
Elastic Stress Distribution Around Circular Tunnels-01

Hello everyone. In the previous class we discussed about various ground conditions, the suitable method of excavation in those ground conditions as far as tunnelling is concerned, appropriate support system and what are the precautions that one needs to keep in mind while going for the excavation in such ground conditions. Then we also discussed about the difference between swelling and squeezing phenomena.

So, that was all about the ground conditions and related issues. Today we will start a new topic that is elastic stress distribution around circular tunnels, since we will be having the derivation for the elastic analysis for the circular tunnels. So, this may stretch over 2 lectures. So, to start with let us first try to understand the geometry of the problem.

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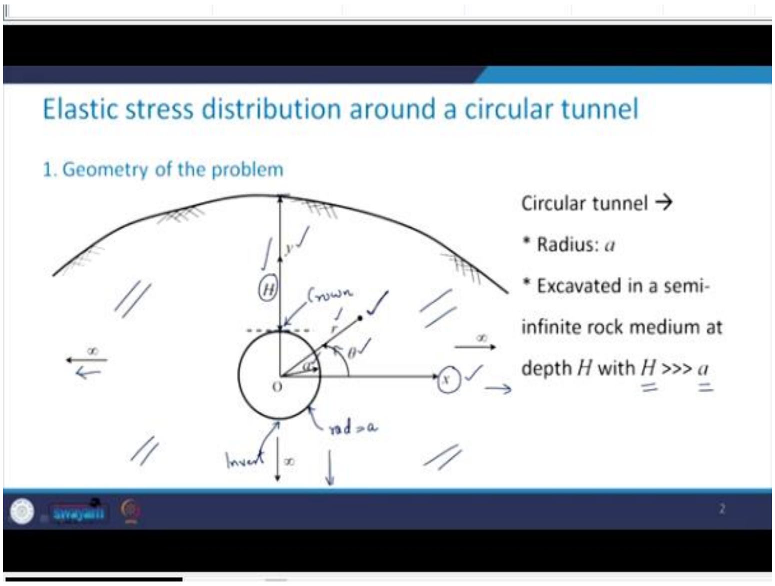

So, to start with look at this figure and here you have this excavation in the form of circular tunnel which has the radius equal to a, that has been shown here, this tunnel has been excavated in a semi-infinite rock medium at a depth *H* which is this, this is the depth between this point and this point, that is on the ground surface. You need to take a look that this over burden depth is much larger than the radius of this circular tunnel.

One more thing which we will be using quite often tunnel and the top most point is called as the crown and the bottom most point is called as invert of the tunnel, here any point in this semi-infinite space is defined by these polar coordinate systems which is the r, θ where this θ has been taken in an anticlockwise direction from this horizontal axis which I am denoting as x axis.

The vertical axis is the y axis and this rock has infinite extent in this direction, this direction and in this direction as well and this H is much larger than the radius of the circular tunnel.

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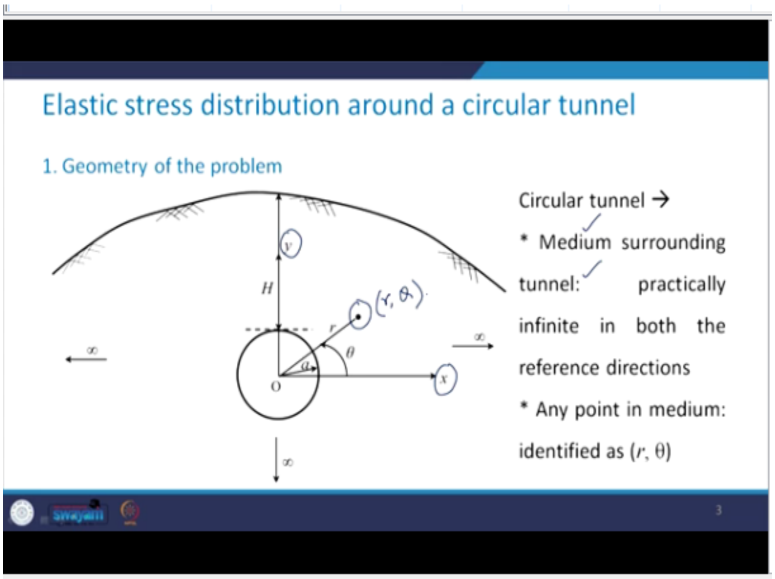

As I mentioned the medium surrounding the tunnel is practically infinite in both the reference directions which are x and y. Any point in the media that is say this point will be defined by the point called r, θ.

