**FEM and Constitutive Modelling in Geomechanics**
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**Lecture: 7**
**Introduction to Rayleigh-Ritz Method**

So, hello students welcome back again let us continue the discussions to develop some continuous functions for our nodal variables. And in that context we have the Rayleigh-Ritz procedure I will introduce you to this procedure it is a beautiful way of expressing the continuity of the nodal variables. And in fact this is a precursor to the current finite element analysis based on the numerical methods where the Rayleigh-Ritz method is a mathematical one and later with the advent of digital computers the entire things is digitized and let us see.

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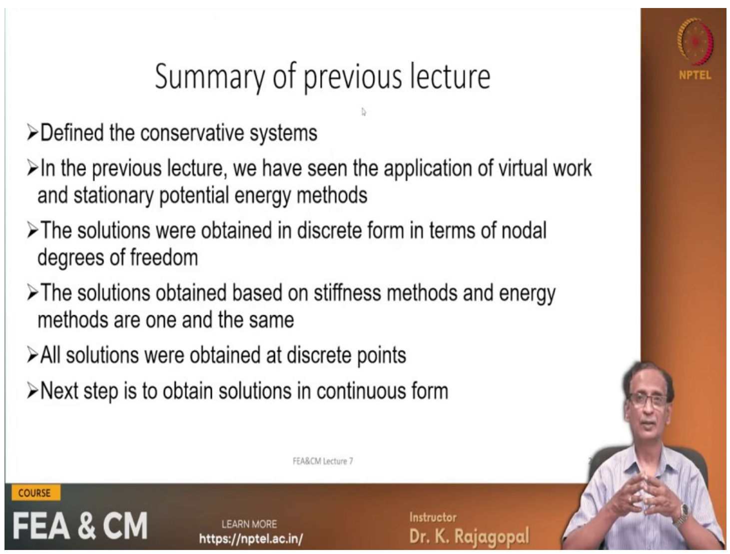


And before we go into the Rayleigh-Ritz procedure let us see what we did in the previous lectures. So, in the previous lectures we had seen the; we have defined the conservative system and conservative system is where in which the work done is independent of the path taken by the load. It is only based on the distance between the 2 points the least distance between the 2 points.

And we have seen the application of virtual work and stationary potential energy methods for solution of some simple problems both one dimensional and 2 dimensional. And we got the solutions in discrete form in terms of nodal degrees of freedom we got a solution but these displacements are only particular to the nodes that we have and the solutions that we had obtained both by the stiffness method and also by the energy methods we found that they are one and the same.

It is because both are solving the same equations and then as I mentioned earlier all these Solutions were obtained at discrete points and our next target should be to get a continuous variation because we want to move it to continuum because our object is to is to determine the response of soil as a medium. The soil is a continuous medium unlike your bar and beam element and the spring elements. So, that we will we will do in this today's class.

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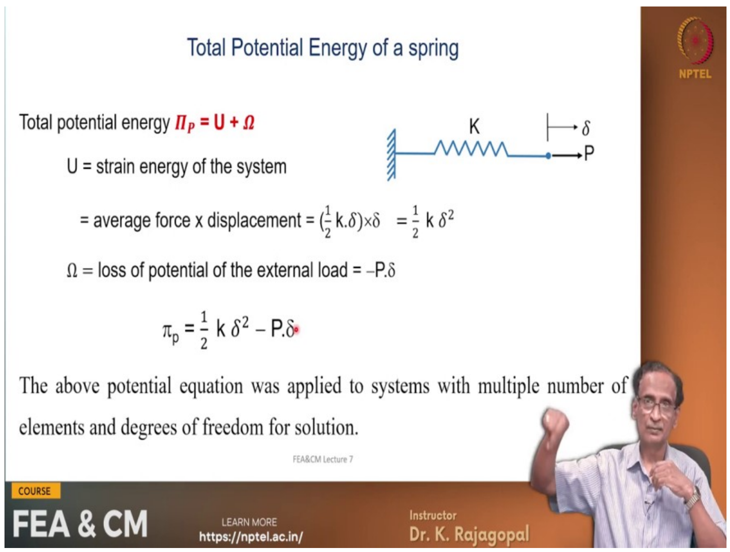
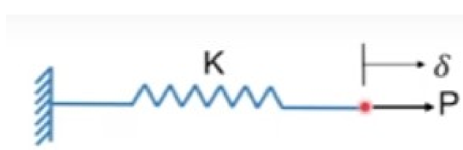



And before that just let me recap our total potential energy of the system. And although the concept is applicable universally for one dimensional 2-dimensional and 3-dimensional and continuum discrete and all types of systems we had only seen it in the context of uniaxial elements in a status class. We write the total potential energy of the system pi as u + Omega where u is the strain energy of the system.




See here what we mean by the system is we have a spring and then there is a load P and the P is the applied load whereas the spring is the system that we have and that can store the energy like if we elongate or compress it can store the energy. And so, if you gradually apply a displacement of Delta the average force multiplied by displacement is the energy because to start with Delta was 0. So, the spring is not loaded.

So, there is no energy in the system but as your gradual elongating or compressing it, it is developing some force and then it is developing some energy that is force times displacement. So, the average force is one half K Delta multiplied by your displacement there is one half K Delta Square. And then the Omega is the loss of potential of the external load it is actually by moving the load by that much it is losing the potential to do the work.

Total potential energy $\Pi_P = U + \Omega$

U = strain energy of the system

$$= \text{average force x displacement} = \left(\frac{1}{2}k.\delta\right)\times\delta \quad = \frac{1}{2}k\,\delta^2$$

$$\Omega = \text{loss of potential of the external load} = -P.\delta$$

Before the application of this deformation the load has so, much potential to move but out of that we have taken out some Delta and that minus P Delta corresponds to the loss of potential of the external load. So, we write it as minus P Delta. So, our total potential is one half K Delta Square minus P Delta and our stationary principle states that any small variation in this total potential should be equal to zero for the system to be in equilibrium.

And that we had seen yesterday with the example of a of a marble moving in a in a in a pan and at the top of the pan the marble has a chance to roll down but then at the bottom of the pan the marble is stable. And even if you slightly deform or move it a little bit it is still stable because it is at the bottom most point within the pan that is what we mean by the by the constant potential energy.

And now this is in a discrete form see we have a Delta that is known at one point and the energy is defined in terms of that but then what happens inside. Like let us say you take some other point here we do not know.

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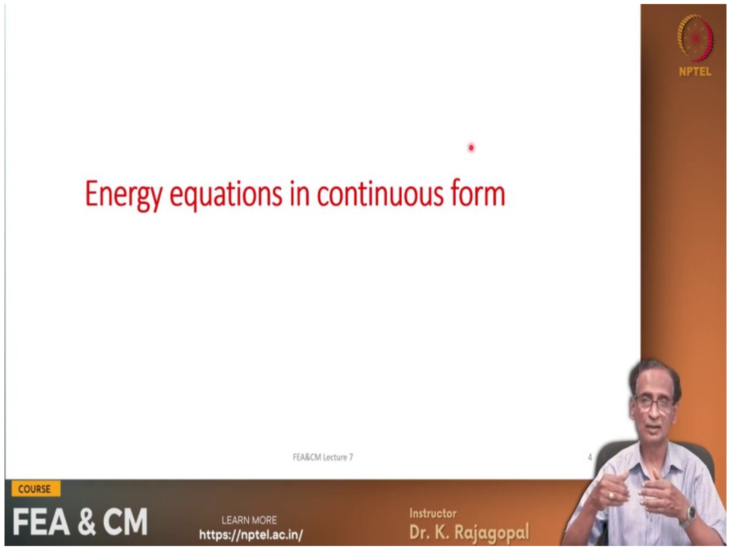


And so, it is good to go in for um continuous forms of the same equation so, that we can apply this method to a continuum.

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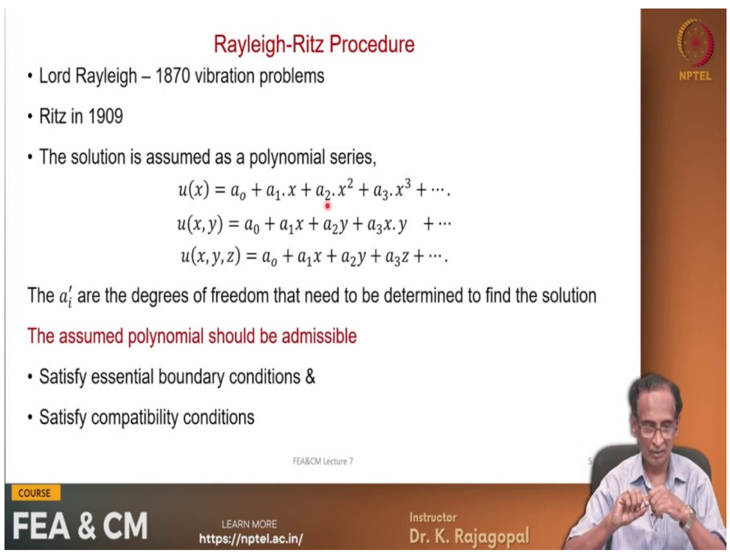


And before that the Rayleigh-Ritz procedure let me introduce the Rayleigh-Ritz procedure. And in 1870 Lord Rayleigh he introduced this concept for vibration problems for determining the eigenvalues and eigenvectors and Ritz in 1909 he also proposed a similar methodology and combining both these names this Rayleigh-Ritz procedure is developed. And they suggested that we can assume the solution in a polynomial series right.

