FEM and Constitutive Modelling in Geomechanics
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Lecture: 6
Virtual work & amp and principle of stationary potential energy

So, hello students let us continue from where we stopped in the previous lectures. In the previous lectures we had seen how to develop our equilibrium equations for bar and beam elements starting from the fundamentals by applying a unit deformations in different directions we got the equilibrium equations. And we have seen the local coordinate system and then global coordinate system and so on.

And now in this lecture let us go a little bit more mathematical let us look at the energy methods for deriving our equilibrium equations and also look at the principle of stationary potential energy that gives us a very good platform for doing a lot of mathematical analysis.

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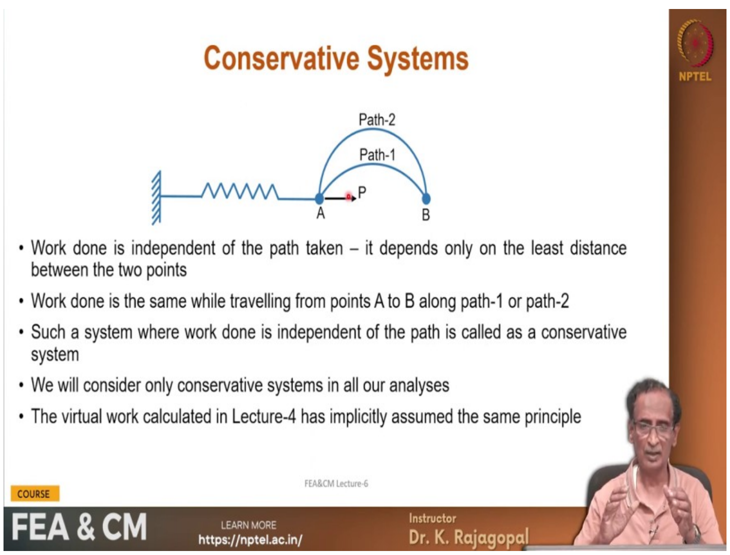
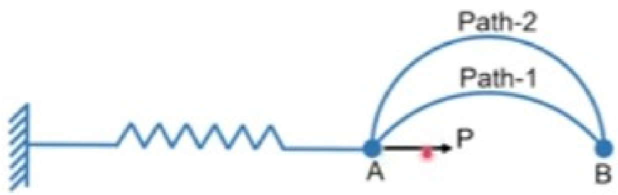


See in this process we are going to look at only the conservative systems.

Conservative Systems



A conservative system is a system in which the work done does not depend on the path. Say for example let us say we move the load from point A to point B and we could move it along the least the length direction directly AB a straight line path or along the curved path one or along curved path 2. Whatever path that you take the work done is exactly the same that is the load multiplied by the least distance.

And so, those systems are called as conservative systems and you might ask why we cannot consider the path dependence that is a different thing because that requires different mathematical formulation. But for now we will only look at this the work done is independent of the path taken and if you consider some other methods maybe it is possible. So, and already we have seen the virtual work calculation in lecture 4.

Where we had derived the transformation matrix for the bar element based on the and the virtual work principle like virtual work done whether you calculate in the local coordinate system or in the global coordinate system both are the same. And through that we have got K global is Lambda transpose K element times Lambda. And there also we had calculated the work done but we did not give any importance to the path taken and so on. But this is a the assumption that we make we consider only the conservative systems where the work done does not depend on the path.

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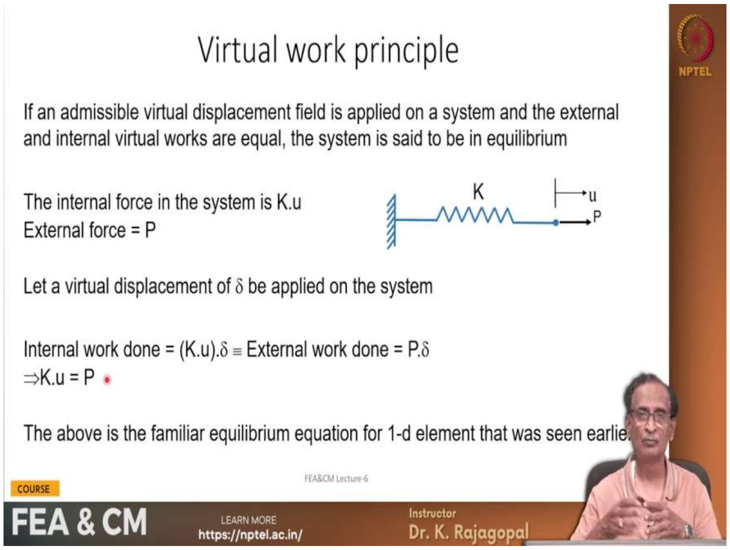
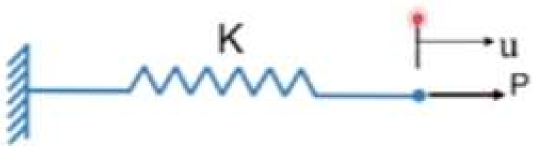


Let us look at one of the oldest principles the virtual work principle. So, if any admissible virtual displacement field is applied on a system and the external and internal virtual worker works are equal system is said to be in equilibrium.



It is let us say that we have a spring under a load of P and it is subjected to a to an elongation of u on top of this let us say we give some virtual displacement d.

And the internal force within the spring is K times u right and the external force is P and let us assume let us apply a virtual displacement of Delta on the system. And the internal work done is the force in the spring multiplied by d K u times Delta that should be exactly equal to the external work done by the applied force that is P times Delta.

$$\text{Internal work done} = (K.u).\delta \equiv \text{External work done} = P.\delta$$

So, if you cancel out Delta both from the left hand side and right hand side we are left with

$$K \,.\, u = P$$

K times u is P that is our familiar equation stiffness times displacement is equal to P now the load.

And so, this is what we have seen for one dimensional element and in fact even the spring is a one dimensional element.

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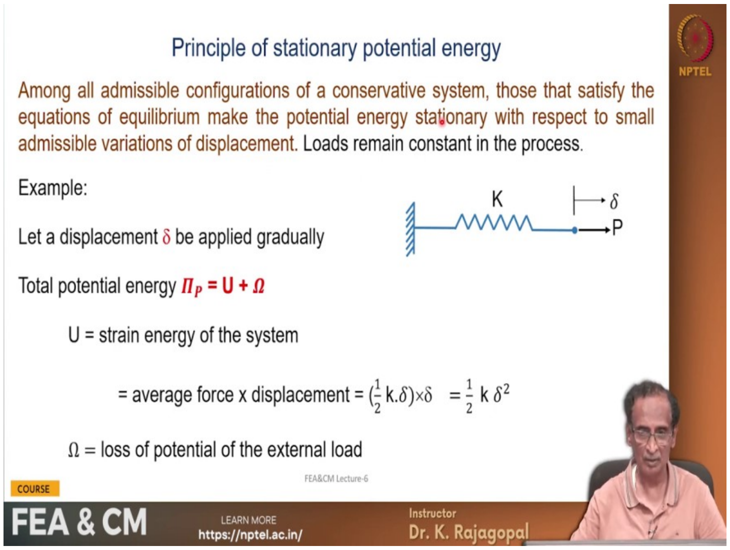

And let us look at the same principle but from a different point of view the principle of stationary potential energy. Actually it is it is very easy to imagine that when you have a system under equilibrium if you move a little bit the potential energy does not change. I will illustrate that a bit later but let us look at the statement of this principle. Say among all admissible configurations of a conservative system.

Those that satisfy the equations of equilibrium make the potential energy stationary with respect to small admissible variations of displacement. Say the potential energy is a constant when we apply some small deformations. Like for example you take it you take some object and keep it at some height if you move it a little bit like apply some deformation there is some change in the potential energy.

So, that means that this is not a stable thing but then say keep it on a Surface like this and move it a little bit to the left or the right and its potential energy is not changing with respect to my hand or to the surface on which it is resting. So, we can say that that is the most stable system. And so, this mathematically we can pose this problem so, that we can derive some useful relations.

So, let us take the same spring K on the gradual apply some displacement of Delta and let us say there is a load of P. And the total potential energy is pi is equal to u plus Omega where u is the strain energy of the system.