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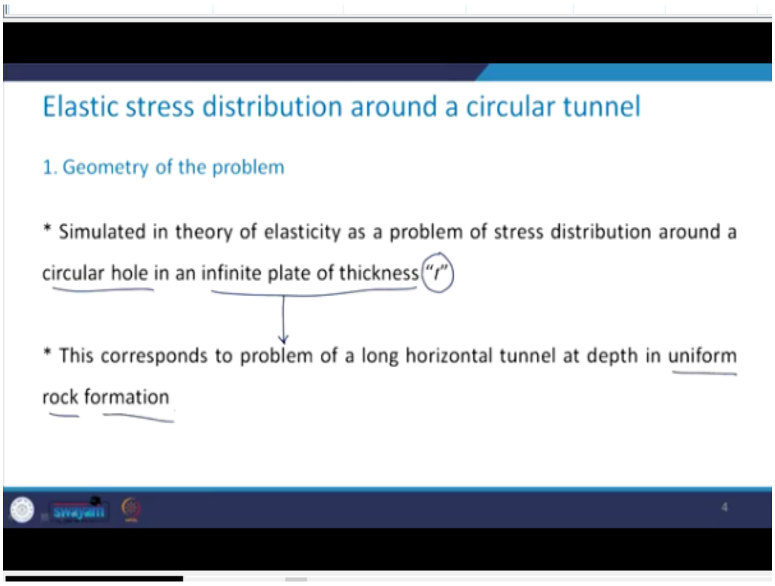

This is simulated in the theory of elasticity as a problem of stress distribution along a circular hole in an infinite plate of thickness *t* and this problem of circular hole in an infinite plate of thickness can correspond to the problem of a long horizontal tunnel at depth in the uniform rock formation.

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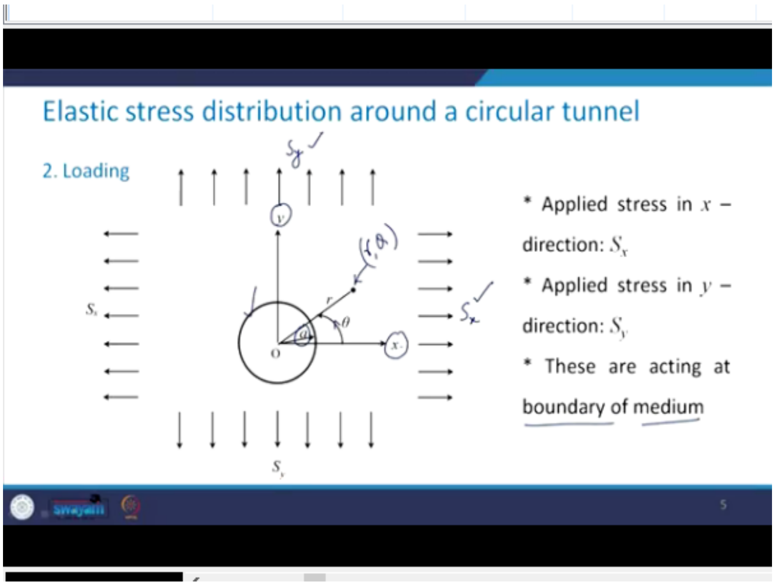

The second aspect is the loading. See this is the tunnel, you have the 2 reference axes x and y. This is the point defined by r, θ, where θ is the angle in the anticlockwise direction from this x axis, the radius of the tunnel is *a*. Now the applied distress in x direction is S_x and applied stress in the y direction is S_y. These stresses they are acting at the boundary of the medium.

And where is the boundary of the medium we saw that we can consider these 2 extents of the medium as an infinite. So, when this r is tending to infinity there these applied loads in the x direction and y direction they are taken as S_x and S_y respectively.

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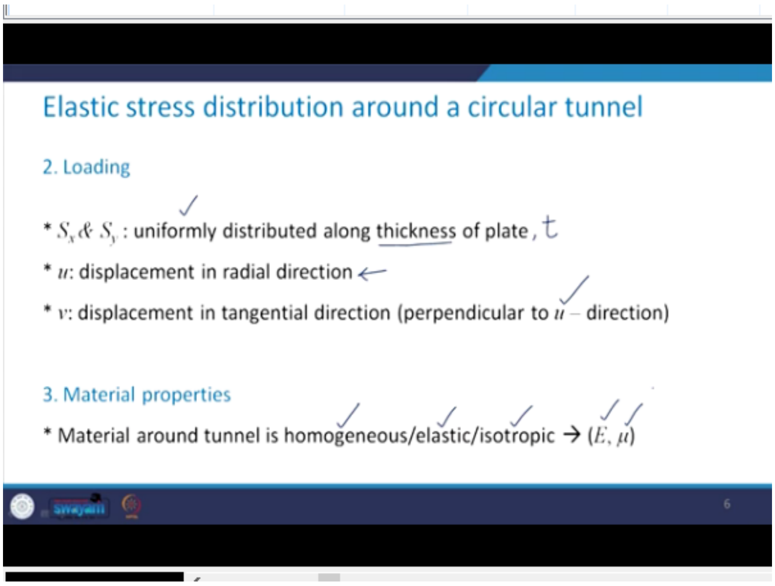

So, this S_x and S_y they are uniformly distributed along the thickness of the plate which is t, u be the displacement in the radial direction and v is the displacement in tangential direction which is perpendicular to u direction. The third aspect is related to material properties, so it has been assumed that material around the tunnel is homogeneous, elastic and isotropic. So, E and μ are going to be the material characteristic of rock around the tunnel.