And let us say u is our variable u is let us say displacement and if it is a one dimensional problem there is only one coordinate x and you can assume any polynomial like a0 + a 1 x + a 2 x square + a 3 x Cube and so on. Like it could be an infinite series of series and in 2 dimensions will have 2 coordinates x and y and the polynomial could be a0 + a 1 x + a 2 y + a 3 xy + so on like we can have any number of terms.

And in 3 dimensional problems u of x y z this could be a0 + a 1 x + a 2 y + a 3 z and so on.

The solution is assumed as a polynomial series,

$$u(x) = a_o + a_1.x + a_2.x^2 + a_3.x^3 + \cdots.$$
$$u(x,y) = a_0 + a_1x + a_2y + a_3x.y \quad + \cdots$$
$$u(x,y,z) = a_o + a_1x + a_2y + a_3z + \cdots.$$

And what they said is these a's a0 a 1 a 2 a 3 and so on. These are the degrees of freedom that need to be determined to find the solution. They actually previously we defined the degrees of freedom at the node points in x direction and y direction and similar to that the Rayleigh and Ritz they called these a's as the degrees of freedom are the generalized coordinates that we will see later in the context of finite element analysis.

And what they also said this assumed polynomial should meet some requirements and the first thing is the polynomial should be admissible and that is it should satisfy all the essential boundary conditions and then satisfies the compatibility conditions. So, it is actually there should not be any break in the shape of the element or a break in the variation. So, it should be a basically it is a continuous function.

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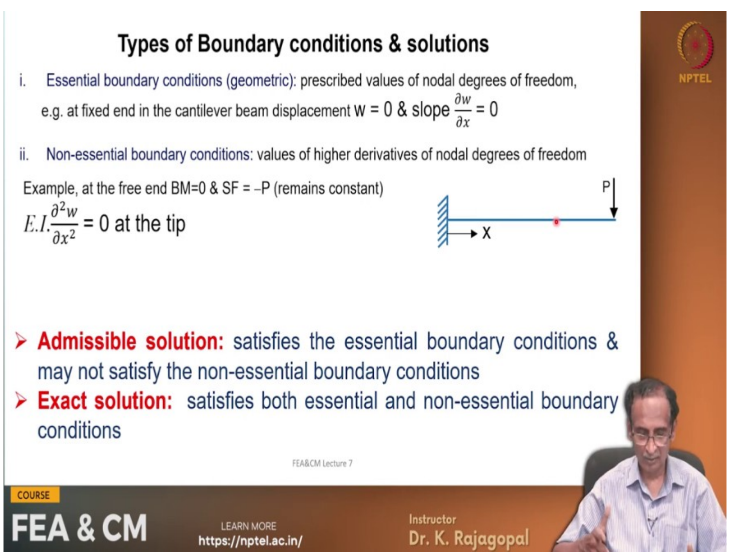


And say these boundary conditions the simplest ones are essential boundary conditions or geometric boundary conditions.


Let us say we take a we take a cantilever beam a fixed end cantilever beam and we can say that at this fixed end your displacement is 0 and then the slope is also zero and these are the minimum boundary conditions that any solution should satisfy so, that we get some reasonable result it may not be accurate but at least it should be able to represent this boundary condition of this fixed end.

$$w = 0 \ \& \ \text{slope} \ \frac{\partial w}{\partial x} = 0$$

So, the beam displacement w is 0 is one boundary condition and then the slope dou w by dou x is also zero and apart from this we may have some non-essential boundary conditions that is let us say if you look at this beam. We have applied a tip load and within this beam your shear force should be constant it is just simply P and then your bending moment should be 0 at the tip because there is no lever arm for this load.

Then the bending moment should be maximum at this point at the fixed end point that should be equal to P times l and these are something that we do not enforce but we need to get out of the solution.

So, our bending moment is

$$E.I.\frac{\partial^2 w}{\partial x^2} = 0 \text{ at the tip}$$

E I dou Square w by dou x square is 0 E I times curvature is your moment and it should be 0 at the tip that that x is equal to l. And if our solution is admissible it will satisfy the essential boundary conditions.

And it may or may not satisfy the non-essential boundary conditions like your shear force is constant along the length or your bending moment is 0 at the tip and so on. And if the solution is exact it will satisfy not only the essential boundary conditions but also non-essential. Like for example your shear force is constant and the bending moment is 0 at the tip. These non-essential boundary conditions are problem dependent this is whatever I explained is only in the context of the of the cantilever beam. And there could be something else for different problems.

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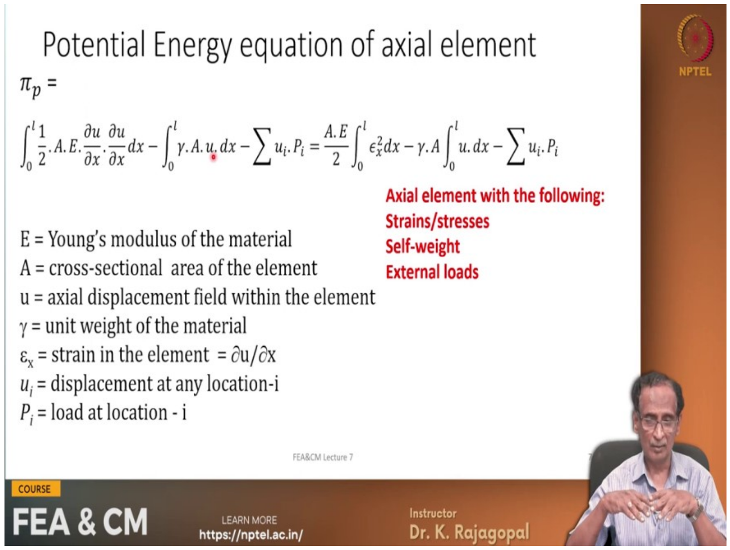


Now let us look at this potential energy if equation in a continuous form and let us apply this for an axial element. And from now on our object is to get the solution in a continuous form like whatever may be the maybe the point we should have a solution. Unlike in the case of the previous examples where we defined the degrees of freedom only at some discrete points but now we want a continuous function.

And we have an axial element and it may have some strain and then some stresses and the strain and stresses are proportional to each other and it may be subjected to its own self weight and then on top of that there could be some external loading and it is an axial element. So, there is only one direction and let us say that it is of length l. And so, there could be some potential energy because of the internal strain.

Say this train is if u is the displacement axial displacement field within the element dou u by dou x is the strain and our Young's modulus times strain is distress and the stress multiplied by strain is the work done and we can integrate this over the full length of l and we are taking the average stress. So, that is one half E times dou E by dou x multiplied by The Strain dou E by dou x and then we have an area cross sectional area A.

$\pi_p =$

$$\int_0^l \frac{1}{2}.A.E.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial x}dx - \int_0^l \gamma.A.u.dx - \sum u_i.P_i = \frac{A.E}{2}\int_0^l \epsilon_x^2 dx - \gamma.A\int_0^l u.dx - \sum u_i.P_i$$

E = Young's modulus of the material
A = cross-sectional area of the element
u = axial displacement field within the element
$\gamma$ = unit weight of the material
$\varepsilon_x$ = strain in the element $= \partial u/\partial x$
$u_i$ = displacement at any location-i
$P_i$ = load at location - i

And we integrate this of the length of l we will get the potential energy of the system and this a could be anything it need not be constant because now we have an integral equation now we can have a A bar with a variable cross section. So, it may be varying along the length. So, in that case we can put an A of x and then define some function and then the loss of potential because of the self weight um.

So, if u is your axial displacement field within the element gamma times dx times A is the force right at any point that multiplied by the displacement at that point is the work done and then because we have a length and this unit weight is acting over the length of the element. So, it is 0 to integrated over 0 to l. So, gamma times a cross-sectional area multiplied by dx is the force multiplied by displacement at that particular Point u and we integrate this over a length of l.

