Course Name: An Introduction to Climate Dynamics, Variability and Monitoring

Professor Name: Dr. Sayak Banerjee

Department Name: Climate Change Department

Institute Name: Indian Institute of Technology Hyderabad (IITH)

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ENERGY BALANCE MODEL AND TEMPERATURE RESPONCE TO RADIATIVE FORCING

Good morning class and welcome to our continuing lectures on climate dynamics, climate variability and climate monitoring. Today we will continue our discussion on the zero-d climate model and look at some solutions of the climate model that we just derived. So let us remind ourselves of the climate model that we have derived from using the zero-d energy balance equation. We have the heat capacity of the atmospheric system which is currently consisting of the troposphere, the surface of the earth and the mixed layer of the oceans. And this heat capacity we have said is mostly contributed by the oceanic heat capacity which is significantly higher than the heat capacity of the land surface or of the air. The heat capacity into the rate of change of a temperature perturbation with time is $\frac{dT'}{dt}$ plus the climate feedback parameter alpha into the temperature perturbation at that given instant of time.

$$C\frac{dT'}{dt} + \alpha T' = R_F(t)$$

$$\alpha = -\frac{\partial f}{\partial T}\bigg|_{x_{ss}}$$

$$R_F(t) = \frac{\partial f}{\partial x}\bigg|_{T_{ss}} x'$$

So, $\alpha T'$ equal to the time dependent radiative force in term which is defined here is the change in the net long-wave radiation at the tropopause level due to a change in the greenhouse gas concentration, like the concentration of CO_2 gas around the steady-state temperature T_{ss} due to a small increase or decrease of the greenhouse gas concentration x'. So, here x is say the mole fraction or the mass fraction that is molar mixing ratio or the mass mixing ratio of the greenhouse gas, x_{ss} the steady state value, x' is the perturbation of the greenhouse gas concentration and $\left.\frac{\partial f}{\partial x}\right|_{T_{ss}} x'$ is the impact this has on the net long wave radiation balance at the level of the tropopause. Around the gradient of that around the steady state temperature. So, this is defined as the radiative forcing parameter. And remember the climate feedback parameter is $-\frac{\partial f}{\partial T}$ for a given greenhouse gas concentration x_{ss}. That is the change in the tropopause radiative energy balance due to the change in temperature of the earth system at a given greenhouse gas concentration. So, this is feedback parameter, and this is radiative, and these two terms come into the 0D energy balance equation model of the climate system. So, precise values of C and alpha are difficult to get. There is a lot of work that goes on estimating these values.

If you look at climatic, climatological papers, these values are estimated, and the estimates get improved through careful measurement and modeling. In terms of the order of magnitude analysis, the alpha is of the order of 1 watt per meter square Kelvin. We will see, we will develop the idea of how to get alpha further as we go along during the lectures, in the next few lectures. And the heat capacity is around 70 percent of the ocean is mixed layer heat capacity because 70 percent of the planet surface is covered with ocean. So, that is 1 gigajoule per meter square Kelvin.

We also defined an important parameter called the relaxation time or feedback response time, which is a ratio of these two terms, C by alpha and its unit is seconds. And for the planetary system, the C by alpha, the feedback response time is around 10 to the power 9 seconds, which is around 30 years. And we said that the feedback response time gives us the approximate time scale that is required for a perturbed climate system to reach a new steady state. So, if you perturb a greenhouse gas concentration for example and as a result the temperature is changing over time, how long does it require for the temperature to reach a new equilibrium case where it is no longer changing with time. So, that is given by this feedback response time which is of the order of 10 to the power 9 seconds of 30 years.

Thus, given an instantaneous perturbation, it takes the climate system approximately 30 years to reach to a steady state. So, these are all approximate estimates. We will actually look at some of the models and see what the feedback response times actually are and how tau plays a role at finding out this feedback response time. So now how do we solve this equation? So it is a first order differential equation with $\frac{dT'}{dt}$ T prime and a term $\frac{R_F(t)}{C}$ which does not depend on t prime. So this equation can be easily solved by the integrating factor method that you may have learnt from usual engineering mathematics.

$$\frac{dT'}{dt} + \frac{\alpha}{C}T' = \frac{R_F(t)}{C}$$

$$\frac{dT'}{dt} + \frac{T'}{\tau} = \frac{R_F(t)}{C}$$

So, I am just going to go over this. So, what we do is, here we take the c to the denominator. So, we get $\frac{dT'}{dt}$ plus $\frac{\alpha}{C}$ which is $\frac{1}{\tau}$, right, into T' is equal to $\frac{R_F(t)}{C}$. So, this becomes the gradient of T' with time plus the temperature perturbation divided by the relaxation time or feedback response time equal to the radiative forcing term divided by the heat capacity of the atmospheric or atmosphere ocean system. Now, we will multiply both sides by the integrating factor which is $exp\left(\frac{t}{\tau}\right)$. T is the time, tau is the relaxation time or the feedback response time. Now, if you multiply both sides as we have done here, then if you can see this expression here, this expression can be directly $\frac{d}{dt}\left(T' exp\left(\frac{t}{\tau}\right)\right)$ So, this is a direct, so you can put these two terms are basically chain rule differentiation of this term with respect to time. So, temperature perturbation exponential of time by feedback response time. and this is $\frac{R_F(t)}{c} exp\left(\frac{t}{\tau}\right)$.