Total potential energy $\Pi_P = U + \Omega$

And that we can calculate is the average force because the Delta is starting from 0 to Delta during this application of this deformation. So, the average force is K Delta by 2 and that multiplied by Delta is your average strain energy.

U = strain energy of the system

$$= \text{average force x displacement} = (\tfrac{1}{2}\,\text{k}.\delta)\times\delta \quad = \tfrac{1}{2}\,\text{k}\,\delta^2$$

That is one half K Delta Square and then Omega is the loss of potential energy of the external load and it is actually I should have had this I will come to this later.

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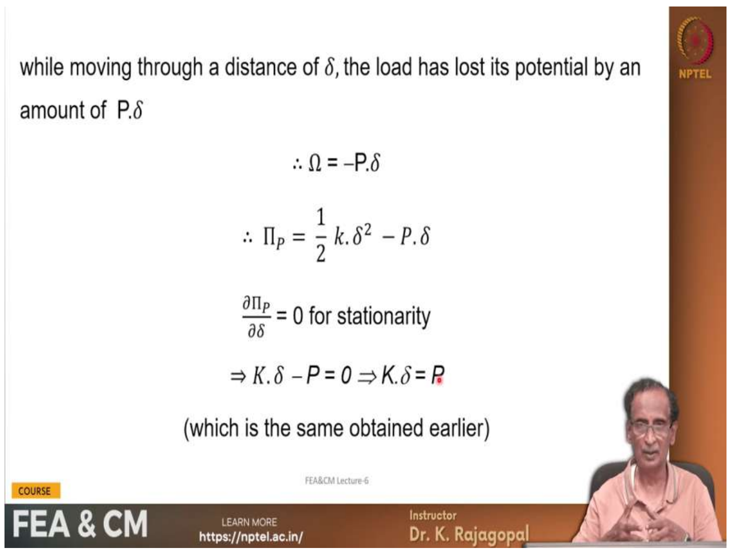

Let us say while moving through a distance of Delta the load has lost its potential by an amount of P Delta. So, we can say the Omega is the loss in potential as minus P times Delta because by that much it has lost the potential.

$$\therefore \Omega = -P.\delta$$

Like for example say if this object is at this height it has got so, much of potential energy but if I move it then it has lost that much potential energy to do the work.

So, our total potential is one half K Delta Square minus P Delta right and the principal states that any small disturbance the total potential with respect to Delta should be zero for the system to be in equilibrium.

$$\therefore \Pi_P = \frac{1}{2}\,k.\delta^2 \; - P.\delta$$

So, mathematically we can say that dou Pi by dou Delta is 0 for stationary.

$$\frac{\partial \Pi_P}{\partial \delta} = 0 \text{ for stationarity}$$

$$\Rightarrow K.\delta - P = 0 \Rightarrow K.\delta = P$$

And so, if you differentiate this we get K times Delta minus P is 0 or K times Delta is P where K is our stiffness Delta is the deformation that is equal to load P.

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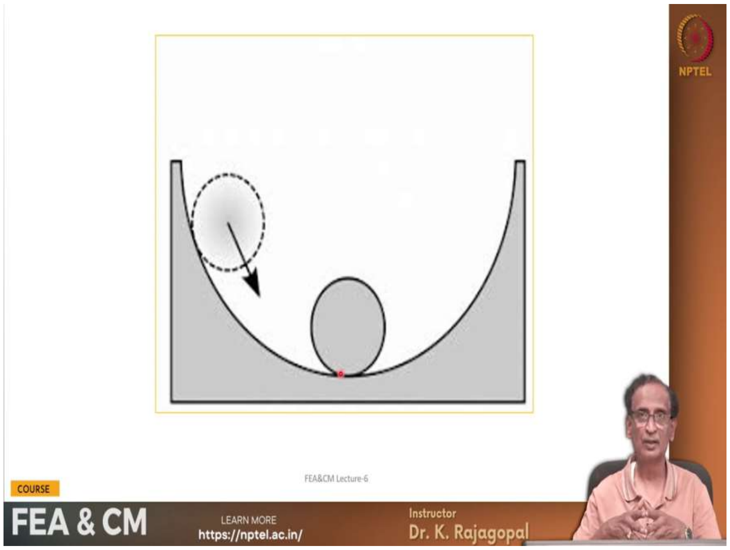
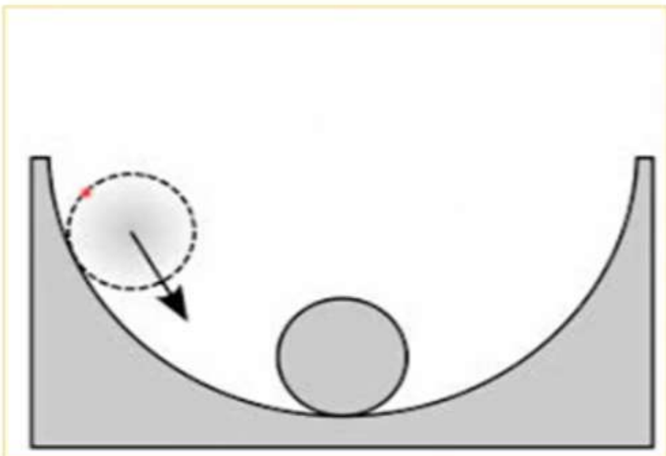


And so, actually this is what we obtained earlier and we can actually illustrate this with a simple example that we must have all of you must have studied in your first year mechanics course.



Let us say you have a marble in a round container and the most stable like if you drop a marble here it will just go on rolling and rolling, rolling until it comes to the bottom. And it is it is not stable here because it can roll or we can say that as you move it its potential energy is changing.

But imagine the same marble at the bottom of this bowl and just move it a little bit now to the left or to the right its energy is not changing its potential energy is not changing. So, we can say that that is the most stable position for the marble with respect to the to the bowl.

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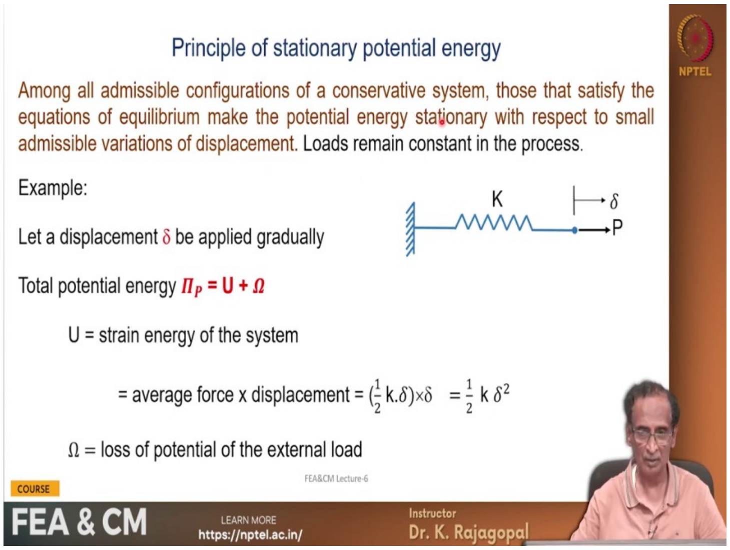

And that is what is meant by the statement when we say that among all the admissible deformation states those that satisfy the equations of equilibrium make the third the potential energy stationary. So, the potential energy if it is if it remains stationary then we can say that the system is in equilibrium our stationary.