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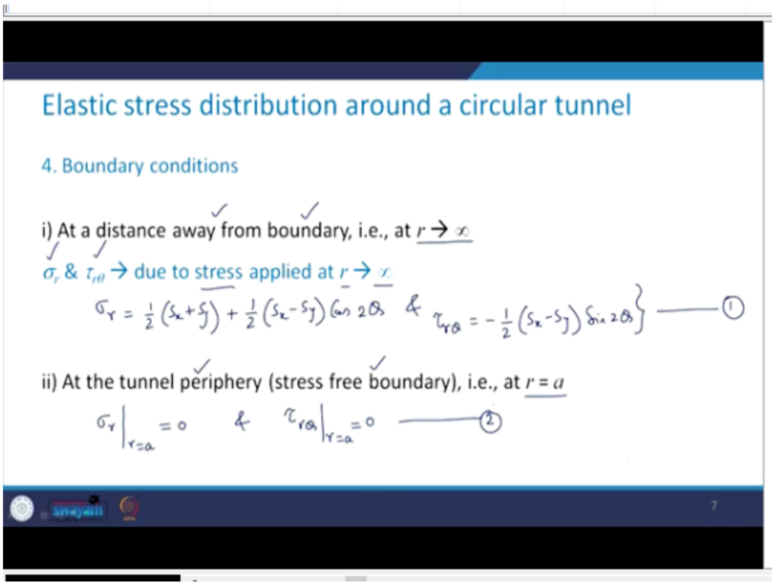

Boundary conditions, as I mentioned but at a distance away from the boundary that is at r tending to infinity, this σ_r and $\tau_{r\theta}$ they will be due to the stress which are applied at r tending to infinity and what are those stresses? They are S_x and S_y in x and y direction respectively. Let us see how can we represent these boundary conditions mathematically.

$$\sigma_r = \frac{1}{2}(S_x + S_y) + \frac{1}{2}(S_x - S_y)\cos 2\theta \ and\ \tau_{r\theta} = -\frac{1}{2}(S_x - S_y)\sin 2\theta$$

I write this equation as equation number 1. Now at the panel periphery which is the stress-free boundary and represented by $r = a$ you will have the boundary conditions as

$$\sigma_r|_{r=a} = 0 \ and\ \tau_{r\theta}|_{r=a} = 0$$

Make this equation as equation number 2.

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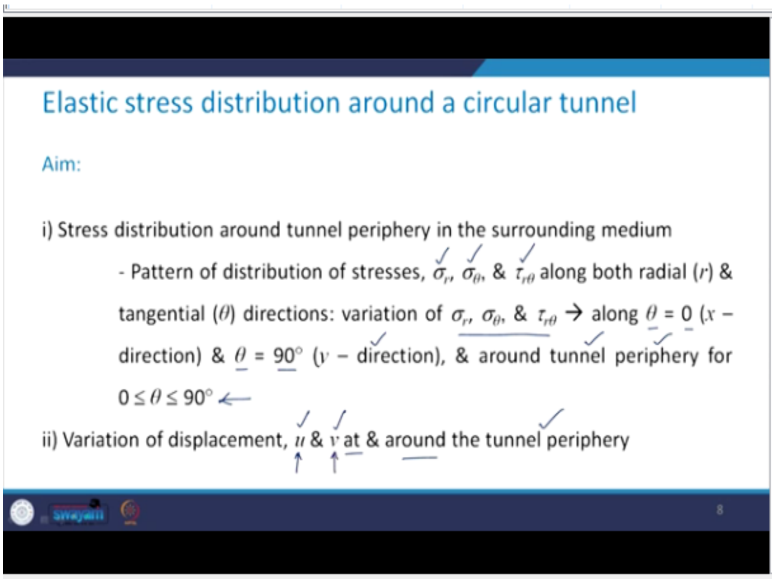

Now what exactly is the aim of this study is to obtain the stress distribution around tunnel periphery in the surrounding media, what do we mean by this? That is, we want to know that what is a pattern of distribution of stresses which are σ_r, σ_θ and $\tau_{r\theta}$ along both the radial as well as the tangential directions, and we want to know the variation of these stresses along θ=0 means it is going to be the x direction, and θ = 90° means along the y direction.

