Rock Engineering
Prof. Priti Maheshwari
Department of Civil Engineering
Indian Institute of Technology, Roorkee

Lecture No -30
Concept of instantaneous c and ϕ Balmer approach

Hello everyone. In the previous class, we discussed about the physical interpretation of Mohr-Coulomb failure criterion and then we learnt about the Coulomb Navier failure criterion. Today we will learn something which is interesting which is the concept of instantaneous c and ϕ and this is also called as Balmer's approach because it was given by a person named Balmer. This is interesting in the sense that when you use the Mohr-Coulomb criterion.

And for that the parameters are c and ϕ and you have seen that the Mohr-Coulomb failure envelope is linear. In the case of the rocks the failure envelope may not be linear. It is going to be non-linear. Now is there any way that we can find out the equivalent linear parameters for that non-linear relationship? So, this Balmer's approach gives us the answer towards this question. Let us see how?

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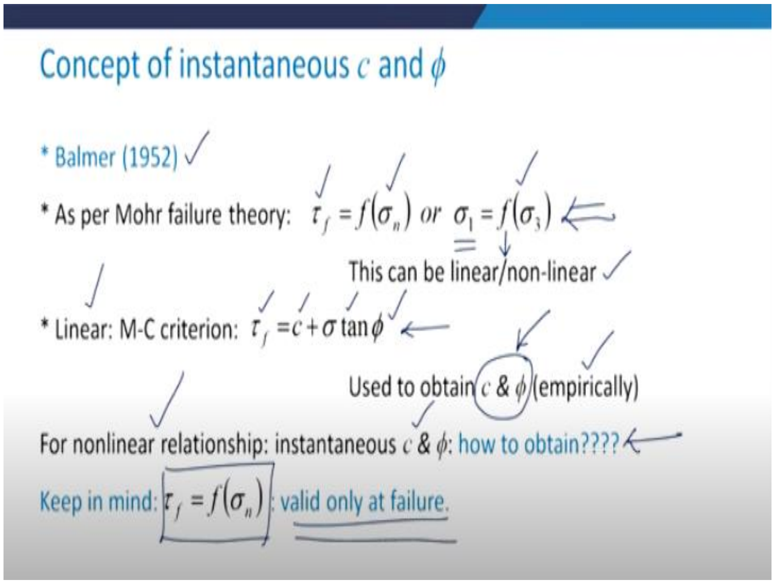

So, it was given by Balmer in 1952 as per the Mohr's failure theory we have seen that the shear stress at the failure is a function of the normal stress at the plane of failure or σ_1 is some function of σ_3. Now this functional relationship can be linear or it can be nonlinear. Mohr-Coulomb criterion is represented by a straight line which is given by this equation

$$\tau_f = c + \sigma \tan\phi$$

We conduct the triaxial tests and we fit the data to this equation and empirically we try to obtain these two parameters c and ϕ. Now for non-linear relationship can we find out something similar to this c and ϕ as it was there in case of the linear or the straight-line criteria? Yes, we can do that and that is called as instantaneous c and ϕ. The question is how to obtain? So, to start with one needs to keep in mind that this expression or the functional relationship is valid only at the failure. Not before not after.

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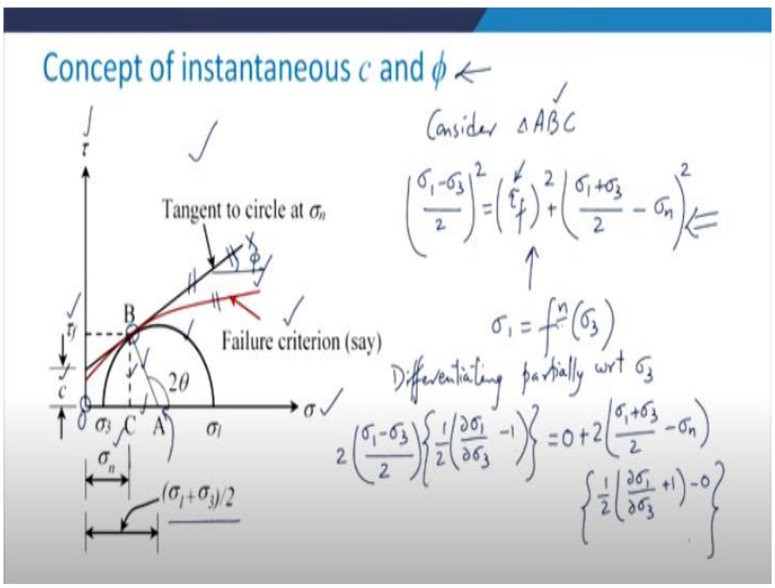