Then on top of that there could be some externally applied forces along the length of the bar and let us say that Vector is P i, P i is the load at location one and location i multiplied by the corresponding displacement u i right. And so, this we can equate like if you are area of cross section is constant along the length we can bring it out of the integral A E by 2 integral 0 to l Epsilon x square Epsilon x is nothing but dou u by dou x.

And here we are assuming that our displacements and the strains are so, small that our strains can be the first order variation of the of your displacements dou u by dou x the rate of change of displacement is called as this strain and minus gamma a integral u dx minus of the sum total of u i and P i and this is our total potential energy of the system. And what Rayleigh and Ritz have proposed is that we can assume u in terms of some polynomial expansion as a0 + a 1 x + a 2 Y and so on. And our a0 a 1 a 2 and so on. These are the degrees of freedom that are similar to our previous degrees of freedom.

Then any variation in this total potential with respect to any of these degrees of freedom should be zero for the system to be in equilibrium.

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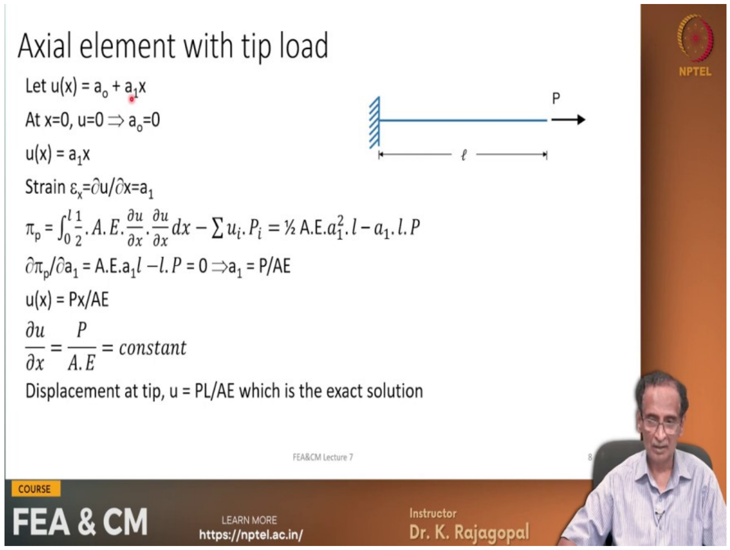


And so, what we do is we can assume some solution in terms of a polynomial and then and then work out this total potential and then determine the constants a0 a 1 a 2 and so on. So, let us as an example illustration let us take a bar and it is applied a load of P at the tip and this bar is of length l and previously we had defined these bars with some certain nodes node 1 node 2 and so on.


But here we do not have any such notation this is just a simple bar it is a continuous bar of length l then at the tip there is a load of P applied and we want to find a solution for this and let us say that our solution is to start with u of x is a0 + a 1 x right is just a simple assumption like you can assume anything you want you can assume a0 + a 1 x + a 2 x square and so on. And to start with I have assumed this and our boundary condition is that at x is equal to zero our displacement is 0 because this is a fixed end.

Let $u(x) = a_o + a_1 x$
At $x=0$, $u=0 \Rightarrow a_o=0$
$u(x) = a_1 x$

So, our u is 0 at x is 0. So, that means that a0 is zero. So, u of x is a 1 x and this is called as an admissible solution because it is already automatically applied the required boundary condition. And instead of 0 let us say we put some constraint that the tip the displacement at this end is some delta that a0 could be equal to Delta right and the strain within the element is dou u by dou x and that is equal to a 1 and our total potential Pi is integral 0 to l one half A E dou u by dou x times dou E by dou x it is actually basically one half a times Epsilon x square and minus of u i P i.

Strain $\varepsilon_x = \partial u/\partial x = a_1$

$$\pi_p = \int_0^l \frac{1}{2}.A.E.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial x}dx - \sum u_i.P_i = ½\, A.E.a_1^2.l - a_1.l.P$$

$$\partial\pi_p/\partial a_1 = A.E.a_1 l - l.P = 0 \Rightarrow a_1 = P/AE$$

And if we integrate this you substitute u of x here and integrate you get one half A E a 1 square because we have dou u by dou x times dou u by dou x and then l because the strain does not have any x we just have an l and the displacement at the tip is a 1 times l multiplied by the load at that point P right and if you apply the stationary principle dou Pi by dou a 1 should be 0 for our equilibrium. And so, if you differentiate with respect to a one we get A E times a 1 l minus LP that is 0.

$$u(x) = Px/AE$$

$$\frac{\partial u}{\partial x} = \frac{P}{A.E} = constant$$

So, our a 1 is P by A E and our u is a P x by A E right and then at the tip the displacement is P l by AE that is our familiar equation from the strength of materials and how do we know that this is a this is an exact solution. So, for that we can determine the strain dou u by dou x and if you do that that comes to P by A E P by A is the stress and divided by E is your strain and that is constant along the length.

Displacement at tip, u = PL/AE which is the exact solution

And we know that if you apply a tip load the displacement at the tip is PL by AE and the strain is constant. So, we can say that this polynomial that we had assumed for the displacement is will give us exact solution for the tip loaded the problem. And now let us look at some other problem.

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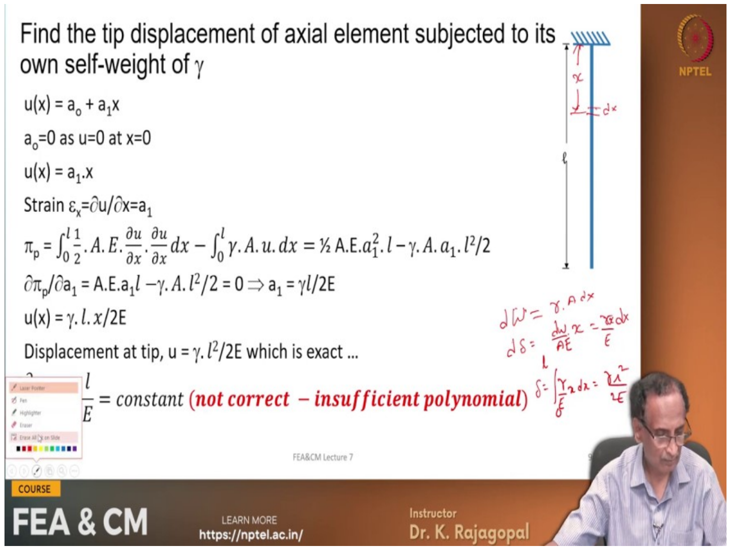
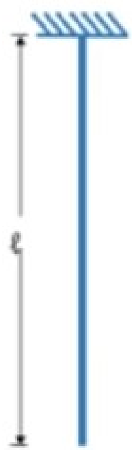



Let us say that we have a bar of length l and then it is subjected to its own self weight see it is hung from the from the ceiling like this right and then because of its own self weight it will elongate and we want to get the displacement field under this self weight.



And so, let us see that minimum conditions that we have are the displacement at this point is zero. And then along the length our strain is going to vary linearly right.

Then at this point at this tip we know that this strain is zero because that the free end and there cannot be any strain associated with this right and at this point the strain should be the maximum right. And then we do not know what is your tip displacement that we can easily determine let me just try this let us say we take any arbitrary length of dx and our weight oops sorry our weight is a gamma times A times dx right.

$$dw = \gamma \, . \, A \, . \, dx$$

And our d d Delta is actually it is let us say dw d Delta is dw by A times x let us say this is at some distance x from the from the origin.

$$d\delta = ( dw \,.\, x ) / A$$

And so, we have this as oh sorry d Delta right we should have a Young's modulus E that is the this the stress divided by E times x. So, gamma dx times x by E and our Delta is we can we can integrate from 0 to l gamma by E x dx. So, that comes to gamma l Square by 2E.

This is what we can easily derive theoretically and let us see what happens with our solution. Let us let me go back to my lesson. Let us assume A first order polynomial let us say u is a0 + a 1 x and at x is equal to 0 our a0 is zero and server our u is zero. So, a0 is zero. And so, our u of x is a 1 x and the strain is dou u by dou x that is a one and we see that we can say at this stage itself that this polynomial is not sufficient.

$$u(x) = a_o + a_1 x$$
$$a_o = 0 \text{ as } u = 0 \text{ at } x = 0$$
$$u(x) = a_1.x$$

Because our strain is constant but then because of the increasing weight your strain is going to vary linearly with the distance but let us anyway let us do the problem and then find out later. But at this stage itself even at the time of assuming the polynomial itself we can say that our solution is not going to be accurate. So, our total potential pi is integral 0 to l one half AE Epsilon x square dx minus gamma A dx that is the force multiplied by u integrated over 0 to l that is the total loss of potential.

$$\text{Strain } \varepsilon_x = \partial u/\partial x = a_1$$
$$\pi_p = \int_0^l \frac{1}{2}.A.E.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial x}dx - \int_0^l \gamma.A.u.dx = ½\, A.E.a_1^2.l - \gamma.A.a_1.l^2/2$$
$$\partial \pi_p/\partial a_1 = A.E.a_1 l - \gamma.A.l^2/2 = 0 \Rightarrow a_1 = \gamma l/2E$$
$$u(x) = \gamma.l.x/2E$$

And if you integrate if you in substitute your E u as a1x and integrate we get one half A E a 1 square l minus gamma a 1 l Square by 2 right. And so, we can apply the principle of the stationary T and the dou pi by dou a 1 should be 0. And if you go through this integral we get our a 1 is gamma l by 2 E and our displacement at any point is a gamma l x by 2E right. And our displacement at the tip if you if you substitute x is equal to l the displacement at the tip u is gamma l Square by 2E.