And also, I need to study the pattern of the distribution of these stresses around the periphery for θ varying between 0- and 90-degree, variation of displacement also we would like to see at and around the tunnel periphery. And this variation of displacement, we would like to see in the radial as well as in the tangential direction. In the radial direction, this displacement is represented by u and tangential direction it is *v*.

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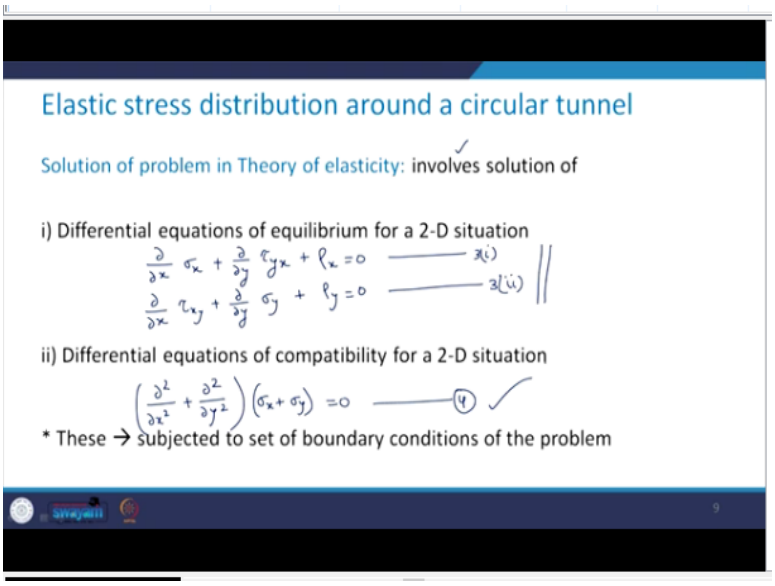

Now, let us try to see how to obtain the solution of this problem using the theory of elasticity? So, basically this involves the solution of first is the differential equation of equilibrium for a 2-D situation, let us see what are these equations? So, we have

$$\frac{\partial}{\partial x}\sigma_x + \frac{\partial}{\partial y}\tau_{yx} + \rho_x = 0$$

Mark it as equation number 3(i) and other equation is

$$\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\sigma_y + \rho_y = 0$$

Then, we need to have the differential equations of compatibility for a 2-D situation as well. So, that is given as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$$

Make it equation number 4. Now, when these set of equations that is equilibrium equation and the compatibility condition, when these are subjected to the set of appropriate boundary conditions pertaining to the problem, these gives us the solution of the problem using the theory of elasticity. Now, what should be the approach towards the solution?

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So, in this case Airy's stress function approach to be used. Now, this Airy's stress function is represented by ϕ and we need to assume that there is the absence of body forces, these are ρ_x and ρ_y in the previous equations we have seen or when these are constant. So, in that case we need to apply this Airy's stress function approach in order to get the solution. So, this Airy's stress function

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}\ , \sigma_y = \frac{\partial^2\phi}{\partial x^2}\ , \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

Now, just substitute all these 3 quantities to equation number 3(i) what we are going to get as left-hand side of the equation as

$$LHS = \frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial}{\partial x}\left(-\frac{\partial^2\phi}{\partial x\partial y}\right) = \frac{\partial^3\phi}{\partial x\partial y^2} - \frac{\partial^3\phi}{\partial x\partial y^2} = 0 = RHS$$

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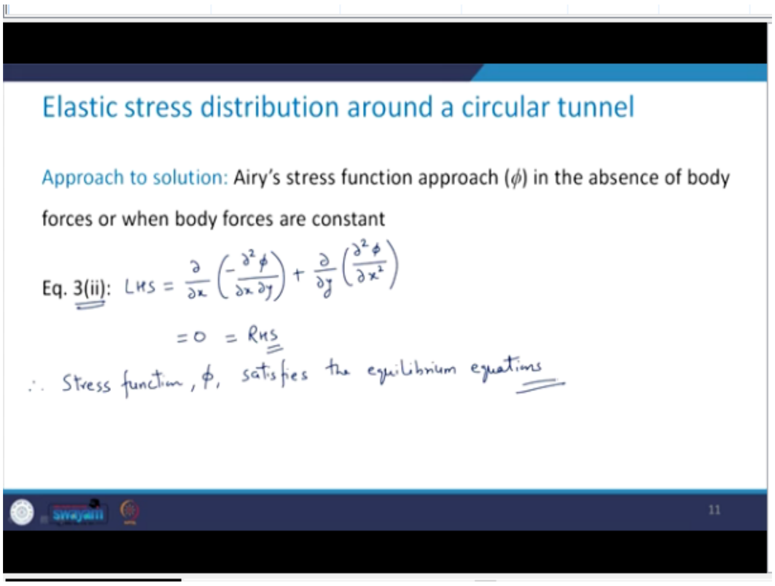

Similarly, if we substitute this equation number 3 (ii) left hand side of the equation will become equal to

$$\frac{\partial}{\partial x}\left(-\frac{\partial^2\phi}{\partial x\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x^2}\right) = 0 = RHS$$

So, therefore, what we get from here is that the stress function which is ϕ satisfies the equilibrium equations.