So, take a look at this figure now. We have the relationship between τ and σ and the tangent to the Mohr circle which is this tangent to this at the failure point which is represented by this point B is given by this straight-line portion. But then when I conduct a few more tests at different values of σ_3, I see that the actual failure criterion is not this linear one but it is this one which is represented by red color curve here.

The question is this c and this angle ϕ is defined for this linear curve or the straight line, can we find out something equivalent for this failure criterion which is non-linear? So, the process of finding this instantaneous c and ϕ we are going to learn here. So, you consider the triangle ABC here in this figure. So, this ordinate is τ_f corresponding to this point C up to point C from the origin, this is σ_n.

This is the center of the circle. A point A whose distance from the origin O is $(\sigma_1 + \sigma_3)/2$. This is how we draw the Mohr's stress circle.

Consider ΔABC,

$$\left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 = \left(\tau_f\right)^2 + \left(\frac{\sigma_1 + \sigma_3}{2} - \sigma_n\right)^2$$

$$\sigma_1 = f(\sigma_3)$$

Differentiating partially with respect to σ_3

$$2\left(\frac{\sigma_1 - \sigma_3}{2}\right)\left\{\frac{1}{2}\left(\frac{\partial\sigma_1}{\partial\sigma_3} - 1\right)\right\} = 0 + 2\left(\frac{\sigma_1 + \sigma_3}{2} - \sigma_n\right)\left\{\frac{1}{2}\left(\frac{\partial\sigma_1}{\partial\sigma_3} + 1\right) - 0\right\}$$

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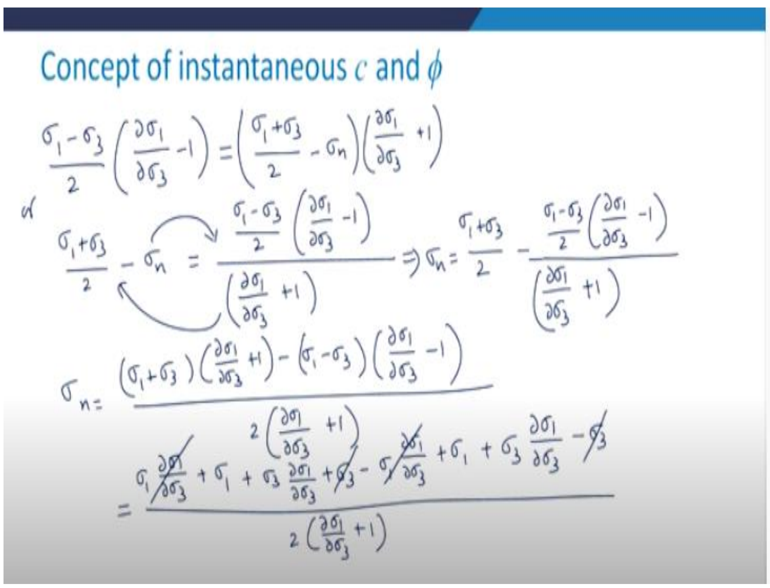

Now, let us try to simplify this further.

$$\left(\frac{\sigma_1 - \sigma_3}{2}\right)\left(\frac{\partial\sigma_1}{\partial\sigma_3} - 1\right) = \left(\frac{\sigma_1 + \sigma_3}{2} - \sigma_n\right)\left(\frac{\partial\sigma_1}{\partial\sigma_3} + 1\right)$$

On rearranging

$$\left(\frac{\sigma_1 + \sigma_3}{2} - \sigma_n\right) = \frac{\left(\frac{\sigma_1 - \sigma_3}{2}\right)\left(\frac{\partial\sigma_1}{\partial\sigma_3} - 1\right)}{\left(\frac{\partial\sigma_1}{\partial\sigma_3} + 1\right)}$$