$$\frac{dT'}{dt} exp\left(\frac{t}{\tau}\right) + \frac{T'}{\tau} exp\left(\frac{t}{\tau}\right) = \frac{R_F(t)}{C} exp\left(\frac{t}{\tau}\right)$$

$$\frac{d}{dt}\left(T' exp\left(\frac{t}{\tau}\right)\right) = \frac{R_F(t)}{C} exp\left(\frac{t}{\tau}\right)$$

So, now you can integrate this directly and we can assume that at time t=0, the temperature perturbation was 0. So, what we are assuming is the radiative forcing started at time t=0. At that time t=0, the temperature was steady state temperature. So, $T' = 0$. As the time increases, T' starts to have a positive or a negative value as due to the effect of radiative forces. So, the initial conditions are time t=0, $T' = 0$. So, for our kind of planetary system, if you are looking at anthropogenic global warming effects, then this t will be pre-industrial time which is sometimes taken as 1850 or 1800s.

$$T'(t) = \frac{e^{-t/\tau}}{C} \int_0^t R_F(t') e^{-t/\tau} dt'$$

So, now if we take this initial condition and integrate, then we get here. This into t' as you can see and this exponential term goes on that side. So, you get $\frac{e^{-t/\tau}}{c}$ integral of 0 to t the relative forcing term into $e^{-t/\tau}dt'$. Here t' is a dummy variable. So, just be do not get confused. Capital T prime is the temperature perturbation with time. $e^{-t/\tau}$ is the exponential, exponentially decaying term here. C is the heat capacity of the atmosphere ocean system. R_F is the radiative forcing term which is time dependent. And this small t prime is basically the dummy time variable because we are integrating from 0 to t, and this is $e^{-t/\tau}dt'$. So, this becomes the final expression if we get the functional values of the radiative forcing term, then we can exactly evaluate how the temperature perturbation is changing with time, all right. Given that you know C, tau, etcetera, remember tau is C by alpha.

So, the alpha term, the climate feedback parameter term remains in this equation, all right. So, now we will look at some solutions to this problem to understand how the physics is changing because of the different types of relative forcing that can exist. So, we have found this expression. This is the temperature perturbation of temperature change. So, next we will look at a few simple models where different types of relative forcing terms are assumed and then we can find plot the temperature perturbation. So, one of the simplest forms of radiative forcing is what is called step function forcing.

$$R_F(t) = \begin{cases} 0 \; for \; t \leq 0 \\ R_{F1} \; for \; t > 0 \end{cases}$$

$$T'(t) = \frac{e^{-t/\tau}}{c} \int_0^t R_{F1} e^{t\prime/\tau} dt'$$

So, radiative forcing term is 0 for all t less than or equal to 0 as you would expect, then it becomes a constant sudden constant value R_{F1} at all times t greater than 0. So, radiative forcing is a constant and it is applied on the climate system at $t > 0$. So, there then of course this term becomes time independent. So, it just becomes $\frac{e^{-t/\tau}}{c} \int_0^t R_{F1} e^{t\prime/\tau} dt'$. And we can because this is constant we can take it out of the integral and just integrate this expression. So, when we do that we get an expression like this. So, firstly at time t equal to 0, this term becomes e to the power 0, so 1. So, this just becomes R_{F1} by c e to the power minus t by tau. At time t prime equals to t, this term and this term will cancel out and we will get a tau term coming out. So, this is what we are getting here,