Displacement at tip, u = $\gamma . l^2/2E$ which is exact ...

$$\frac{\partial u}{\partial x} = \frac{\gamma . l}{2.E} = constant\ (not\ correct\ -insufficient\ polynomial)$$

And if you just look at this result it looks accurate. Now that is what the theoretical result is also gamma l Square by 2E. So, our tip displacement is also exactly matching with the exact result. So, can we say that the solution is accurate but then we have to look at other things like we can evaluate the displacement at some other point and see. And so, let us go for higher order term that is dou u by dou x. And so, if you do this gamma l by 2E and that is constant that is coming out as constant.

Actually here itself we could have seen that the strain is constant a 1 and we know that the strain should vary linearly but then our solution is saying that the strain is constant. So, that means that whatever solution that we have obtained is not sufficient it is it is admissible like because it is it is satisfying the boundary conditions but then it is not sufficiently accurate we need to improvise it. And let me just now let us include one more term in the polynomial and see what happens.

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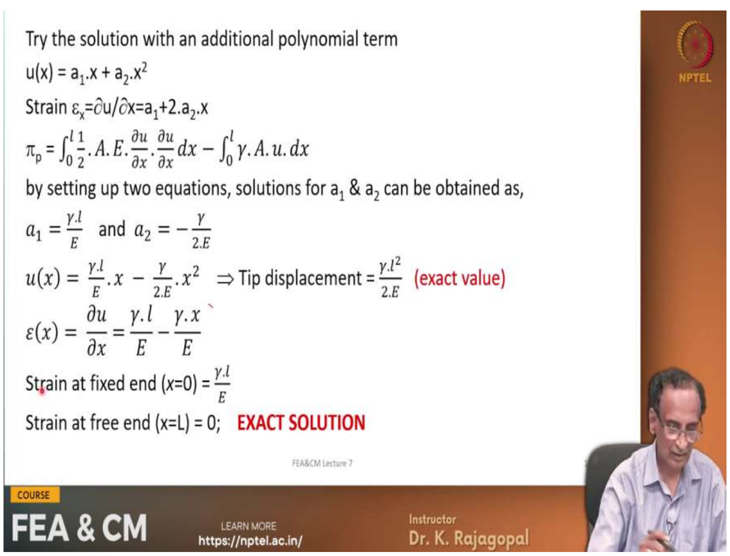


Now we are saying u is a 1 x + a 2 x square we have added one more polynomial term and our strain is Epsilon x is dou u by dou x that is a 1 + 2 a 2x and our Phi is integral 0 to l one half A E dou u by dou x and dou u by dou x minus integral 0 to l gamma A u dx and if you do the integration. And then set up 2 simultaneous equations by taking dou Pi by dou a 1 and dou Pi by dou a 2 to 0 and set them to 0.

Try the solution with an additional polynomial term

$u(x) = a_1.x + a_2.x^2$

Strain $\varepsilon_x = \partial u/\partial x = a_1 + 2.a_2.x$

$\pi_p = \int_0^l \frac{1}{2}.A.E.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial x}dx - \int_0^l \gamma.A.u.dx$

by setting up two equations, solutions for $a_1$ & $a_2$ can be obtained as,

$a_1 = \frac{\gamma.l}{E}$ and $a_2 = -\frac{\gamma}{2.E}$

$u(x) = \frac{\gamma.l}{E}.x - \frac{\gamma}{2.E}.x^2$ $\Rightarrow$ Tip displacement = $\frac{\gamma.l^2}{2.E}$ (exact value)

$\varepsilon(x) = \frac{\partial u}{\partial x} = \frac{\gamma.l}{E} - \frac{\gamma.x}{E}$

And then solve we will see that a 1 is gamma l by E and a 2 is minus gamma by 2E these are the 2 constants that we determine. So, our solution u is a 1 x + a 2 x square and a 1 is gamma l by E server solution u is gamma l by E x minus gamma by 2E x square right. So, if you substitute x is equal to l our tip displacement is comes out as gamma l Square by 2E which is equal to the exact theoretical result.

But then before we conclude whether it is a good or bad our strain along the length is dou u by dou x and that comes out as a gamma l by E minus gamma x by E right. So, our strain is going to change along the length and that at the free end let us look at the free and that x is equal to l if you substitute your strain comes out as zero that is what we have and then at x is equal to 0 your strain is gamma l by E gamma l is the is the total stress and the divided because of the total length of the element and divided by E is your strain.

And so, basically if you plot your strain variation will come out like this. So, our strain is maximum at the top and then it is in decreasing to zero let us say this is zero and here it is gamma l by E and that is natural. Because we can easily see the variation because as your moving along the length you are stress is changing the stress is maximum here gamma times l and the stress is 0 here. And so, our strain is 0 here and Gamma l by E.

Strain at fixed end (x=0) = $\frac{\gamma.l}{E}$

Strain at free end (x=L) = 0; EXACT SOLUTION

So, that means that whatever solution that we got this is our solution this is a exact solution. So, we can conclude that this is this is it like we do not need to increase any further this polynomial need not be changed any further. And we can we can stop at this polynomial let me just see I think I am getting better at writing because if you write anything at one shot you will get a it will get erased easily.

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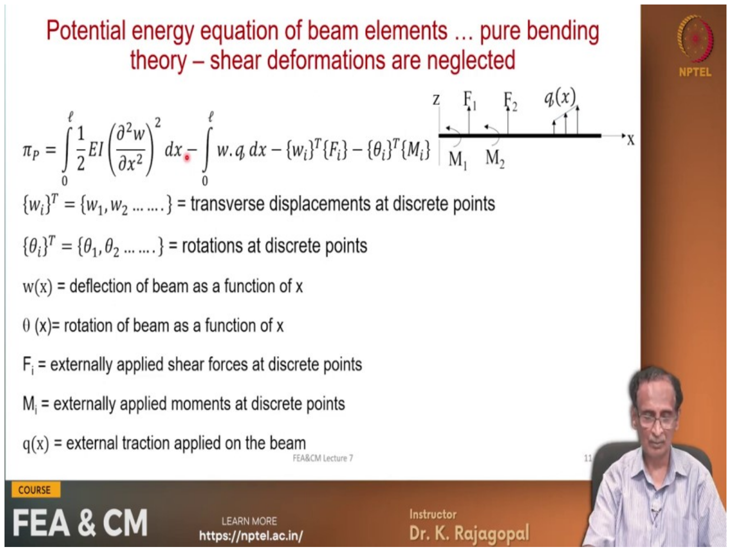
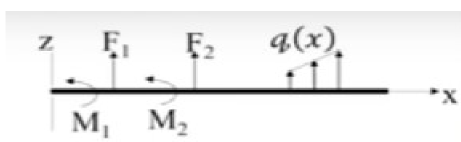



Now let us move on to other problem like let us go on to the flexural problem beam element. And let us consider a beam element like this and the letters neglect axial displacements because axial we have already done with the bar elements with the axial elements and let us also assume that it is a it is a Pure flexural Element and there will not be any Shear induced bending.




And the shear deformations are neglected and we can write our total potential energy equation as integral one half E i dou Square w by dou x square whole square is actually dou Square w by dou x square is the curvature and your moment times the sorry E i times this curvature is your moment. And the moment times curvature in the integral form is the work done right an integral of curvature is your Theta that is the rotation.

$$\pi_P = \int_0^\ell \frac{1}{2} EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx - \int_0^\ell w.q\, dx - \{w_i\}^T\{F_i\} - \{\theta_i\}^T\{M_i\}$$

And so, this is your integral expression for the work done by the moment and then the rotation. And let us say that w is the transverse deformation and let us have some applied force q that is varying along the length and let there be some Shear forces applied at along the length. And then some moments and we can calculate the loss of potential corresponding to these like this let us say our q is a continuous function.