$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2} - \frac{\left(\frac{\sigma_1 - \sigma_3}{2}\right)\left(\frac{\partial \sigma_1}{\partial \sigma_3} - 1\right)}{\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)}$$

$$\sigma_n = \frac{(\sigma_1 + \sigma_3)\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right) - (\sigma_1 - \sigma_3)\left(\frac{\partial \sigma_1}{\partial \sigma_3} - 1\right)}{2\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)}$$

$$= \frac{\sigma_1 \frac{\partial \sigma_1}{\partial \sigma_3} + \sigma_1 + \sigma_3 \frac{\partial \sigma_1}{\partial \sigma_3} + \sigma_3 - \sigma_1 \frac{\partial \sigma_1}{\partial \sigma_3} + \sigma_1 + \sigma_3 \frac{\partial \sigma_1}{\partial \sigma_3} - \sigma_3}{2\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)}$$

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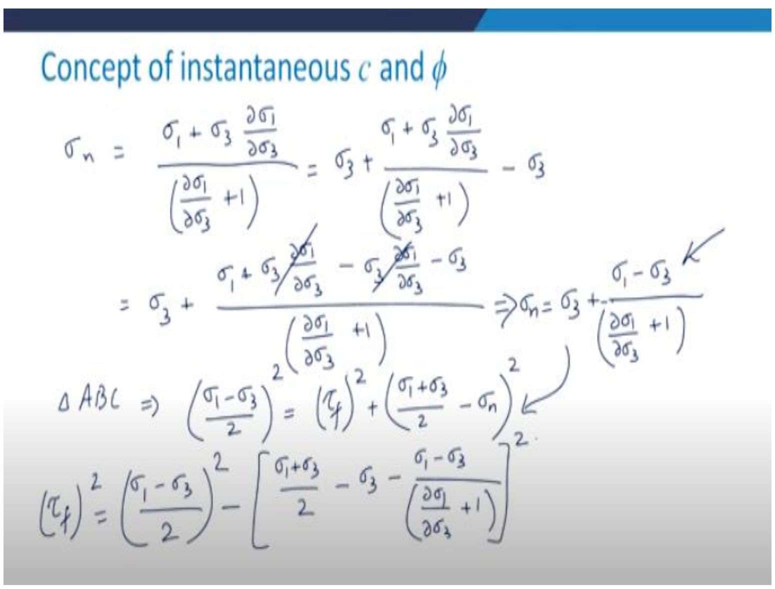

So, ultimately what we are going to get here is by adding and subtracting σ_3

$$\sigma_n = \frac{\left(\sigma_1 + \sigma_3 \frac{\partial \sigma_1}{\partial \sigma_3}\right)}{\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)} = \sigma_3 + \frac{\left(\sigma_1 + \sigma_3 \frac{\partial \sigma_1}{\partial \sigma_3}\right)}{\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)} - \sigma_3$$

$$= \sigma_3 + \frac{\left(\sigma_1 + \sigma_3 \frac{\partial \sigma_1}{\partial \sigma_3}\right) - \sigma_3 \frac{\partial \sigma_1}{\partial \sigma_3} - \sigma_3}{\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)}$$

$$\sigma_n = \sigma_3 + \frac{\sigma_1 - \sigma_3}{\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)}$$

Consider ΔABC,

$$\left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 = \left(\tau_f\right)^2 + \left(\frac{\sigma_1 + \sigma_3}{2} - \sigma_n\right)^2$$

$$\left(\tau_f\right)^2 = \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 - \left(\frac{\sigma_1 + \sigma_3}{2} - \sigma_3 - \frac{\sigma_1 - \sigma_3}{\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)}\right)^2$$