Because equation 3 and equation 3 (i) and equation 3 (ii) they were corresponding to the equation of equilibrium and we can see that whatever is this stress function it satisfies these.

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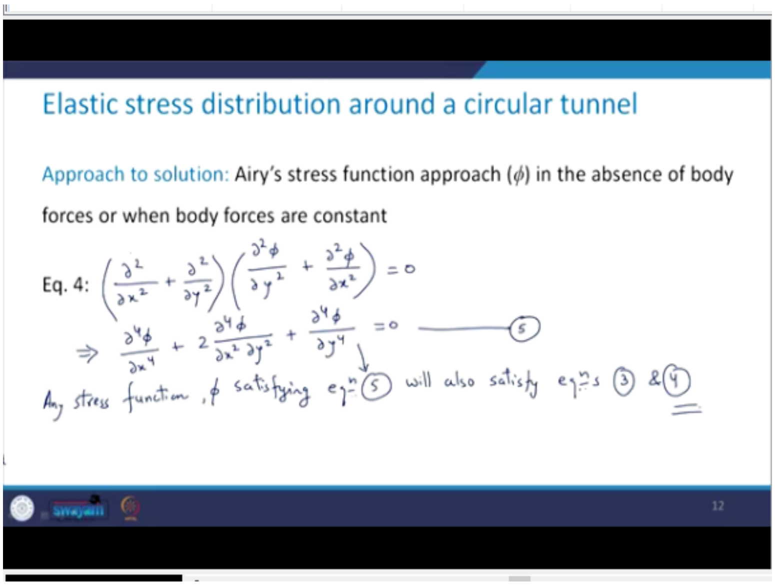

Now, what will happen to equation number 4, which is the compatibility equation. Just substitute the expression for ϕ in equation number 4 and you will get an expression like this.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial x^2}\right)=0$$

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=0$$

I mark this equation as equation number 5. So, any stress function ϕ satisfying equation number 5, that is this equation will also satisfy equations 3 and so, that is the thing we need to keep in mind. Now, what can be the choice of this stress function for this problem of finding the stress distribution around a circular tunnel in the elastic rock?

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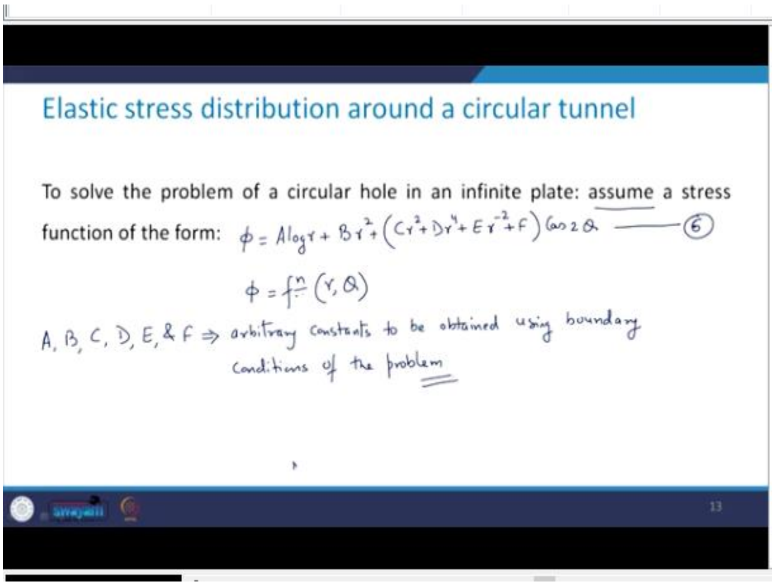

So, we need to assume a stress function of the form like this

$$\phi = A\log r + Br^2 + (Cr^2 + Dr^4 + Er^{-2} + F)\cos 2\theta$$
$$\phi = f(r, \theta)$$

I will make this equation as equation number 6. Now, you will see that this ϕ is a function of r and θ where these constants A, B, C, D, E and F. These are arbitrary constants which are to be obtained using boundary conditions which are relevant to the problem.