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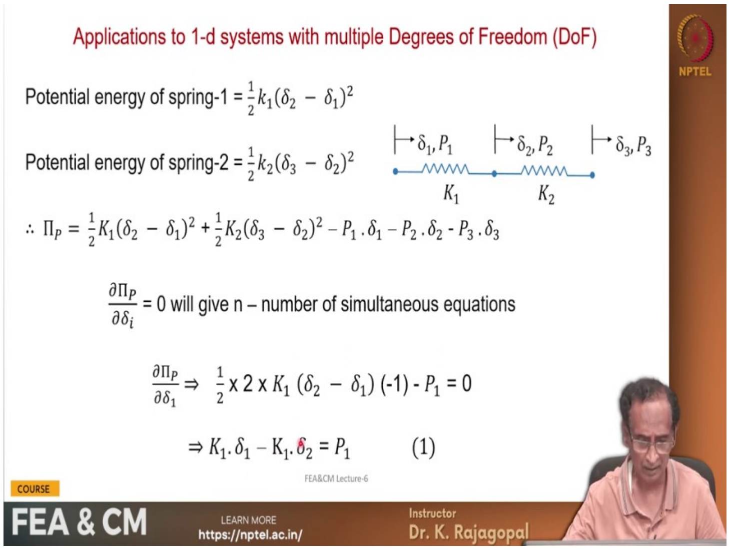
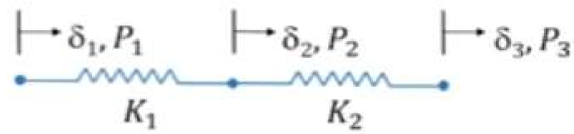


So, we can apply this for all our problems whatever we have come across and let us apply that to a multi degree of freedom system let us say we have 2 Springs K 1 and K 2 and 3 deformations Delta 1 Delta 2 and Delta 3 and the 3 external loads P 1, P 2 and P 3.



And we need to first estimate the total potential of the system in an internal energy of the system that is a the energy in Spring K 1 and energy in Spring K 2.

$$\text{Potential energy of spring-1} = \frac{1}{2}k_1(\delta_2 - \delta_1)^2$$

So, the energy in Spring K 1 is the one half K 1 Delta 2 minus Delta 1 whole Square and same thing the potential energy of spring 2 is one half K 2 Delta 3 minus Delta 2 whole Square and Delta 2 minus Delta 1 is the relative deformation only when you have a relative deformation the spring will get strained. Say if Delta 2 and Delta 1 are the same then there will not be any strain developed in the spring or the force.

$$\text{Potential energy of spring-2} = \frac{1}{2}k_2(\delta_3 - \delta_2)^2$$

So, we are more interested in the relative deformation and previously we said it is only Delta because one end is fixed. So, whatever deformation that you apply to the front end that is the relative deformation that is caused in the spring. So, our total potential pi is the potential energy in Spring 1 plus spring 2 minus the loss of potential of P 1 P 2 and P 3.

$$\therefore\ \Pi_P = \frac{1}{2}K_1(\delta_2 - \delta_1)^2 + \frac{1}{2}K_2(\delta_3 - \delta_2)^2 - P_1 . \delta_1 - P_2 . \delta_2 - P_3 . \delta_3$$

So, the total potential is one half K 1 Delta 2 minus Delta 1 whole square plus one half K 2 Delta 3 minus Delta 2 whole Square minus P 1 Delta 1 minus P 2 Delta 2 minus P 3 Delta 3.

And so, we can take this total potential and differentiate with respect to different degrees of freedom Delta 1, Delta 2 and Delta 3 are the different degrees of freedom.

$$\frac{\partial \Pi_P}{\partial \delta_i} = 0 \text{ will give n} - \text{number of simultaneous equations}$$

So, we take a differential of this pi and if it is 0 then we can say that our system is is a stationary or in equilibrium. And so, by differentiating with different degrees of freedom we get sufficient number of equations simultaneous equations for solving for our nodal unknowns displacements.

$$\frac{\partial \Pi_P}{\partial \delta_1} \Rightarrow \quad \frac{1}{2} \times 2 \times K_1 \ (\delta_2 \ - \ \delta_1) \ (-1) - P_1 = 0$$

So, dou Pi by dou Delta 1 will give you this and we get the first equation K 1 Delta 1 minus K 1 Delta 2 is P 1 right.

$$K_1.\delta_1 - K_1.\delta_2 = P_1$$

It is actually this is a similar to what we had seen earlier we applied unit deformations and then tried to find the corresponding forces. So, if I apply unit deformation at Delta 1 force will be developed at this node or at this degree of freedom and at this degree of freedom the force is not developed at this degree of freedom because it is not connected. So, that is what we see here Delta 3 is not in this equation.

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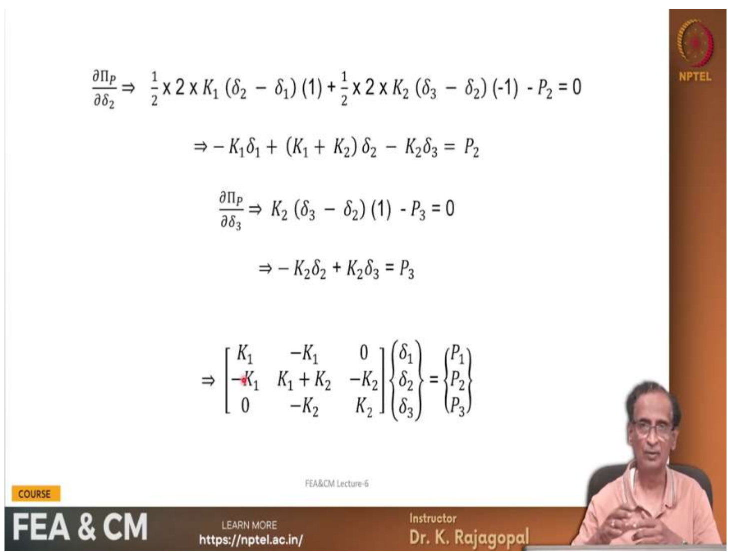

And then when we differentiate this with respect to Delta 2 we get an equation in terms of all the 3 degrees of freedom Delta 1 Delta 2 and Delta 3 that is because the Delta 2 is connected to both Delta 3 and the delta 1.

$$\frac{\partial \Pi_P}{\partial \delta_2} \Rightarrow \quad \frac{1}{2} \times 2 \times K_1 \ (\delta_2 \ - \ \delta_1) \ (1) + \frac{1}{2} \times 2 \times K_2 \ (\delta_3 \ - \ \delta_2) \ (-1) \ - P_2 = 0$$

$$\Rightarrow -K_1\delta_1 + (K_1 + K_2)\,\delta_2 - K_2\delta_3 = P_2$$

So, if you apply some deformation Delta 2 there will be a force developed in this at this degree of freedom and this degree of freedom and also at this same at its own degree of freedom.

$$\Rightarrow -K_1\delta_1 + (K_1 + K_2)\,\delta_2 - K_2\delta_3 = P_2$$

And so, that is minus K 1 Delta 1 plus K 1 plus K 2 Delta 2 minus K 2 Delta 3.

And similarly if you differentiate with respect to Delta 3 we get only in terms of Delta 2 and Delta 3 because Delta 3 is not connected to Delta 1 right.

$$\frac{\partial \Pi_P}{\partial \delta_3} \Rightarrow K_2\,(\delta_3 - \delta_2)\,(1)\ - P_3 = 0$$

$$\Rightarrow -K_2\delta_2 + K_2\delta_3 = P_3$$

So, if you express these 3 equations in the form of Matrix you will get like this K 1 minus K 1 0 minus K 1 K 1 plus K 2 minus K 2 0 minus K 2 K 2 Delta 1 Delta 2 Delta 3 that is equal to P 1 P 2 and P 3 right.

$$\Rightarrow \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1 + K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

And we see this assembled stiffness Matrix is symmetric because the upper diagonal terms are equal to the lower diagonal terms and then it is banded it is actually there is a zero here there is a zero here. So, we have a small band around which we have non-zero terms and the same equation we have seen earlier even when we derived these equilibrium equations by using the fundamental definition for the stiffness.

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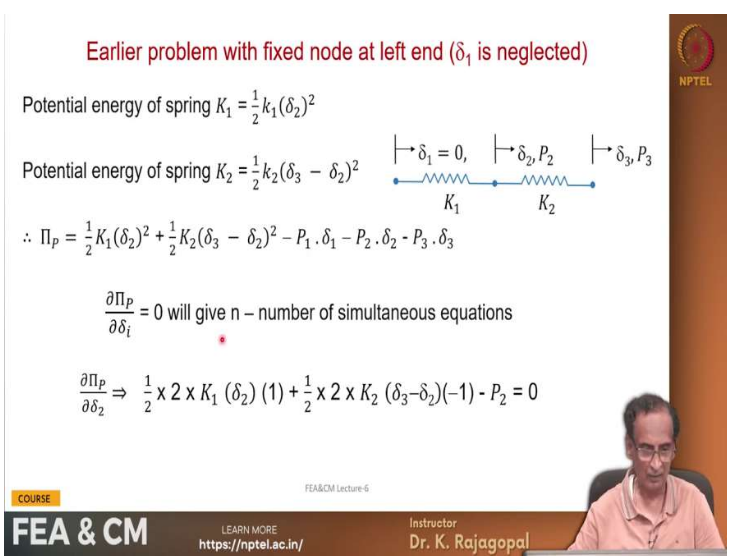

So, let us slightly modify this and let us apply some deformation different sorry the boundary condition.

$$\Rightarrow \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1 + K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

So, actually this equation it is not obvious but the the determinant of this equation this Matrix is zero. So, I will not be able to solve for the deformations as Delta is equal to K inverse P because your determinant is zero. And so, we need to apply some boundary condition.