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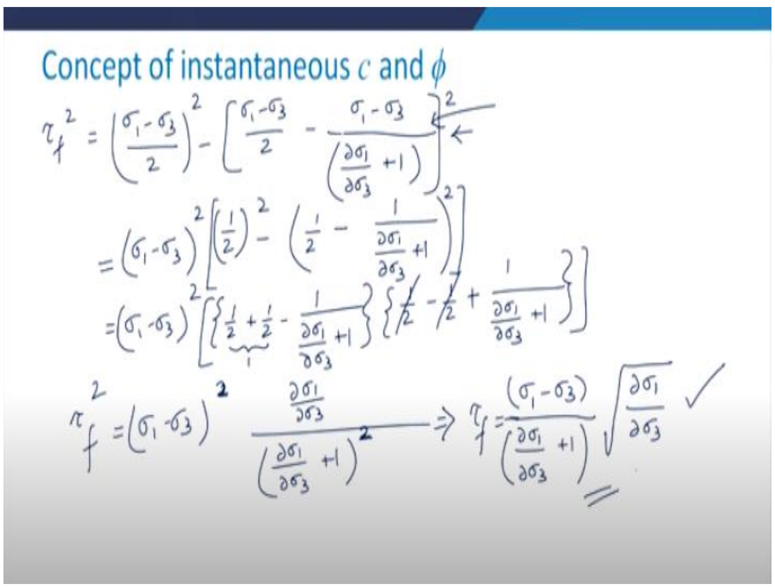

So, further simplifying this equation what we are going to get is

$$\left(\tau_f\right)^2 = \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 - \left(\frac{\sigma_1 - \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)}\right)^2$$

$$= (\sigma_1 - \sigma_3)^2 \left[\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2} - \frac{1}{\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)}\right)^2\right]$$

$$= (\sigma_1 - \sigma_3)^2 \left[\left\{\frac{1}{2} + \frac{1}{2} - \frac{1}{\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)}\right\}\left\{\frac{1}{2} - \frac{1}{2} - \frac{1}{\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)}\right\}\right]$$

$$\left(\tau_f\right)^2 = (\sigma_1 - \sigma_3)^2 \frac{\frac{\partial \sigma_1}{\partial \sigma_3}}{\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)^2}$$

$$\tau_f = \frac{\sigma_1 - \sigma_3}{\frac{\partial \sigma_1}{\partial \sigma_3} + 1} \sqrt{\frac{\partial \sigma_1}{\partial \sigma_3}}$$

So, this is going to be the expression for the τ_f. So, earlier we got the expression for a σ_n in this manner and now we got the expression for this τ_f.

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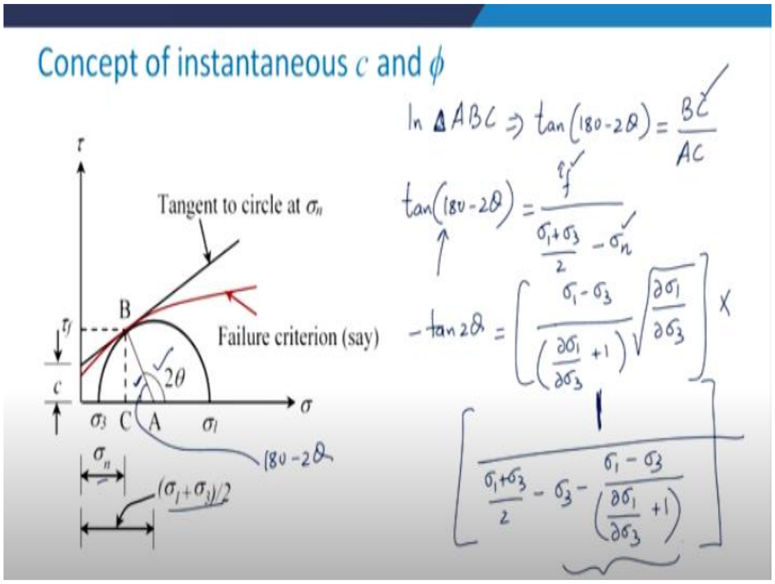

Now again, take a look at this triangle ABC.