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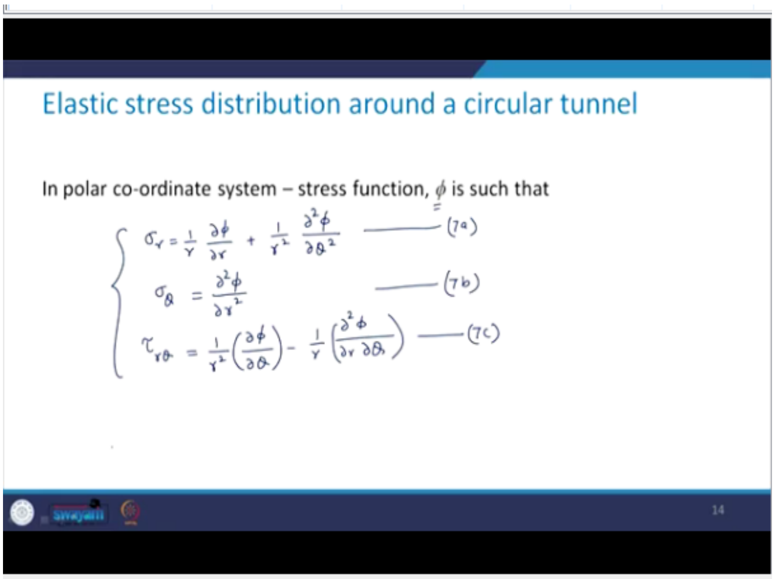

Now, to proceed further in the polar coordinate system, this stress function ϕ is such that

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$

$$\tau_{r\theta} = \frac{1}{r^2}\left(\frac{\partial \phi}{\partial \theta}\right) - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}$$

See these equations are the standard equations in the theory of elasticity and can be obtained from any textbook on the theory of elasticity.

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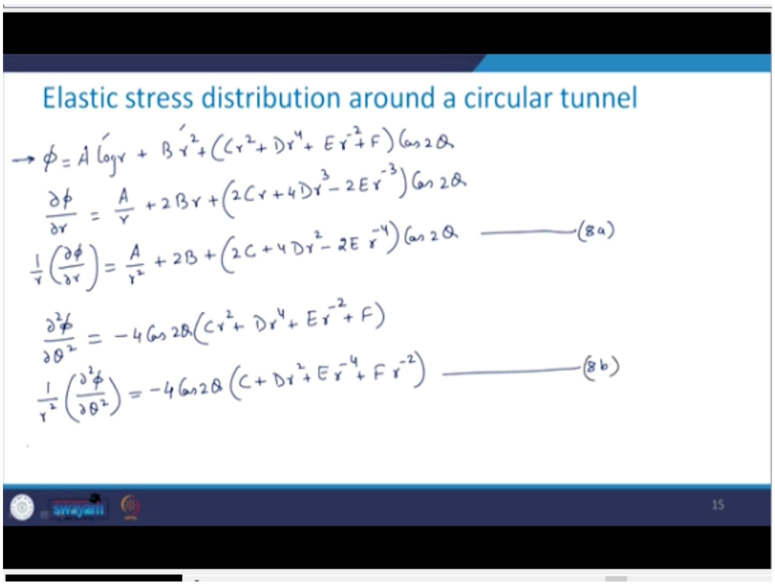

Now, let us proceed further. So, what we have is

$$\phi = A \log r + Br^2 + (Cr^2 + Dr^4 + Er^{-2} + F)\cos 2\theta$$

This is what that we have assumed. Now, differentiate this with respect to *r* see what we get.

$$\frac{\partial \phi}{\partial r} = \frac{A}{r} + 2Br + (2Cr + 4Dr^3 - 2Er^{-3})\cos 2\theta$$

$$\frac{1}{r}\left(\frac{\partial \phi}{\partial r}\right) = \frac{A}{r^2} + 2B + (2C + 4Dr^2 - 2Er^{-4})\cos 2\theta \quad \Rightarrow (8a)$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = -4\cos 2\theta\,(Cr^2 + Dr^4 + Er^{-2} + F)$$

$$\frac{1}{r^2}\left(\frac{\partial^2 \phi}{\partial \theta^2}\right) = -4\cos 2\theta\,(C + Dr^2 + Er^{-4} + Fr^{-2}) \quad \Rightarrow (8b)$$

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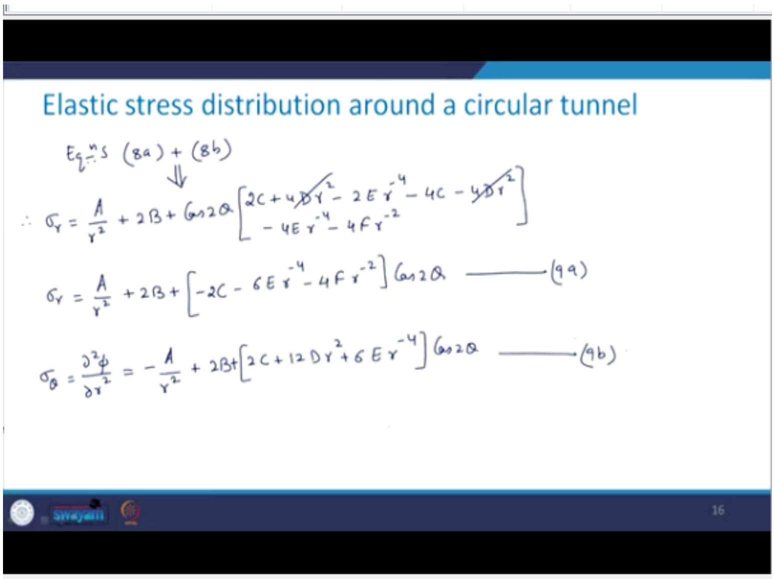