And for that we have seen 2 methods one is the exact method and the other is the boundary spring method.

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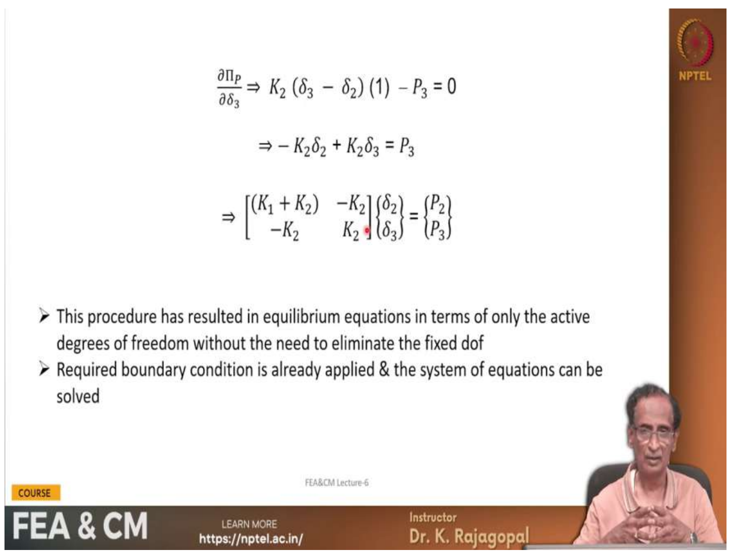
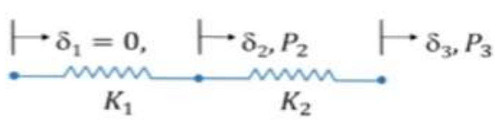


And let us say that let us not bother about applying the boundary conditions but right now itself let us say you want to apply some constraint that this particular node does not move Delta 1 is 0. Then the potential energy in K 1 will be one half K 1 Delta 2 square because Delta 2 itself is the is the relative deformation. So, if not let us say if it is not 0 but let us say some point one or something.



So, the relative deformation in the spring will be Delta 2 minus 0.1 and we are not using the Delta 1 because it is not a variable anymore it is a fixed value at 0 or 0.1 or something.

Then the energy in Spring 2 is one half K 2 Delta 3 minus Delta to whole Square. And so once again we can differentiate this with respect to different degrees of freedom.

$$\text{Potential energy of spring } K_1 = \frac{1}{2}k_1(\delta_2)^2$$

$$\text{Potential energy of spring } K_2 = \frac{1}{2}k_2(\delta_3 - \delta_2)^2$$

$$\therefore \Pi_P = \frac{1}{2}K_1(\delta_2)^2 + \frac{1}{2}K_2(\delta_3 - \delta_2)^2 - P_1 . \delta_1 - P_2 . \delta_2 - P_3 . \delta_3$$

And so, our Del dou Pi by dou Delta 2 Delta 1 is not and is not a variable anymore like actually P 1 Delta 1 you can set it to zero because it is it is known quantity and so our dou Pi by dou Delta 2.

$$\frac{\partial \Pi_P}{\partial \delta_i} = 0 \text{ will give n – number of simultaneous equations}$$

$$\frac{\partial \Pi_P}{\partial \delta_2} \Rightarrow \frac{1}{2} \times 2 \times K_1 \; (\delta_2) \; (1) + \frac{1}{2} \times 2 \times K_2 \; (\delta_3 - \delta_2)(-1) - P_2 = 0$$

And then dou Pi by dou Delta 3 will give us some other equation and now we get only a 2 by 2 equation in terms of Delta 2 and Delta 3 and P 2 and P 3.

$$\frac{\partial \Pi_P}{\partial \delta_3} \Rightarrow K_2 \; (\delta_3 - \delta_2) \; (1) - P_3 = 0$$

$$\Rightarrow -K_2\delta_2 + K_2\delta_3 = P_3$$

$$\Rightarrow \begin{bmatrix} (K_1 + K_2) & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{Bmatrix} P_2 \\ P_3 \end{Bmatrix}$$

And this Matrix has got an inverse because its determinant is not 0. And so, directly we were able to get our equilibrium equations without needing to modify the equations for a fixed degree of freedom.

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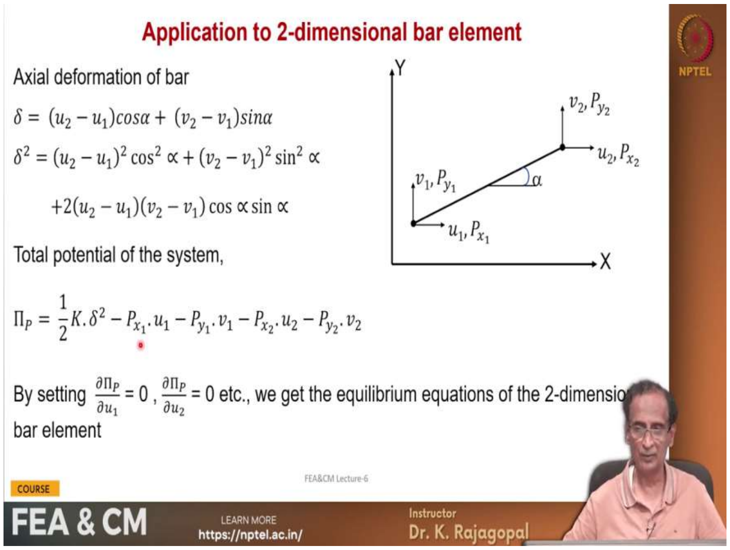
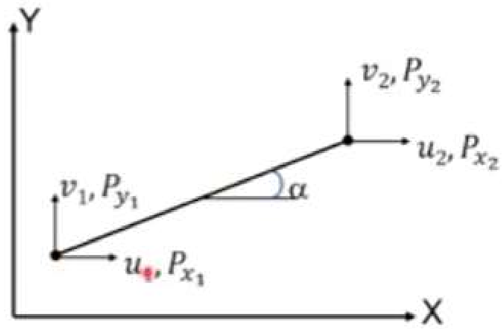


And now let us apply this principle to 2 dimensional bar element.



Let us consider a bar element with the 2 notes node 1 and node 2 and the deformations are u 1 v 1 u 2 v 2 and then the forces are P x 1 P y 1 P x 2 P y 2 and the axial deformation is u 2 minus u 1 times cosine Alpha plus v 2 minus v 1 sine Alpha and Delta square is this whole thing and our potential equation for the system Pi can be one half K Delta Square minus P x 1 u 1 P y 1 V 1 minus P x 2 u 2 minus P by 2 V 2 right.

Axial deformation of bar

$$\delta = (u_2 - u_1)cos\alpha + (v_2 - v_1)sin\alpha$$

$$\delta^2 = (u_2 - u_1)^2 \cos^2 \propto + (v_2 - v_1)^2 \sin^2 \propto$$

$$+2(u_2 - u_1)(v_2 - v_1) \cos \propto \sin \propto$$

Total potential of the system,

$$\Pi_P = \frac{1}{2} K.\delta^2 - P_{x_1}.u_1 - P_{y_1}.v_1 - P_{x_2}.u_2 - P_{y_2}.v_2$$

And so, with the with respect to different degrees of freedom u 1 v 1 u 2 v 2 we can differentiate and set it to zero and we will get our equilibrium equations.