And let and this could continue with the full length 0 to l w times q dx where w is our transverse displacement and the q is the traction and then say at this point your displacement is w 1 the loss of potential is F 1 times w1 and corresponding to F 2 let us say the displacement is w2. So, the loss of potential is F 2 times w 2. Similarly the loss of potential because of the rotation and the moment is Theta times M at some discrete point i right.

$\{\breve{w}_i\}^T = \{\breve{w}_1, w_2 \dots \dots\}$ = transverse displacements at discrete points

$\{\theta_i\}^T = \{\theta_1, \theta_2 \dots \dots\}$ = rotations at discrete points

w(x) = deflection of beam as a function of x

θ (x)= rotation of beam as a function of x

And our w's are the transverse displacements at the discrete points like 0.1.2.3 and so on. And our w of x is the deflection of the beam as a function of length l x and Theta of x is the rotation of beam as a function of x.

$F_i$ = externally applied shear forces at discrete points

$M_i$ = externally applied moments at discrete points

q(x) = external traction applied on the beam

And F i is the externally applied the shear Force at some discrete points. And m i is the external applied moments at discrete points and the q is the applied the attraction it is it is a surface force or surface pressure.

And this is your total potential of the flexural element this is coming from the internal moment and then the rotation integral curvature is rotation. And so, we can actually apply this total potential energy equation and then get some limit solutions for simple problems as a demonstration but it is applicable for any problem any like let us say even if you have a building frame you should be able to solve it using this methodology.

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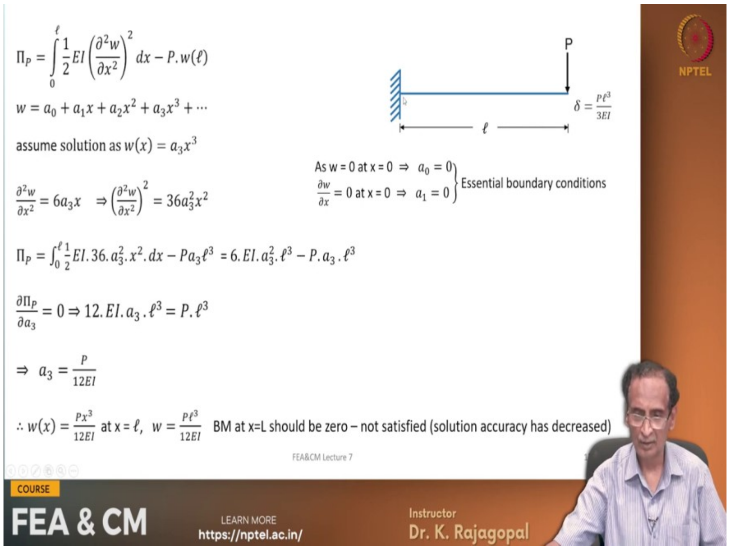



Let us apply for a simple problem of cantilever beam with the tip load and since we do not have this q and other things we can write the total potential is one half E i dou Square w by dou x square the whole Square integrated over 0 to l minus P times at the tip displacement and what is the polynomial that we can assume. See this polynomial can be in the form of a0 + a 1 x + a 2 x squared + a 3 x Cube and so on.

$$\Pi_P = \int_0^{\ell} \frac{1}{2} EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx - P.w(\ell)$$

$$w = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

But the essential boundary conditions are at x is equal to zero our displacement is 0 and then our rotation is zero the slope should be 0 at this point. So, if you do that w 0 at x is equal to 0 means a0 is 0 and dou w by dou x is 0 at x is equal to 0 means a 1 is 0. So, our admissible polynomial can start with a 2 x square and so on. And let us assume only a single term just for Simplicity and let us say that our w is A 2 x square.

assume solution as $w(x) = a_2 x^2$

$$\frac{\partial^2 w}{\partial x^2} = 2.a_2 \Rightarrow \left(\frac{\partial^2 w}{\partial x^2}\right)^2 = 4.a_2^2$$

And we can see how good the solution is with just a single term and dou square w by dou x square is 2 A 2 and dou Square w by dou x square whole square is a 4 A 2 square and we can substitute these terms dou Square w by dou x square and then w of l this equation and then the total potential energy is is this and then variation with respect to A 2 should be 0 so, that means that our A 2 is a P l by 4 E i and we can substitute that back here.

$$\Pi_P = \int_0^{\ell} \frac{1}{2} EI.4.a_2^2\, dx - Pa_2\ell^2 = 2.EI.a_2^2.\ell \quad -P.a_2.\ell^2$$

$$\frac{\partial \Pi_P}{\partial a_2} = 0 \Rightarrow 4EIa_2\ell \quad = P\ell^2$$

$$\Rightarrow \quad a_2 = \frac{P.\ell}{4EI}$$

$\therefore w(x) = \frac{P\ell x^2}{4EI}$ , at x = $\ell$, $w = \frac{P\ell^3}{4EI}$ ; BM≠0 & SF≠–P Solution is Not accurate

So, our w is Pl by 4 E i times x square. And so, our w of x is a P l x square by 4 E i. So, at x is equal to l our w is P l cube by 4 E i which is not correct because our displacement tip displacement is P l cube by 3 E i and the so, actually if you look at this solution w of x is P l x square by 4 E i you are the shear force should be constant with with within the beam section and that is not coming out and w by dou x is q because actually it is coming out as a zero which is which is not correct.

Sorry I think it should be only and dou square w by dou x square is x gets disappeared and your bending moment should be 0 at x is equal to l and that is not coming out and your bending moment should be increasing linearly along the length. So, in fact at the fixed end it should be P times l which is not coming like your bending moment is is constant and then the shear force is zero.

So, that means that this polynomial is not able to give us the required solution and we can try to improve the solution by assuming a higher the polynomial A 3x Cube. And once again we see that our solution is not good enough because we are not able to get a constant Shear Force and then the bending moment is not zero at the tip and it is not equal to P times l at this fixed end.

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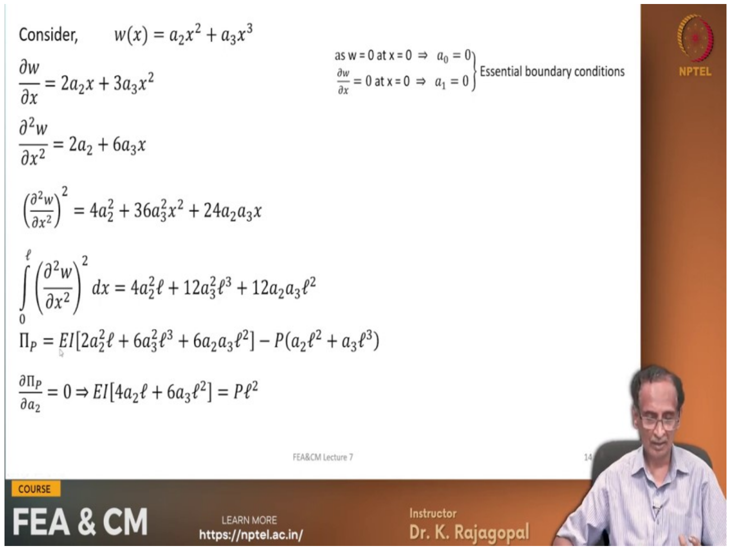


So, now we can actually try another thing let us include the 2 polynomial terms the one quadratic term and one cubic term.

Consider, $w(x) = a_2x^2 + a_3x^3$

$$\frac{\partial w}{\partial x} = 2a_2x + 3a_3x^2$$

$$\frac{\partial^2 w}{\partial x^2} = 2a_2 + 6a_3x$$

a 2 x square + a 3 x cubed and inherently we also have a0 and a 1 x but only thing is a0 and a 1 are 0 but we can technically say that we have those terms also in this polynomial. So, dou w by dou x is 2a 2 x + 3 a 3 x square and dou square w by dou x square is 2 a 2 + 6 a 3x.

And the whole square of that is this and so our total potential of the system is this and we can differentiate this with respect to a 2 and a 3 and set them to zero and find our constants.