Now, let us further simplify this. So, we add equations 8a and 8b. So, this is what is going to give us

$$\sigma_r = \frac{A}{r^2} + 2B + \cos 2\theta\,[2C + 4Dr^2 - 2Er^{-4} - 4C - 4Dr^2 - 4Er^{-4} - 4Fr^{-2}]$$

$$\sigma_r = \frac{A}{r^2} + 2B + \cos 2\theta\,[-2C - 6Er^{-4} - 4Fr^{-2}]$$

That is 9a.

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial \theta^2} = \frac{A}{r^2} + 2B + \cos 2\theta\,[2C + 12Dr^2 + 6Er^{-4}]$$

That is 9b.

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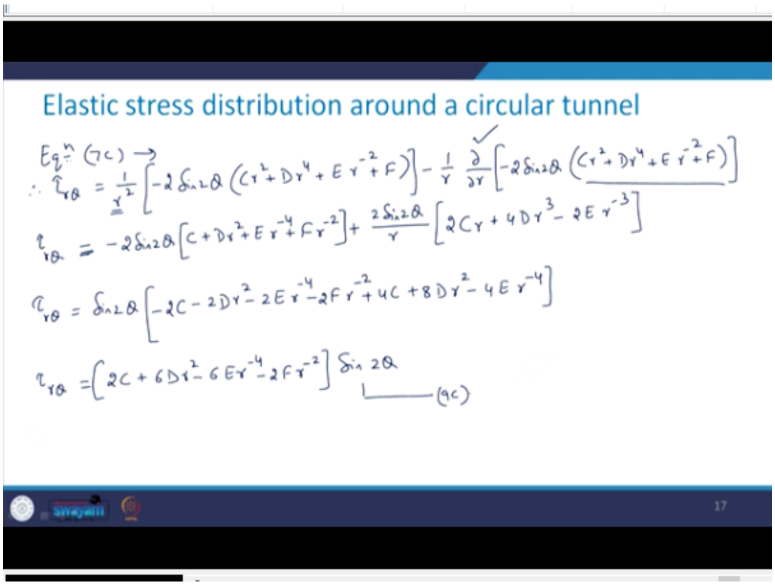

And from the equation 7c, this will give me

$$\tau_{r\theta} = \frac{1}{r^2}[-2\sin 2\theta(Cr^2 + Dr^4 + Er^{-2} + F)]$$

$$-\frac{1}{r}\frac{\partial}{\partial x}[-2\sin 2\theta\,(Cr^2 + Dr^4 + Er^{-2} + F)]$$

$$\tau_{r\theta} = -2\sin 2\theta(C + Dr^2 + Er^{-4} + Fr^{-2}) + \frac{2\sin 2\theta}{r}[2Cr + 4Dr^3 - 2Er^{-3}]$$

$$\tau_{r\theta} = \sin 2\theta[-2C - 2Dr^2 - 2Er^{-4} - 2Fr^{-2} + 4C + 8Dr^2 - 4Er^{-4}]$$

$$\tau_{r\theta} = [2C + 6Dr^2 - 6Er^{-4} - 2Fr^{-2}]\sin 2\theta \;\rightarrow 9c$$

Make this equation as equation number $9c$. So, this is how we can get the expression for σ_r, σ_θ and $\tau_{r\theta}$ in terms of the constants of the assumed stress function ϕ.