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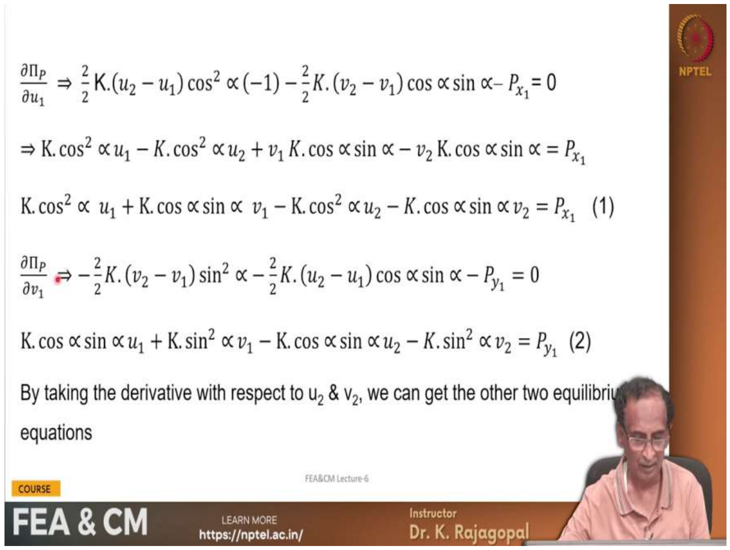

So, our dou Pi by dou u 1 will give you this equation is actually this and then if you sum if we simplify it we will get like this.

$$\frac{\partial \Pi_P}{\partial u_1} \Rightarrow \frac{2}{2} K.(u_2 - u_1)\cos^2 \propto (-1) - \frac{2}{2} K.(v_2 - v_1)\cos \propto \sin \propto - P_{x_1} = 0$$

$$\Rightarrow K.\cos^2 \propto u_1 - K.\cos^2 \propto u_2 + v_1 K.\cos \propto \sin \propto - v_2 K.\cos \propto \sin \propto = P_{x_1}$$

$$K.\cos^2 \propto u_1 + K.\cos \propto \sin \propto v_1 - K.\cos^2 \propto u_2 - K.\cos \propto \sin \propto v_2 = P_{x_1} \quad (1)$$

$$\frac{\partial \Pi_P}{\partial v_1} \Rightarrow -\frac{2}{2} K.(v_2 - v_1)\sin^2 \propto - \frac{2}{2} K.(u_2 - u_1)\cos \propto \sin \propto - P_{y_1} = 0$$

$$K.\cos \propto \sin \propto u_1 + K.\sin^2 \propto v_1 - K.\cos \propto \sin \propto u_2 - K.\sin^2 \propto v_2 = P_{y_1} \quad (2)$$

And then similarly dou Pi by dou v 1 will give you this equation.

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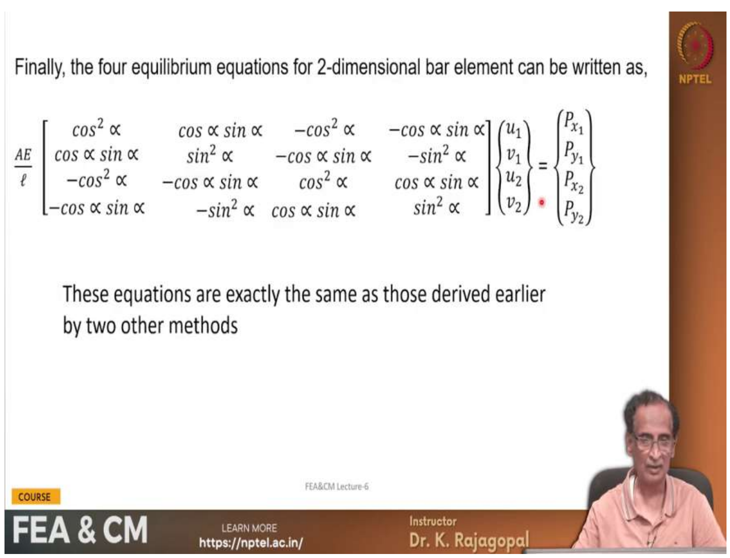

And by taking derivatives with respect to u 2 and v 2 we get 2 more equations.

And then once you assemble all the equilibrium equations you will get an equation a matrix that is very similar to what we had seen earlier.

$$\frac{AE}{\ell}\begin{bmatrix} \cos^2\propto & \cos\propto\sin\propto & -\cos^2\propto & -\cos\propto\sin\propto \\ \cos\propto\sin\propto & \sin^2\propto & -\cos\propto\sin\propto & -\sin^2\propto \\ -\cos^2\propto & -\cos\propto\sin\propto & \cos^2\propto & \cos\propto\sin\propto \\ -\cos\propto\sin\propto & -\sin^2\propto & \cos\propto\sin\propto & \sin^2\propto \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} P_{x_1} \\ P_{y_1} \\ P_{x_2} \\ P_{y_2} \end{Bmatrix}$$

By doing the Lambda transpose K Lambda we got this or by directly applying the degree of the unit deformations in the global coordinates we got similar Matrix. So, this K multiplied by u is equal to 3.

And we can actually it is a it is a general procedure that we can apply for any problem but only thing is if we do like this we have to do hand calculations we cannot use any of our Matrix methods.

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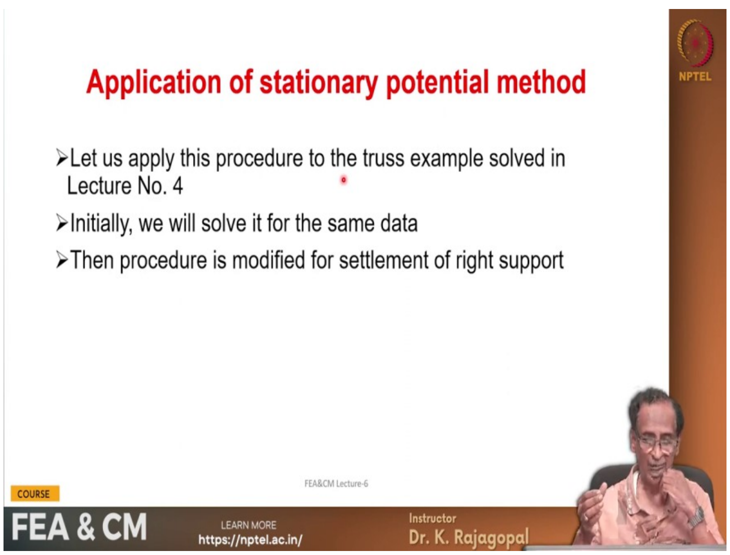

But anyway let us apply this for some problems so, that we can we can do the we can go through the hand calculations and understand it better. And let us apply this stationary potential energy method to the truss problem that we had solved in lecture number four and we will see 2 cases the first one is for the same data that was given and the second one is along with some settlement on the right side right hand side support.

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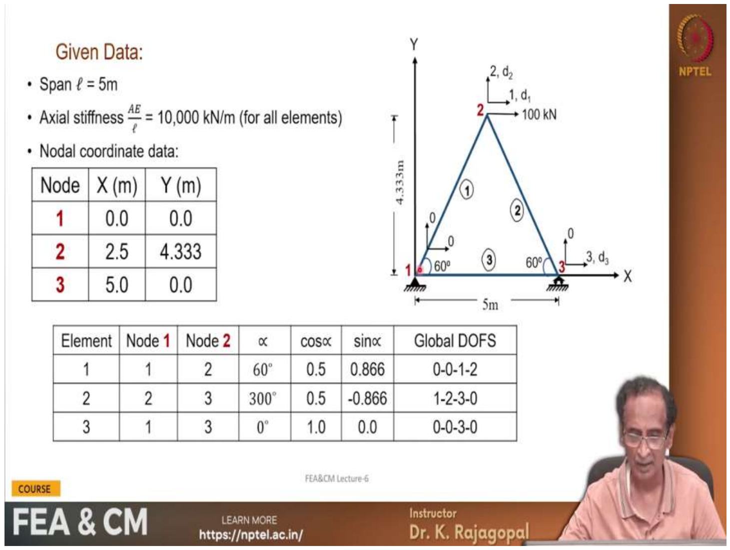

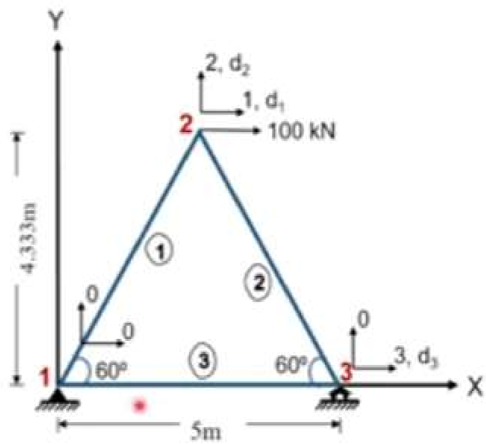

This was the structure that was given it has a 3 nodes node one node 2 node 3 and the 3 elements one 2 and 3 and element one is connected between nodes one and 2 and element 2 is connected between nodes 2 and 3 and element 3 is connected between nodes 1 and 3. And the direction of the element is from node 1 to node 2.