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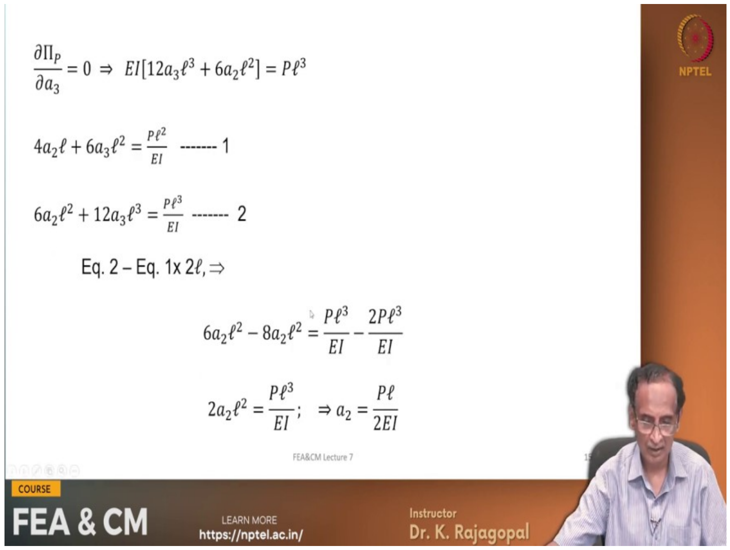


So, if you go through this your a 2 comes out as P l by 2 E i and a 3 is minus P by 6 E i. So, if you substitute these 2 constants a 2 and a 3 in our equation for w; w is P l by 2E i x square minus P by 6 E I x Cube. So, if you substitute x is equal to l your w is P l cube by 2 E i minus P l cube by 6 E i that is P l cube by 3i which is the same as what we get from the flexural theory. And let us calculate the bending moment at the tip of the beam dou Square w by dou x square at x is equal to l is a P l by E i minus P x by E i.

Eq.2 – Eq. 1x1.5$\ell$,

$$3a_3\ell^3 = -\frac{1}{2}\frac{P\ell^3}{EI} \quad \Rightarrow a_3 = -\frac{P}{6EI}$$

$$\therefore w = \frac{P\ell}{2EI}x^2 - \frac{P}{6EI}x^3$$

$$\text{at } x = \ell \qquad \Longrightarrow w = \frac{P\ell^3}{2EI} - \frac{P\ell^3}{6EI} = \frac{P\ell^3}{3EI}$$

And if you substitute x is equal to l that is 0. So, at x is equal to l it is 0 and then at x is equal to 0 your bending moment is P l because actually it is E i times dou Square w by dou x square then as you are moving inside your your bending moment is going to increase. And then Shear Force within the beam is E i dou Cube w by dou x Cube that comes out as minus P and that is constant along the length right.

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=\ell} = \frac{P\ell}{EI} - \frac{Px}{EI} = 0 \text{ at x=L} \quad \therefore \textit{Bending Moment} = 0 \ \textit{at } x = \ell$$

$$\text{Shear force in beam section} = \text{EI.}\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=\ell} = -P \ \text{(constant over length)}$$

$$\text{BM at fixed end} = \text{P.}\,\ell \text{ (at x=0)}$$

So, we see that considering 2 terms a 2 x square + a 3 x Cube is able to give us the exact solution it is predicting the tip displacement correctly then it is giving us the constant Shear

Force and then bending moment of 0 at the tip and then bending moment of P times l at x is equal to l. So, we can say that this polynomial is able to satisfy both essential and non-essential boundary conditions and so, it must be exact.

And what is the difference between the earlier polynomials under this polynomial actually earlier we had considered only one term a 2 x square or a 3 x Cube and then with a 2 x square our solution was not correct because the so, this polynomial order was not sufficient but then if you if you assume x Cube term by neglecting the lower ordered term also we have a problem. We had in fact the solution is has become worse now it is a P l cube by 12E i.

Previously it was P l cube by 4 E i which is at least close enough to P l cube by 3i but now the solution has become even worse and our bending moment and the shear Force are not what we expect. And so, the reason for this is we are not including the complete polynomial and what Rayleigh and Ritz have suggested is you include a complete polynomial for any problem to improve the solution accuracy.

And so, by including this and in fact this is a complete polynomial because this is also including a0 + a 1 x but only thing is a0 and a one not zero. So, we do not see those terms here. So, we have a cubic polynomial complete polynomial and we are able to get the exact solution.

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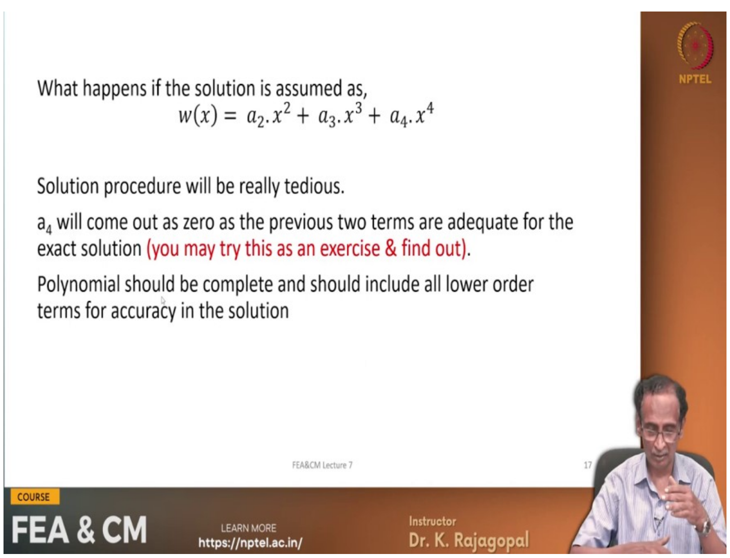


And let us say that we assume one more term a 4 x to the power 4 because let us say we are not sure of the solution that we get with the previous polynomial and if we assume one more term and continue what will happen. So, actually the solution is a bit more tedious because now we have 3 simultaneous equations to solve for 3 unknowns a 2 a 3 and a 4 it is a very tedious problem but then if you do that you will see that ever a 4 will come out as 0.

What happens if the solution is assumed as,

$$w(x) = a_2.x^2 + a_3.x^3 + a_4.x^4$$

Because already the solution is accurate or exact with these 2 terms a to x square and the a 3 x Cube I will not do it here but you can try it out on your own it takes a long time it might take about 2 hours if you do it systematically and then you will find that a 4 is 0. And so, if you have access to any of the programs like Matlab you can set it up in the Matlab program and then do all these problems like you can solve simultaneously questions in a symbolic form.

Because in a calculator you can only solve in the numerical form but in Matlab you can do it in the symbolic form and get your a 2 a 3 a 4 and so on whatever may be the equation that you have. So, the lesson that we have to learn is when we assume any polynomial it should be complete and it should include all the lower ordered terms for better accuracy. So, when we had a2 x square the solution was more accurate compared to the solution that we got with a 3 x Cube as the single term.

Because a 2 x square had included the 2 other lower ordered terms the constant and then the linear term and but when we can included the cubic term we have neglected the square term the quadratic term. So, the solution um the accuracy has come down.

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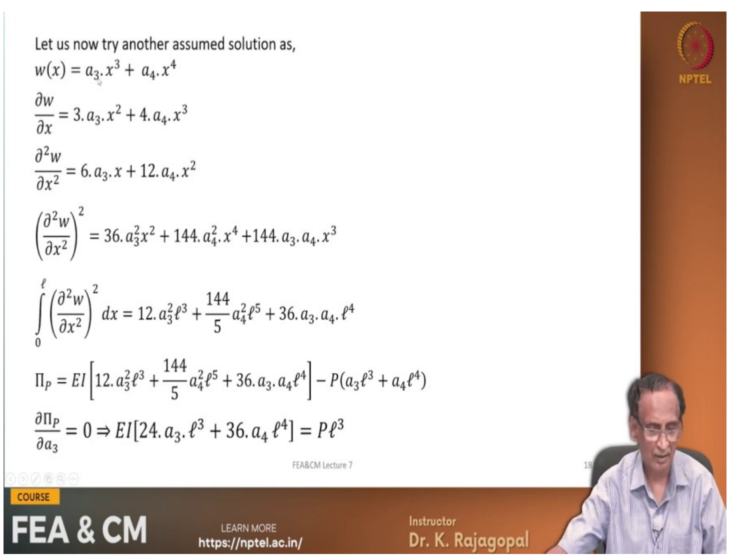

So, now let us do one more term one more polynomial and then see what happens let us assume w of x is a 3 x Cube + a 4 x to the power 4 and we have neglected the x square term.

$$w(x) = a_3 . x^3 + a_4 . x^4$$

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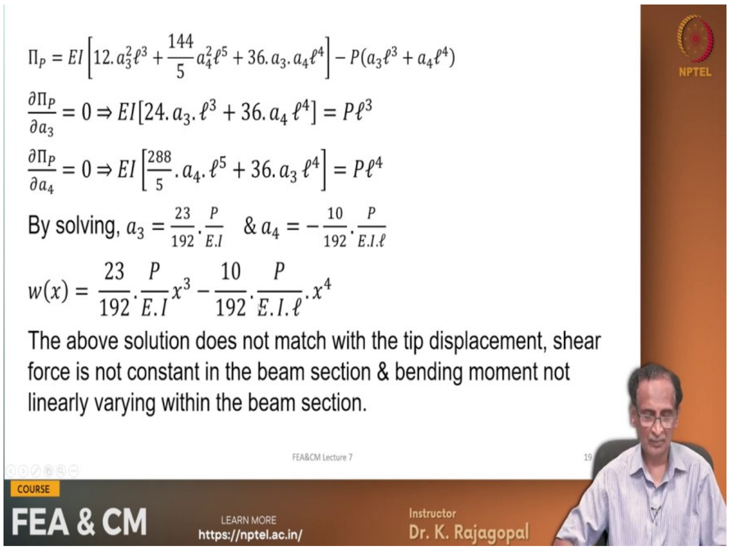