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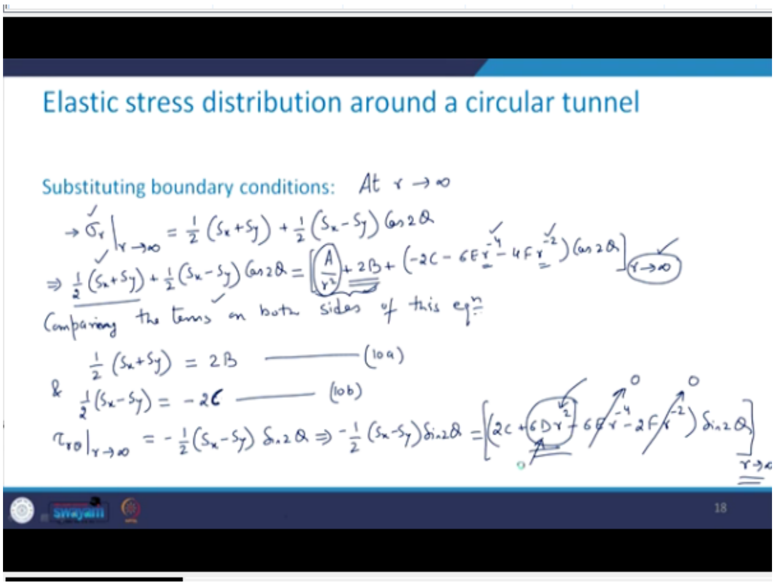

Now, we need to substitute the boundary conditions. So, we have the boundary conditions one was at r tends to infinity and another one was at r tends to 1. So, first let us apply that at r tending to infinity, the boundary conditions are going to be that is

$$\sigma_r|_{r\rightarrow\propto} = \frac{1}{2}(S_x + S_y) + \frac{1}{2}(S_x - S_y)\cos 2\theta$$

$$\frac{1}{2}(S_x + S_y) + \frac{1}{2}(S_x - S_y)\cos 2\theta = \left[\frac{A}{r^2} + 2B + (2C + 4Dr^2 - 2Er^{-4})\cos 2\theta\right]_{r\rightarrow\propto}$$

$$\frac{1}{2}(S_x + S_y) = 2B \rightarrow (10a)$$

$$\frac{1}{2}(S_x - S_y) = 2C \rightarrow (10b)$$

$$\tau_{r\theta}|_{r\to\propto} = -\frac{1}{2}(S_x - S_y)\sin 2\theta$$

$$-\frac{1}{2}(S_x - S_y)\sin 2\theta = (2C + 6Dr^2 - 6Er^{-4} - 2Fr^{-2})\sin 2\theta|_{r\to\propto}$$

Now again in this equation if you just compare these terms and obviously this is subjected to the condition that is applicable when *r* tending to infinity. So, when this *r* is tending to infinity this term will become equal to 0 and this term will also become equal to 0, but what about this term? See here you have positive power of *r* and when *r* tends to infinity this term will also tends to infinity which is physically not possible.

So, that gives me the condition that this constant D has to be equal to 0, if this equation needs to have the physical interpretation, then the constant D should be equal to 0.

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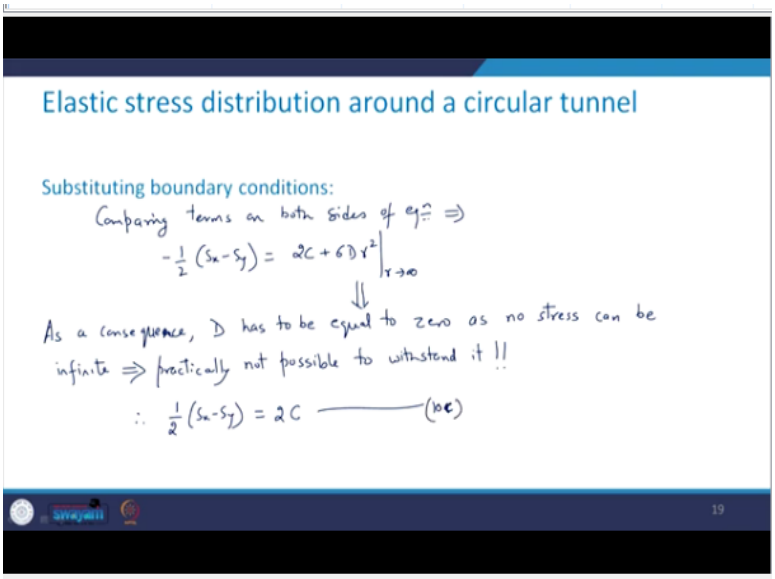

So, what we are going to get if we just compare the terms once again for this equation. So, comparing the terms on the both sides of the equation what we will get is

$$-\frac{1}{2}(S_x - S_y) = 2C + 6Dr^2|_{r\to\propto}$$

Now what will happen as a consequence of this as I explained to you that as a consequence this D has to be equal to 0. Why?

Because no stress can be infinite otherwise it will lose the practical relevance. So, this is practically not possible to withstand it. So, therefore what we are going to get is

$$\frac{1}{2}\left(S_x - S_y\right) = 2C$$

This is our equation number 10c. So, this way we saw that how using the theory of elasticity we can approach towards the problem related to the determination of elastic stress distribution around a circular tunnel.

So, here we have applied the boundary condition at r tending to infinity, we still have few more boundary condition and we still need to find out the complete solution of the problem. So, we will continue with this in the next class. Thank you.