Element	Node 1	Node 2	∝	cos∝	sin∝	Global DOFS
1	1	2	60°	0.5	0.866	0-0-1-2
2	2	3	300°	0.5	-0.866	1-2-3-0
3	1	3	0°	1.0	0.0	0-0-3-0

So, we get this Alpha and these are the coordinates node 1 is at origin zero zero node 2 is at 2.5 and 4.333 x is equal to 5 and y is 0.

And for these are 3 elements we have 3 different Alphas depending on their orientation and then we have this cosine Alpha and sine Alpha. And unlike in the previous case let us not number the fixed degrees of freedom let us assign 0 to the to the fixed degrees of freedom at node 1 both x and y direction displacements are constrained so 0 0. At node 2 we have the degrees of freedom 1 and 2 let us call them as d 1 and d 2 node 3 the degree of freedom along the x axis is free; that is active degree of freedom d 3.

And then let us say that along y axis it is fixed zero because it is on roller and so, the connected degrees of freedom or like this 1 2 1 2 3 0 0 3 0 and so on.

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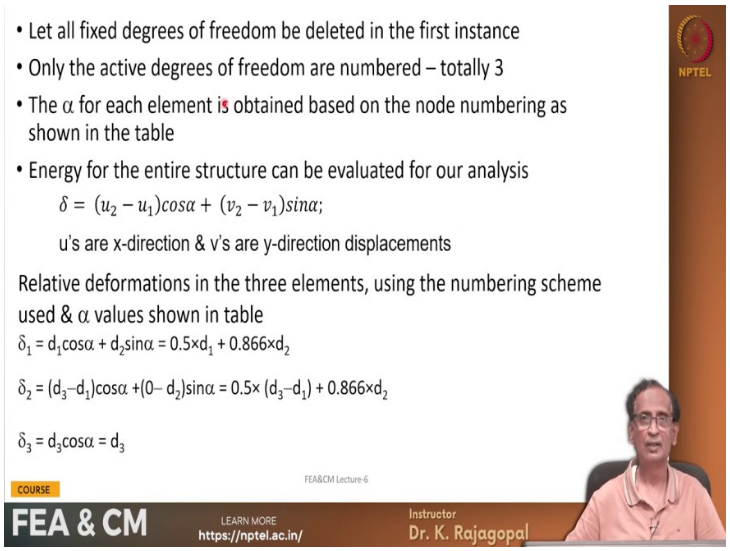

And we have deleted all the fixed degrees of freedom and the numbered only the active degrees of freedom those are free to deform either in the x-axis or along the y-axis. So, we have totally 3 active degrees of freedom and the alpha for each element is obtained based on the node numbering scheme that we have adopted.

Energy for the entire structure can be evaluated for our analysis

$$\delta = (u_2 - u_1)cos\alpha + (v_2 - v_1)sin\alpha;$$

u's are x-direction & v's are y-direction displacements

And energy for the entire structure can be evaluated in terms of our Delta axial deformation u 2 minus u 1 cosine Alpha plus v 2 minus v 1 sine Alpha this is the generic equation.

And the particular for each of these elements we can get your relative deformation Delta. And so, for element 1 it is connected between node 2 and the node 1 and node 2 it is it has 3 degrees of freedom d 1 and d 2 whereas node 1 is fixed server Delta 1 is d 1 minus 0 times cosine Alpha that is x direction displacement is d 1 times cosine alpha y direction displacement v 2 is a is d 2 times sine Alpha and cosine Alpha is a 0.5 and the sine Alpha is 0.866 for element 1.

$$\delta_1 = d_1\cos\alpha + d_2\sin\alpha = 0.5\times d_1 + 0.866\times d_2$$

$$\delta_2 = (d_3 - d_1)\cos\alpha + (0 - d_2)\sin\alpha = 0.5\times (d_3 - d_1) + 0.866\times d_2$$

$$\delta_3 = d_3\cos\alpha = d_3$$

So, Delta 1 is this and our Delta 2 is actually connected between node 2 and the node 3. and the node one is a node 2 and node 2 is a 3 and at node 3 we have d 3 along x axis and along y axis it is fixed zero server Delta 2 the relative deformation in element 2 is d 3 minus d 1 the times cosine Alpha that is the relative x direction displacement plus 0 minus d 2 sine Alpha that is the 0 is the displacement at node 3 in the y direction data is the y direction displacement at node 2 sine Alpha.

And it just so, happens that sine Alpha is minus 0.866 for this element 2. So, this is the the relative displacement in element 2 and Delta 3 is a d 3 minus 0 because the left hand side is fixed. So, d 3 times cosine Alpha that is simply d 3 because Alpha is 0 for this horizontal element.

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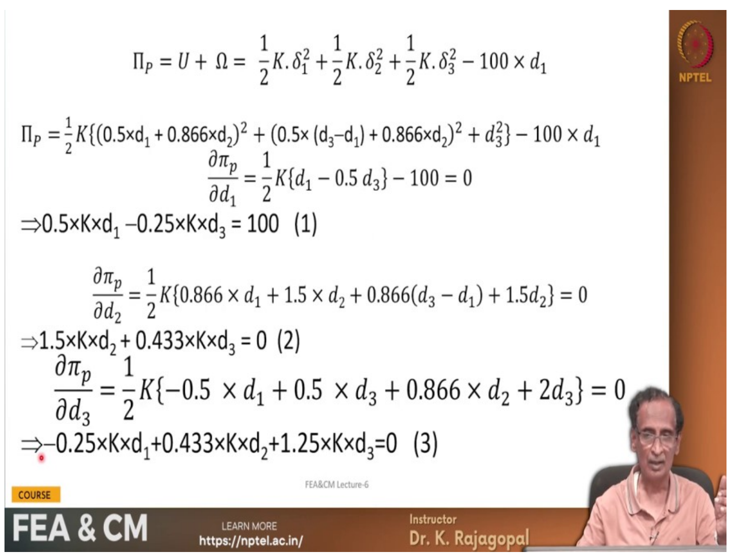

And so, our total potential is one half K Delta 1 square plus one half K Delta 2 squared plus one half K Delta 3 Square minus a load of 100 applied at in x direction at node 2 the deformation is d 1. So, this is your total potential.

$$\Pi_P = U + \Omega = \frac{1}{2}K.\delta_1^2 + \frac{1}{2}K.\delta_2^2 + \frac{1}{2}K.\delta_3^2 - 100\times d_1$$

The total potential is this and now we can take derivatives with respect to d 1 d 2 and d 3. So, if you take a difference if you differentiate y with respect to d 1 you get this 0.5 K d 1 minus 0.25 KD 3 is 0 and it is a hundred.

$$\Pi_P = \frac{1}{2}K\{(0.5\times d_1 + 0.866\times d_2)^2 + (0.5\times (d_3 - d_1) + 0.866\times d_2)^2 + d_3^2\} - 100\times d_1$$

$$\frac{\partial \pi_p}{\partial d_1} = \frac{1}{2}K\{d_1 - 0.5\,d_3\} - 100 = 0$$

$$\Rightarrow 0.5\times K\times d_1 - 0.25\times K\times d_3 = 100 \quad (1)$$

And similarly differentiation with respect to d 2 we have this equation and then differentiation with respect to d 3 we will get one more equation.

$$\frac{\partial \pi_p}{\partial d_2} = \frac{1}{2}K\{0.866\times d_1 + 1.5\times d_2 + 0.866(d_3 - d_1) + 1.5 d_2\} = 0$$

$$\Rightarrow 1.5\times K\times d_2 + 0.433\times K\times d_3 = 0 \quad (2)$$

$$\frac{\partial \pi_p}{\partial d_3} = \frac{1}{2}K\{-0.5\times d_1 + 0.5\times d_3 + 0.866\times d_2 + 2d_3\} = 0$$

$$\Rightarrow -0.25\times K\times d_1 + 0.433\times K\times d_2 + 1.25\times K\times d_3 = 0 \quad (3)$$

So, these are the 3 equilibrium equations that we have and we can assemble them or we can just put them in a matrix form like this.