Then if we go through the solution that we get is this and the solution really does not match any of the theoretical solution then it does not give us the required boundary conditions and the higher order derivatives like our Shear Force should be constant along the beam at the P and your bending moment is 0 at the tip that does not happen. So, you can the solution is given and you can see that the tip the displacement does not match your equation P l cube by 3 i and then along the length your Shear force is not constant.

$$\Pi_P = EI\left[12.a_3^2\ell^3 + \frac{144}{5}a_4^2\ell^5 + 36.a_3.a_4\ell^4\right] - P(a_3\ell^3 + a_4\ell^4)$$

$$\frac{\partial \Pi_P}{\partial a_3} = 0 \Rightarrow EI[24.a_3.\ell^3 + 36.a_4\,\ell^4] = P\ell^3$$

$$\frac{\partial \Pi_P}{\partial a_4} = 0 \Rightarrow EI\left[\frac{288}{5}.a_4.\ell^5 + 36.a_3\,\ell^4\right] = P\ell^4$$

By solving, $a_3 = \frac{23}{192}.\frac{P}{E.I}$ & $a_4 = -\frac{10}{192}.\frac{P}{E.I.\ell}$

$$w(x) = \frac{23}{192}.\frac{P}{E.I}x^3 - \frac{10}{192}.\frac{P}{E.I.\ell}.x^4$$

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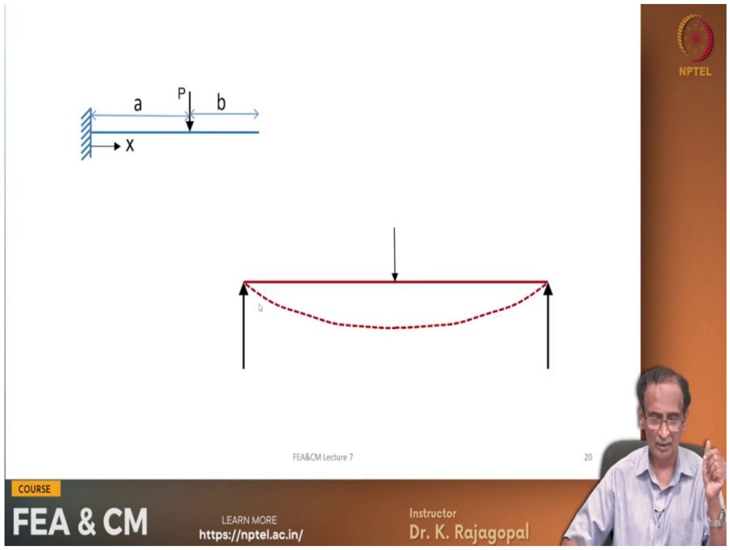


So, and this procedure can be continued for other problems like let us say you have a cantilever Beam with a loading at some other point not at the tip but somewhere else or let us say you have a simply supported beam with a central load. And we can apply it for any problem.

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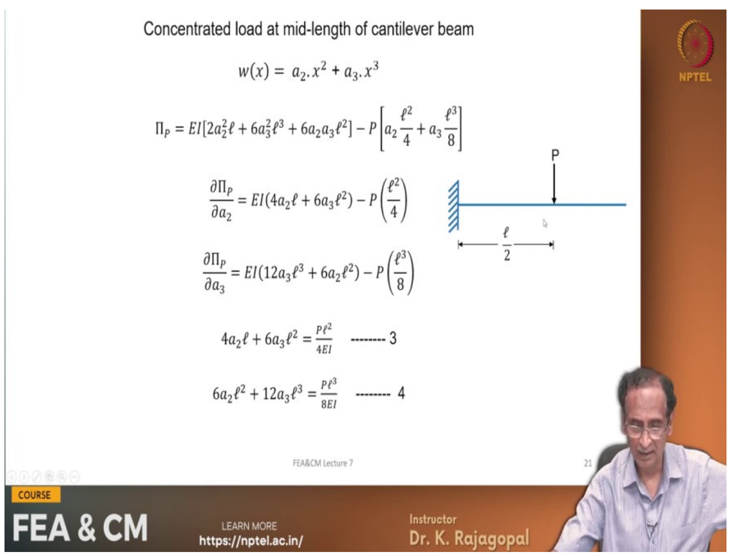
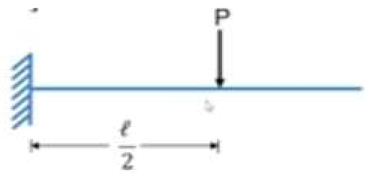



But only thing is we do not know how many terms polynomial terms we need to include previously for the tip loaded problem. Our solution with the 2 polynomial terms or an exact a 2 x square + a 3 x Cube and let us see whether this will work out for this problem of the cantilever Beam with a load applied at the mid length midpoint l by 2.

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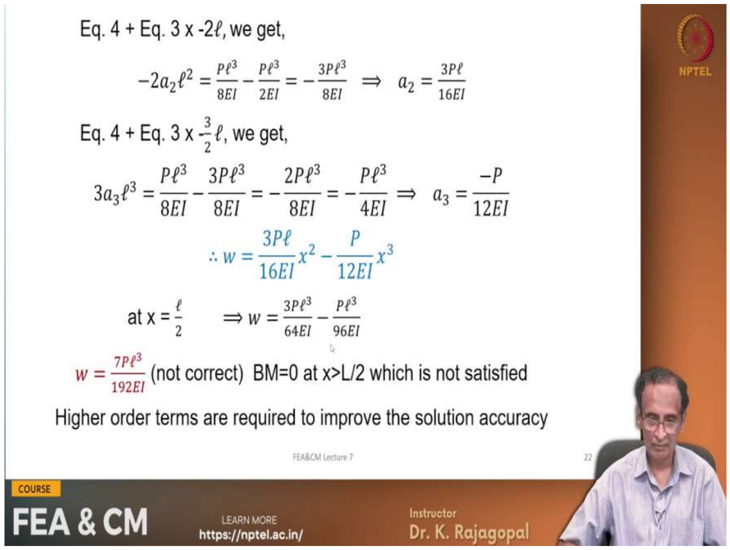


$$\Pi_P = EI[2a_2^2\ell + 6a_3^2\ell^3 + 6a_2a_3\ell^2] - P\left[a_2\frac{\ell^2}{4} + a_3\frac{\ell^3}{8}\right]$$

$$\frac{\partial \Pi_P}{\partial a_2} = EI(4a_2\ell + 6a_3\ell^2) - P\left(\frac{\ell^2}{4}\right)$$

$$\frac{\partial \Pi_P}{\partial a_3} = EI(12a_3\ell^3 + 6a_2\ell^2) - P\left(\frac{\ell^3}{8}\right)$$

$$4a_2\ell + 6a_3\ell^2 = \frac{P\ell^2}{4EI} \quad \text{-------- } 3$$

$$6a_2\ell^2 + 12a_3\ell^3 = \frac{P\ell^3}{8EI} \quad \text{-------- } 4$$

So, if you do this your solution is is coming out as a 3P l cube sorry a 3P l by 16 EI x square minus P by 12 EI x Cube. And so, actually see when we do this problem beyond this length of l by 2 your bending moment should be zero Beyond this length but our solution that we get does not give that.

Eq. 4 + Eq. 3 x $-2\ell$, we get,

$$-2a_2\ell^2 = \frac{P\ell^3}{8EI} - \frac{P\ell^3}{2EI} = -\frac{3P\ell^3}{8EI} \implies a_2 = \frac{3P\ell}{16EI}$$

Eq. 4 + Eq. 3 x $-\frac{3}{2}\ell$, we get,

$$3a_3\ell^3 = \frac{P\ell^3}{8EI} - \frac{3P\ell^3}{8EI} = -\frac{2P\ell^3}{8EI} = -\frac{P\ell^3}{4EI} \implies a_3 = \frac{-P}{12EI}$$

$$\therefore w = \frac{3P\ell}{16EI}x^2 - \frac{P}{12EI}x^3$$

at $x = \frac{\ell}{2}$ $\implies w = \frac{3P\ell^3}{64EI} - \frac{P\ell^3}{96EI}$

$w = \frac{7P\ell^3}{192EI}$ (not correct) BM=0 at x>L/2 which is not satisfied

Higher order terms are required to improve the solution accuracy

And so, that means that we need to include one more polynomial term and redo the problem and we can continue until we get the solution that satisfies both the essential and non-essential boundary conditions.