In ΔABC,

$$\tan(180 - 2\theta) = \frac{BC}{AC}$$

$$\tan(180 - 2\theta) = \frac{\tau_f}{\frac{\sigma_1 + \sigma_3}{2} - \sigma_n}$$

$$-\tan(2\theta) = \left[\frac{\sigma_1 - \sigma_3}{\frac{\partial \sigma_1}{\partial \sigma_3} + 1} \sqrt{\frac{\partial \sigma_1}{\partial \sigma_3}}\right] \times \left[\frac{1}{\frac{\sigma_1 + \sigma_3}{2} - \sigma_3 - \frac{\sigma_1 - \sigma_3}{\left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1\right)}}\right]$$

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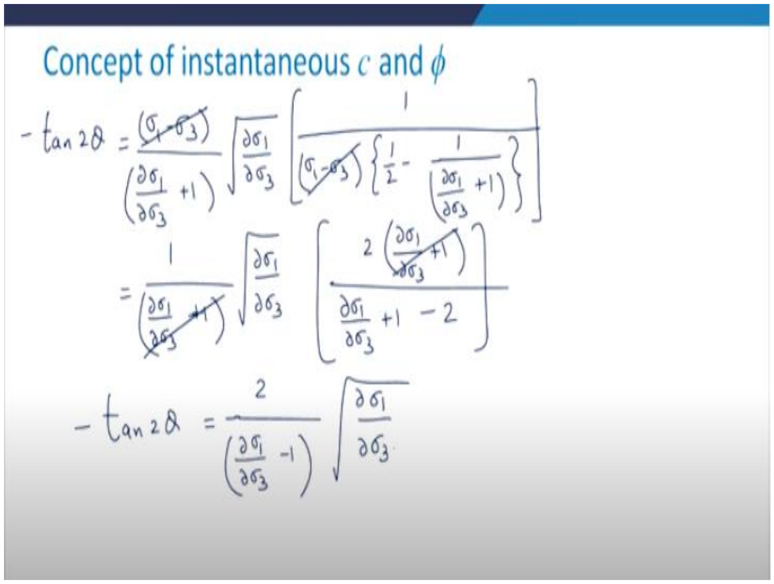

So, now we further simplify the earlier expression.

$$-\tan(2\theta) = \left[\frac{\sigma_1 - \sigma_3}{\frac{\partial\sigma_1}{\partial\sigma_3} + 1}\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}\right] \times \left[\frac{1}{(\sigma_1 - \sigma_3)\left\{\frac{1}{2} - \frac{1}{\left(\frac{\partial\sigma_1}{\partial\sigma_3} + 1\right)}\right\}}\right]$$

$$-\tan(2\theta) = \left[\frac{\sigma_1 - \sigma_3}{\frac{\partial\sigma_1}{\partial\sigma_3} + 1}\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}\right]\left[\frac{2\left(\frac{\partial\sigma_1}{\partial\sigma_3} + 1\right)}{\frac{\partial\sigma_1}{\partial\sigma_3} + 1 - 2}\right]$$

$$-\tan(2\theta) = \frac{2}{\left(\frac{\partial\sigma_1}{\partial\sigma_3} - 1\right)}\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}$$

This is what that we are going to have now.

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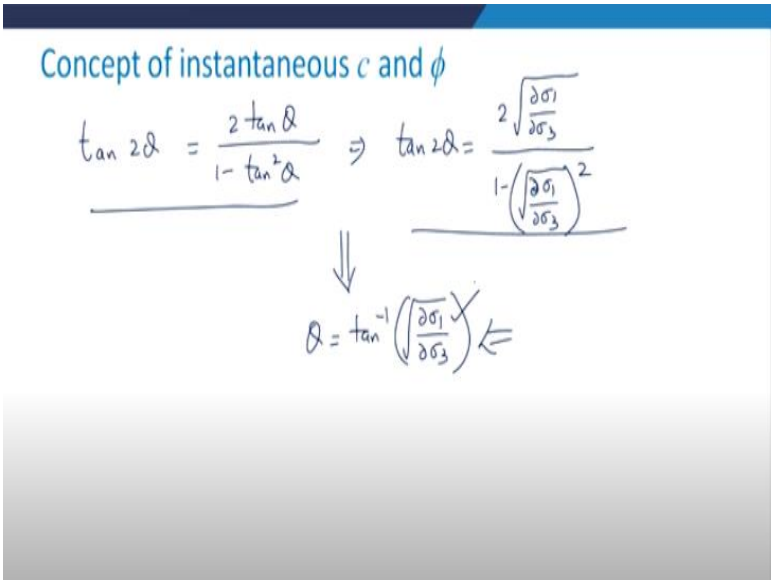

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

Rearranging the previously obtained equation of tan2θ

$$\tan(2\theta) = \frac{2\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}}{1-\left(\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}\right)^2}$$