$$T'(t) = S\left(1 - e^{-t/\tau}\right)$$

$$where, S = \frac{R_{F1}}{\alpha} \; and \; t > 0$$

t prime as a function of time become this factor s which is $\frac{R_{F1}}{\alpha}$, alright. So, basically this tau term and this c term combine together to get us back the alpha term, ok. That tau is basically C by alpha. This C and this tau term combined to produce the alpha term is only one that remains. So you can integrate this and see for yourself. The final expression looks like this. The temperature perturbation with time starting from P greater than 0 is this factor S which is the constant relative forcing by alpha. So here remember alpha is watt per meter square Kelvin. And radiative forcing is watt per meter square. So, these two together produces the unit of S to be kelvins. And this S is called the climate sensitivity. So, the ratio of the radiative forcing constant term in this case divided by the climate feedback parameter alpha is called the climate sensitivity. So, we will see why this is happening, why we are calling it climate sensitivity soon. And at any given time, the total temperature of the climate system is the steady state temperature which is the temperature it was at t equal to 0 plus the temperature perturbation $T(t) = T_{ss}(0) + T'(t)$ which is given by this expression here, all right. S, the climate sensitivity is the asymptotic value that the temperature perturbation t prime approaches as time tends to increase. So, what happens when time is tending to infinity? This becomes e to the power minus infinity. So, this becomes 0. So, the perturbation term tends towards S, which is $\frac{R_{F1}}{\alpha}$ term as the time becomes longer and longer under this kind of step function forcing time kind of relative forcing. That is why it is called the climate sensitivity because this is the final value that the temperature perturbation will reach as the time tends to increase. So, the final new steady state temperature under this condition will be the old steady state temperature plus the climate sensitivity value S. And here remember S has the same unit as temperature because that is what is giving us, we can write this expression because of that. So, then how does the plot looks like? So, it is easier to plot all of this in non-dimensional format. So, the relative forcing is very clear. You have zero relative forcing till t =0 and at exactly at t equal to 0, there is a certain step function, It goes to R_F equals to R_{F1} and it stays constant at that value.

We can non-dimensionalize all the parameters. So, the temperature perturbation the non-dimensionalized form $\overline{T}' = \frac{T'}{S}$. So, we are dividing the temperature perturbation by the sensitivity, climate sensitivity and the non-dimensional time we are using the actual time by the relaxation time tau. Relaxation time is second, so it also helps us to non-dimensionalize the time. With these new values you can see here T prime by S is T bar prime, t by tau is t bar. So, this becomes $\overline{T}' = 1 - e^{-\bar{t}}$. This becomes the expression, where tau is of course equal to C by alpha. So, now we can plot this expression clearly as t bar tends to infinity, the t bar prime tends to 1. So, the final expression when we are plotting t bar prime with t bar, capital t bar prime with t bar, it is going from 0 to 1. At t bar equal to 0, we have 1 minus 1, so it is 0. At t bar tending to infinity, it is 1 minus 0, so 1. So, the y axis varies from 0 to 1. It is an exponentially decaying function, so it kind of goes like this. We can also show that T prime reaches 95 percent of the final predestined perturbation value S at T equal to T tau. So, when t bar equal to 3, so here t bar is equal to 3, this is equal to 0.95. So, the perturbation has reached 95 percent of its final value by the time the total time is 3 times the relaxation time or the climate feedback response time. So, what this shows is that the climate feedback response time tau is intimately linked to when a climate system will reach steady state under this kind of a step function forcing term. It is usually three times or little bit more, three times that of the climate relaxation time. So, if the climate relaxation time is 30 years, then because of a constant relative forcing that has been added to the system, it will take around 90 years to reach a new steady state, 90 to 100 years. So, that will be for our planet, that will be the kind of time horizon we are looking at to reach steady state, new steady state conditions given a constant relative forcing effect.

Now here we have a constant radiative force. What if the radiative force is increasing linearly with time? This is very, this is the kind of function that our planet is currently undergoing due to the anthropogenic greenhouse effect. We will show later that the relationship of the radiative forcing due to CO_2 with the concentration of CO_2 is logarithmic. So, currently CO_2 is increasing exponentially with time. So, the rate of increase of the CO_2 concentration is more or less exponential currently due to the human activity. Because the relative forcing for CO_2 is log of the concentration increase, therefore an exponential increase in concentration results in a linear increase in the relative forcing term due to CO_2. So, here we have a radiative forcing case which is increasing linearly with time from t for t>0. So, $R_F(t)$ was 0 till the time equal to 0 and less and it becomes equals to gamma t where gamma is the slope into t, t is the time for any time t>0. So, if you plot $\frac{R_F}{\gamma}$. So, $\frac{R_F}{\gamma}$ equals to t. So, the relative forcing plot becomes $\frac{R_F}{\gamma}$ by t and we have a 45 degree line that is going like this.

$$R_F(t) = \begin{cases} 0\ for\ t \leq 0 \\ \gamma t\ fot\ t > 0 \end{cases}$$
$$\gamma = constant$$

So, 0 till here then going up as 45 degree. This is a good approximation, as I told you before, of the radiative forcing due to current increase in CO_2. The CO_2 concentration is rising exponentially due to human-caused emissions and radiation balance models show that CO_2 radiative forcing is proportional to log of the CO_2 mass fraction in the atmosphere. Hence, an exponential increase in CO_2 concentration results in a linearly increasing R_F term. So since CO_2 concentration is increasing exponentially right now and radiative forcing is proportional to the log of this CO_2 concentration, the radiative forcing is increasing linearly. So here then we again go back to the original expression if you remember. Here this expression and here instead of $R_F(t)$ we put gamma into T prime, correct? is the expression. So, we put gamma into T prime here and then we integrate this by parts. If we do a by parts integration then we get the final expression is gamma the slope of the relative forcing term into the relaxation time tau by the climate feedback parameter alpha into t by again the relaxation time tau minus 1 plus e to the power minus t by tau for t greater than 0 of course. This kind of an expression we are getting three individual terms. But we can again simplify this by non-dimensionalizing.