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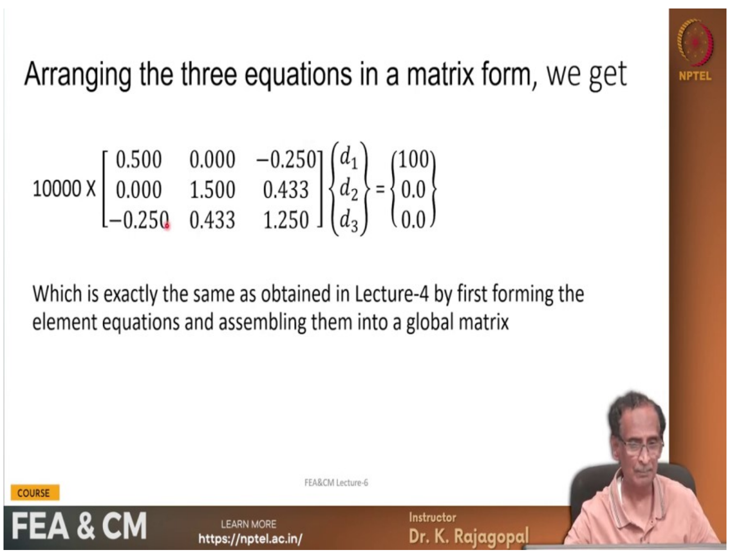

$$10000 \text{ X} \begin{bmatrix} 0.500 & 0.000 & -0.250 \\ 0.000 & 1.500 & 0.433 \\ -0.250 & 0.433 & 1.250 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 0.0 \\ 0.0 \end{Bmatrix}$$

So, K is a 10000.50 minus 0.25 d 1 d 2 d 3 is 100 and then that is actually see this is 0.5 times 10000 is 5000 minus 2500 now because 0.25 and the K is a 10000 and so on. So, it is and once you get this the solution is a is very simple. And in fact if you see this is the same equation that we got earlier in lecture number four by deleting rows and Columns of 1 2 and 6. Because these 3 degrees of freedom are fixed and by deleting those corresponding rows and columns we were left with this. And we can invert this Matrix and get our d 1 d 2 d 3 that was what was done in lecture 4.

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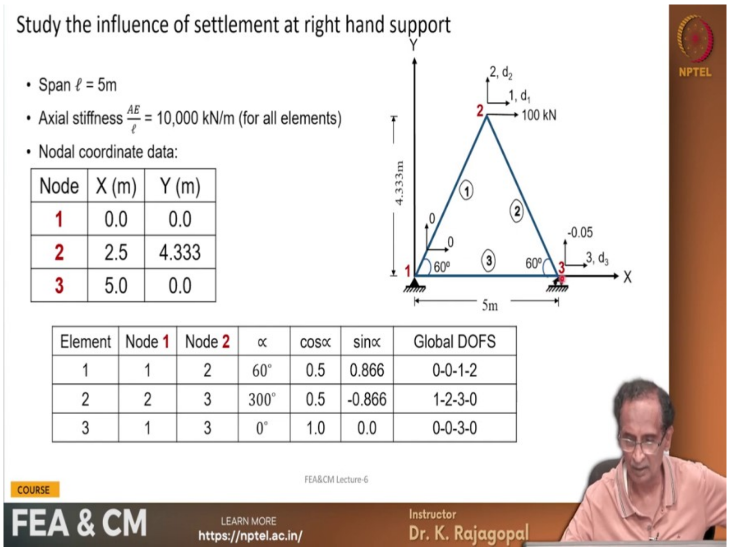

And now let us consider the influence of settlement at right hand support see the supported 3 it is calculated that it will settle down by 50 millimeters this deformation in vertical direction is minus .05 is the same data.

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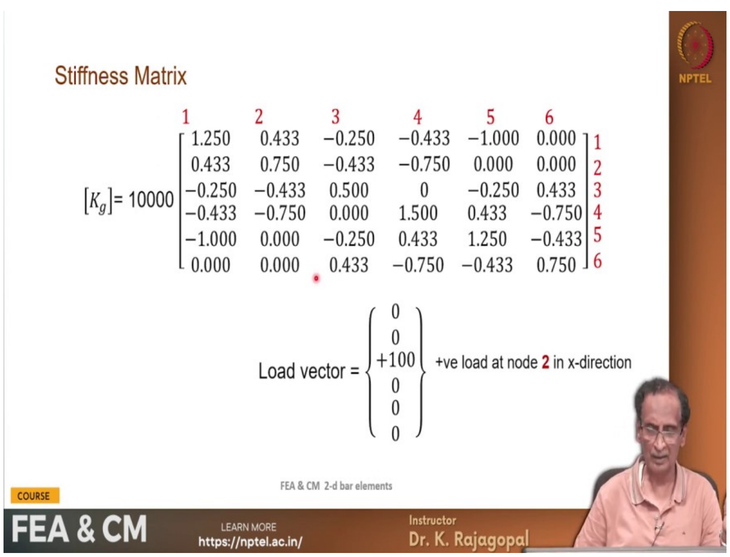

And so, this was the assembled stiffness Matrix and then the load Vector for that structure. We had done this by hand calculations and this is what we have a six by six Matrix.

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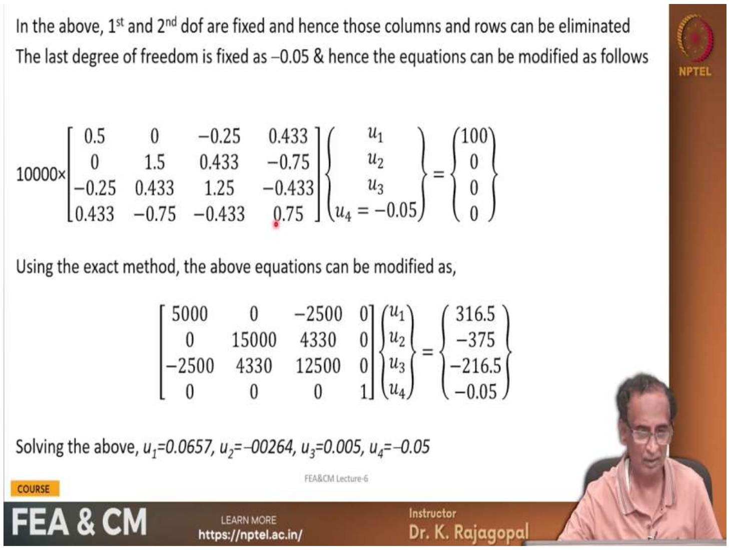

Then out of this our the first and second degrees of freedom are fixed. So, we can delete the first row second row first column second column and keep the keep the last one because the sixth row because we need that to apply a displacement of minus 0.05. So, in u 4 that is in the degree of freedom a y direction degree of freedom at node 3 we want to apply a displacement of minus 0.05.

$$10000\times\begin{bmatrix} 0.5 & 0 & -0.25 & 0.433 \\ 0 & 1.5 & 0.433 & -0.75 \\ -0.25 & 0.433 & 1.25 & -0.433 \\ 0.433 & -0.75 & -0.433 & 0.75 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 = -0.05 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

So, here we know this stiffness coefficient and that stiffness coefficient multiplied by this u 4 is a known quantity. So, we can send it to the right hand side this 10000 times 0.433 times minus 0.05 and you send it to the right hand side you will get and since you already have a load of 100 you add it. Then in the second row also we have this quantity and minus and minus is positive on this left hand side when it goes to the right side we get a minus minus 375.

Using the exact method, the above equations can be modified as,

$$\begin{bmatrix} 5000 & 0 & -2500 & 0 \\ 0 & 15000 & 4330 & 0 \\ -2500 & 4330 & 12500 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 316.5 \\ -375 \\ -216.5 \\ -0.05 \end{Bmatrix}$$

And then the third row also the same thing minus 216.5 and then the fourth row is actually it is a trivial row trivial equation because u 4 is a minus point zero for nothing else. And by solving this we get the u 1 of 0.0657 and u 2 is minus 0.0264 and u 3 is 0.005 and u 4 is minus 0.05 that is what was applied.