On Comparing

$$\tan\theta = \sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}$$

$$\theta = \tan^{-1}\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}$$

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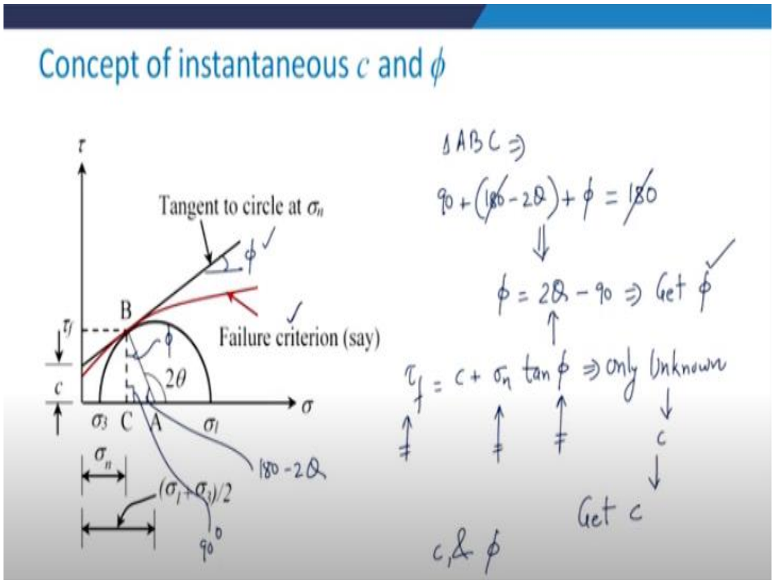

Take a look at this figure again because still we need to find out c and ϕ corresponding to this failure criterion, which is non-linear. So, if you take a look at this figure and concentrate on the triangle ABC once again this angle is ϕ you see and this angle is 90°. This angle is going to be 180 - 2θ. Now since this angle is ϕ simple trigonometry would say that this angle is also equal to ϕ. So, in triangle ABC we can write

$$90 + (180 - 2\theta) + \phi = 180$$

$$\phi = 2\theta - 90 \Rightarrow Get\phi$$

So, from here we can get ϕ. Once we know this ϕ, we have the expression like

$$\tau_f = c + \sigma \tan \phi$$

What all things are known here, what all that we have obtained here following the procedure which was given by Balmer?

We already have obtained τ_f, we have σ_n, here we obtained ϕ. So, the only unknown in this equation is going to be c. So, the only unknown is c in this equation. So, we just substitute all these three values and one can get the value of c in this case. So, you see that how beautifully we could get the value of c and ϕ for the criterion which is not represented by a straight line. Now this shear strength characteristic which is called as instantaneous c.

And ϕ it will represent the characteristic of the rock as same as it would have been if this was following a straight-line failure criterion. So, what exactly we have tried to do in order to obtain this instantaneous c and ϕ is if instead of a straight line fit to the failure criterion, if we have a non-linear failure criterion, then we can find out the equivalent values of c and ϕ for this nonlinear failure criterion as it was there in case of the Mohr-Coulomb failure criterion.

But we cannot call it as the Mohr-Coulomb shear strength parameters. Therefore, we are calling these as the instantaneous c and ϕ. This is the procedure which you need to follow in order to get this c and ϕ. So, let us summarize this. In the beginning you have a non-linear relationship between σ_1 and σ_3 or between τ and σ. That relationship you have to partially differentiate with respect to σ_3 then follow this procedure to find out its derivative etcetera.

You have already derived the expression for τ_f and σ_n, follow this procedure and you can obtain the value of instantaneous c and ϕ. So, this is what that I wanted to discuss with you in today's class. In the next class, we will learn about other failure criterion which are empirical failure criterion and I have already mentioned in some of the earlier classes that when we talk about the empirical relationships, they are derived on the basis of the experimental data.

So, we have to carry out the regression analysis on that experimental data in order to fit an appropriate model which we are calling as a failure criterion or any equation to that experimental data. So, we are going to learn some concepts related to this regression analysis and then we will learn few aspects related to the empirical criterion followed by the discussion on one empirical criterion, that is Hook and Brown criterion. Thank you very much.