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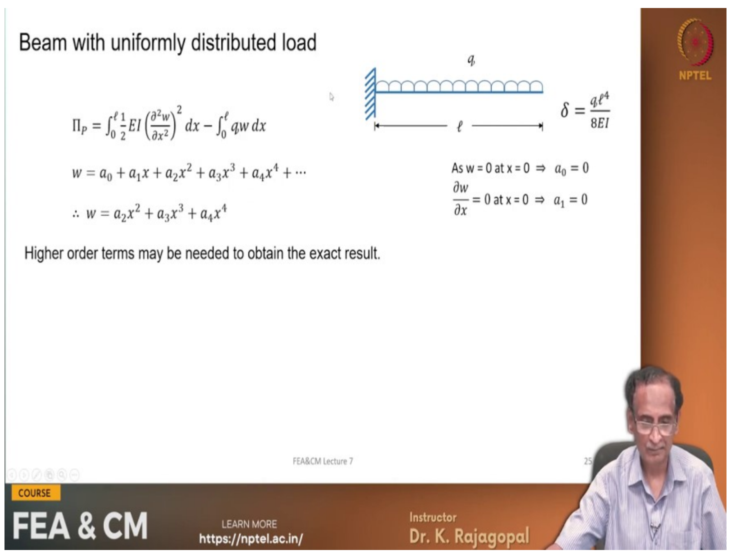
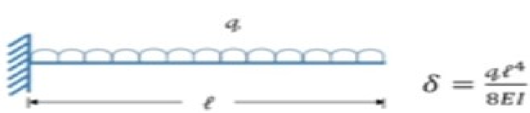



But beforehand we cannot say how many terms we need. And this is the problem with a uniformly distributed load now the potential equation is 2 times w integral 0 to l because your q is varying along or it is continuing along the length.




As w = 0 at x = 0 ⇒ $a_0 = 0$

$\frac{\partial w}{\partial x} = 0$ at x = 0 ⇒ $a_1 = 0$

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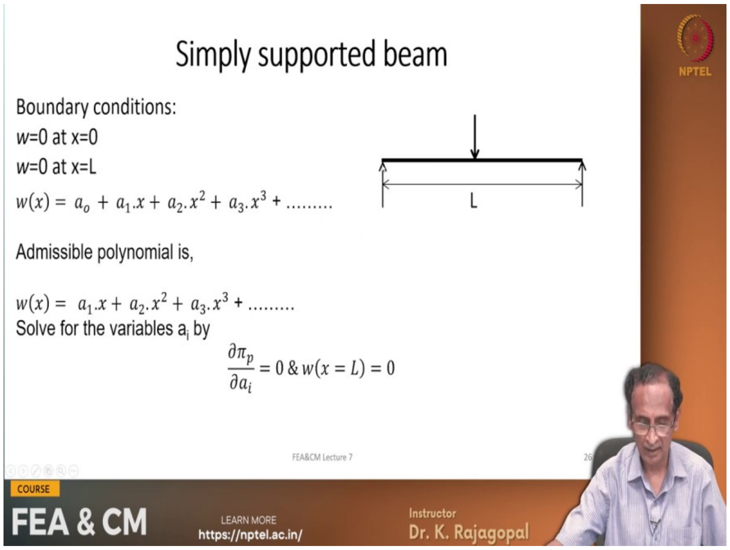
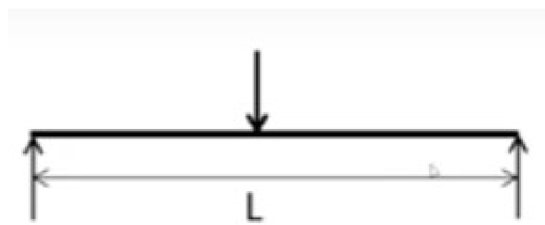



And if you have a simply supported beam the essential boundary conditions are the displacement is 0 at both ends at x is equal to 0 and x is equal to l we need to enforce that apart from all the other required boundary conditions.

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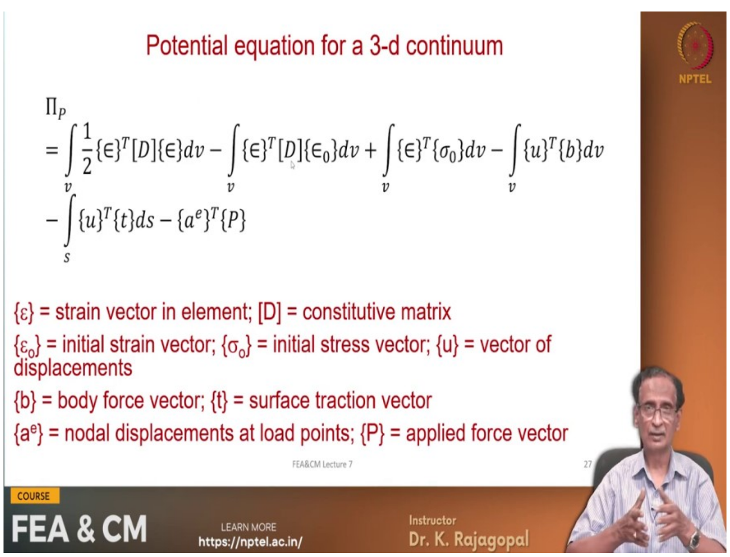


And let us do one more thing like let us look at the potential equation for a 3-dimensional continuum it is for a continuum. And a Continuum is characterized by different stresses Sigma x Sigma y Tau xy and then Sigma z Tau x z and Tau yz and so on. And let us say that our Continuum has got some strain Vector Epsilon and say if D is the constituted Matrix D times Epsilon is stress and that multiplied by strain is your work done.

Potential equation for a 3-d continuum

$$\Pi_P = \int_v \frac{1}{2}\{\epsilon\}^T[D]\{\epsilon\}dv - \int_v \{\epsilon\}^T[D]\{\epsilon_0\}dv + \int_v \{\epsilon\}^T\{\sigma_0\}dv - \int_v \{u\}^T\{b\}dv - \int_s \{u\}^T\{t\}ds - \{a^e\}^T\{P\}$$

And half because the average stress is half d Epsilon that multiplied by your Epsilon is your strain and integrated over the full volume. Now since we have a continuum we are integrating over the volume and then say if your Epsilon naught is the pre-existing strain and the current stress Epsilon transpose d times Epsilon naught is your work done because of your initial strains and then the stresses.

$\{\varepsilon\}$ = strain vector in element; $[D]$ = constitutive matrix
$\{\varepsilon_o\}$ = initial strain vector; $\{\sigma_o\}$ = initial stress vector; $\{u\}$ = vector of displacements
$\{b\}$ = body force vector; $\{t\}$ = surface traction vector
$\{a^e\}$ = nodal displacements at load points; $\{P\}$ = applied force vector

And there could be some initial stresses also and then say if your b is your body Force vector and u is your displacement Vector internal displacement vector you can write this loss of potential as u transpose b dv. And then you may have surface traction and where our u is the surface displacements T is the surface traction and this integrated over the surface area is whereas the others are integrated with volume.

And on top of this you may have some discrete loads concentrated loads at different points and the work done is A E transpose times P where a is your vector of displacements P is the vector of applied loads. So, the potential equation for a 3D Continuum is a bit more complicated. And so, this is our object like we will move on to the to the Continuum from the next class onwards.

So, what we have seen in this lecture is by following the Rayleigh-Ritz procedure we can obtain solution in a continuous form. We do not need to discretize our mesh into certain number of nodal points get our solution but by applying this we will get a continuous solution and we can assume some polynomial and that polynomial should be admissible at the minimum and the admissible polynomial will satisfy the boundary conditions.

And if it is going to be exact it should satisfy even the required boundary conditions and higher order derivatives like your bending moment and Shear Force and see these you should remember that we are not solving for bending moment and Shear first we are only solving for displacements. And from the displacements we are getting your bending moment and the shear Force has E i times dou Square w by dou x square and d i dou Cube w by dou dou x Cube.

And then the same procedure we can continue for 3D Continuum and this we will we will see later in a slightly different form not in the potential equation form but in the virtual work form. So, that is my last slide. So, I hope you understood the Rayleigh-Ritz procedure is actually it is a very beautiful procedure and it is a it is actually it is more mathematical but it has got some relevance to what we are going to do in finite element analysis because even here we assume some polynomial.

And depending on the type of polynomial that we assume our Solutions might be totally different. So, if you have any questions you please write an email to this to this address profkrg@gmail.com and then I will be able to respond back to you. So, thank you very much.