So, we create a non-dimensional radiative forcing term $\frac{R_F}{\gamma\tau}$. Remember here $R_F(t)$ itself is γt. So, γt by $\gamma\tau$ is $\frac{t}{\tau}$, which is equal to the non-dimensional time t bar. So here, we are showing that for this kind of a function the non-dimensional relative forcing value Rf by gamma tau is equal to the time variable itself t bar, the non-dimensional time variable. So, when we do that and we set the non-dimensional temperature perturbation we are writing as T prime the dimensional temperature parameter into alpha by gamma tau.

$$\bar{T}' = \frac{T'(t)\alpha}{\gamma\tau} = (\bar{t} - 1) + e^{-1}$$

So, again remember tau is C by alpha. So, remember that term. So, we are expressing this term T bar into alpha by gamma tau as a non-dimensional temperature T bar in this specific context. So, then if you see what we have done is we have taken alpha this side, gamma tau at the bottom. So, we have taken this factor on the left hand side. Then this becomes T bar minus 1 plus e to the power minus T bar. So, the non-dimensional temperature perturbation T bar prime is equals to the non-dimensional time minus 1 plus e to the power minus of the non-dimensional Alright? Where the relative forcing term itself is the non-dimensional term. Okay? So, we can now plot both the relative forcing and the non-dimensional relative forcing term and the non-dimensional temperature perturbation term T bar prime. So, this will be T bar prime just to be clear. With respect to the non-dimensional time t bar. Since R_F bar is equal to T bar, this is the 45 degree line.

This is our non-dimensional radiative forcing line that we have for our case. And the T bar prime value is like this. So, what it shows is initially the temperature perturbation is rising slower than the radiative forcing term. Then it becomes, so it is kind of a quadratic term. It is kind of going like this and then eventually it is becoming linear. So, it is initially the slope is lower, then it is increasing and then becoming almost parallel to the rf bar term. And we can show this effect here. So, at very small values of t bar, this is almost equal to 0. So, minus 1 plus 1. So, this becomes equal to t bar. And very large values t bar tending to infinity, this term becomes 0 and this just becomes t bar minus 1. So, this term here is basically t bar minus 1. So, if this is, so as we go along, if this is 6, this is 5, this is 7, this is 6. So, that way this is tending to t bar minus 1. So, these are the two functional values. You can plot this yourself and see the plots. So, t bar prime is starting more slowly initially, but starts to accelerate quadrically and then becomes almost equal to 0, almost linear with time at very large values of t bar. So, what does this mean? Let us just understand this from our context of the planet and we will stop our discussion today. As the CO_2, as we are exponentially increasing the CO_2 concentration, the radiative forcing term is increasing linearly, okay.

The temperature perturbation initially is slow. We are seeing this slow growth initially, alright. then it begins to accelerate and again eventually over time becomes linear. So, we will have a slow increase in temperature initially, then it will accelerate up and then becomes almost linearly increasing with time. And the exact values of course depends on this gamma term, this alpha term and this tau term, the relaxation time. is around 30, the value of the feedback parameter alpha which is as for our case it is around 1 and the gamma term which kind of depends on the greenhouse gas, the relationship of the CO_2 increase with the greenhouse gas concentrations which we will model later. But the functional dependence should look like this and this is what we see in our temperature charts as well. The initial emissions of CO_2 picked up from 1850 onwards and really started to increase exponentially from 1900 onwards. But the initial growth in the temperature was slow. Then it started to pick up from 1960-1970 onwards and started to increase very, very fast over the last 4-5 decades. And then it will, as we go along, it will start to linearize and there will be a monotonically increasing temperature with time as we go into the second half of the 21st century, unless there is a decrease in CO_2 emissions due to the IPCC and other by the world. That is what we will expect the temperature signature to look like. So, we will stop here today. In the next class, we will look at another case which is more realistic. We expect the CO_2 emissions and the greenhouse gas emissions to slowly decrease over time and eventually become zero. So, the relative forcing term itself will have a ramp and then a flat term as the CO_2 concentration and the methane concentration stabilizes in the second half of the 21st century.

Hopefully, if the emission control norms are properly implemented. In that case, what would be the expected profile of the temperature perturbation with time? That is something we will look at in the next class. Thank you for listening and see you again in our next lecture.