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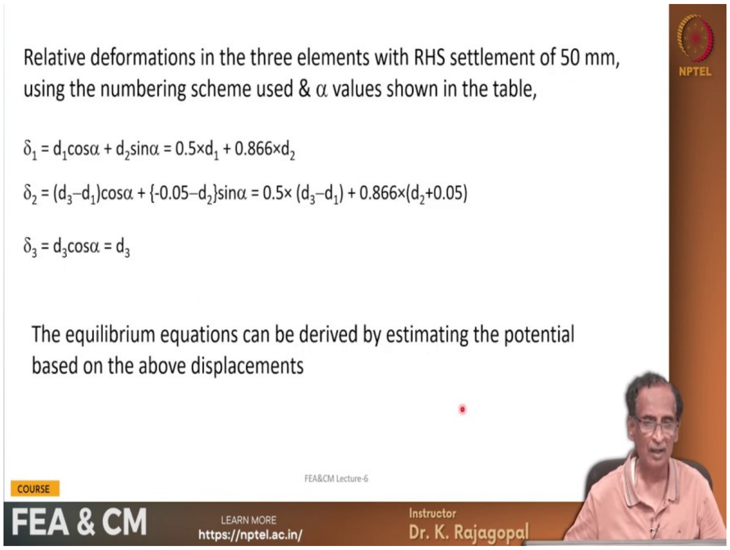

And we can do the same problem by defining our relative displacements and we are saying that on the right hand side the support settles by an amount of 0.05 meters or 50 millimeters. So, Delta 1 is for the element one connected between node 1 and node 2.

$$\delta_1 = d_1\cos\alpha + d_2\sin\alpha = 0.5\times d_1 + 0.866\times d_2$$

$$\delta_2 = (d_3 - d_1)\cos\alpha + \{-0.05 - d_2\}\sin\alpha = 0.5\times(d_3 - d_1) + 0.866\times(d_2 + 0.05)$$

$$\delta_3 = d_3\cos\alpha = d_3$$

And there is no change in this. So, it is the same and Delta 2 d 3 minus d 1 is the relative deformation in the x axis along the x axis multiplied by cosine Alpha.

And then at node 3 the displacement is minus 0.05 and at node 2 it is d 2. So, this is our relative deformation multiplied by sine Alpha and sine Alpha is minus 0.866. So, if you substitute that you get this. And in fact the relative deformation is d 2 plus 0.05 because we are taking the unknown displacements initial in the positive direction d 2 is in the along the positive y direction whereas the 0.05 is in the negative y direction. So, the relative deformation is d 2 plus 0.05.

And Delta 3 the relative deformation in the in the elementary that is on the horizontal element is just simply d 3 cosine Alpha because it is coinciding with the x axis and there is no effect of settlement on element 3 and element 1 because they are not directly connected. And element 3 it is connected but then it is a horizontal element whereas the settlement is in the in the vertical direction.

So, now once you have this Delta 1 Delta 2 Delta 3 we can form our total potential energy equation and then take differentiation with respect to Delta 1 Delta 2 and Delta 3 and get our equilibrium equations.

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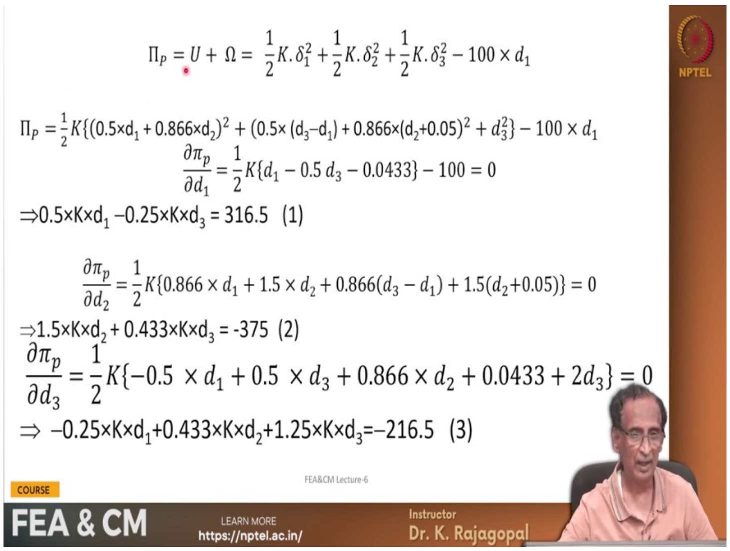

So, let us see that Phi is a u plus Omega and Delta 1 Delta 2 Delta 3 we have we have already seen that and actually I think there is a small mistake I think I made a mistake here yeah now it is got I forgot to do the square for the entire thing plus d 3 Square this is correct. And now we can differentiate this with respect to d 1 d 2 and d 3 and you get a messy equation but then you have to do it by pen and paper method so, that you can get it correctly.

And I have done that I have got 3 equations equation 1, 2 and 3 by differentiation and then by by simplification we get these 3 equations and then in a matrix form.

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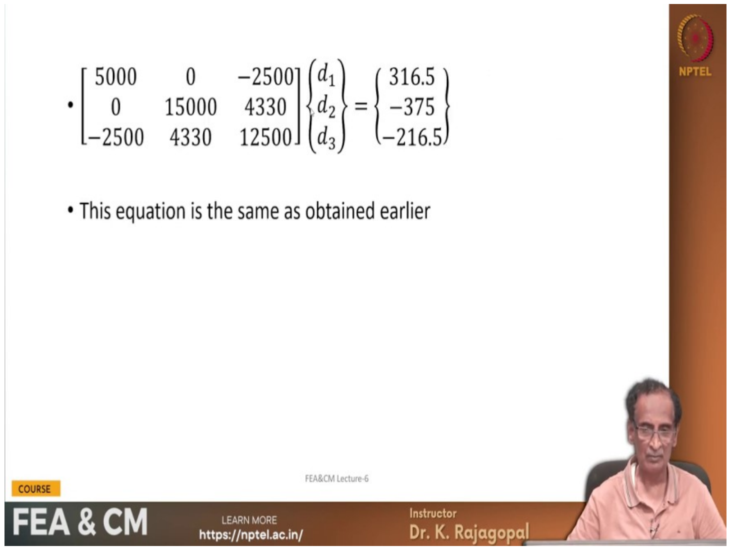

These are the equations d 1 d 2 d 3 and this is exactly same as what we had earlier. So, it is uh. So, we see that even by this stationary potential energy method we could apply a known displacement and then get our governing equations and if you solve it if you solve this system you will get the same displacements that we got earlier.

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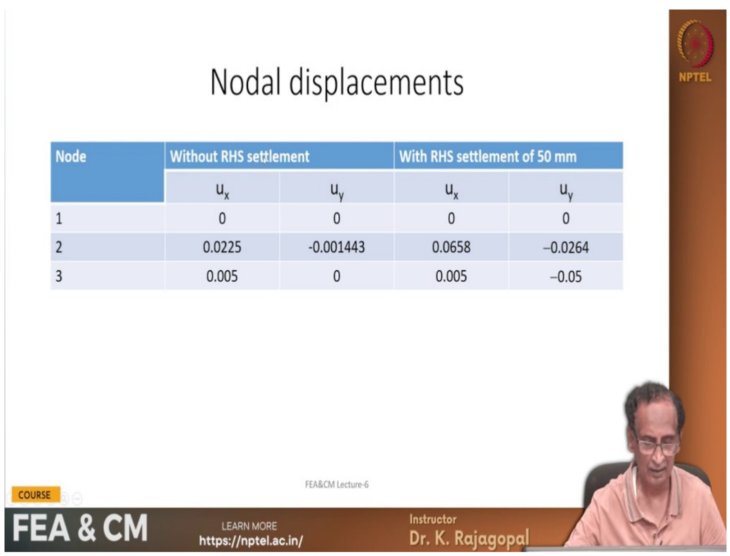

So, here is the tablet and the nodal displacements. This is x direction displacement and y direction displacement this is without right hand side support settlement whereas this is the settlement of 50 mm.

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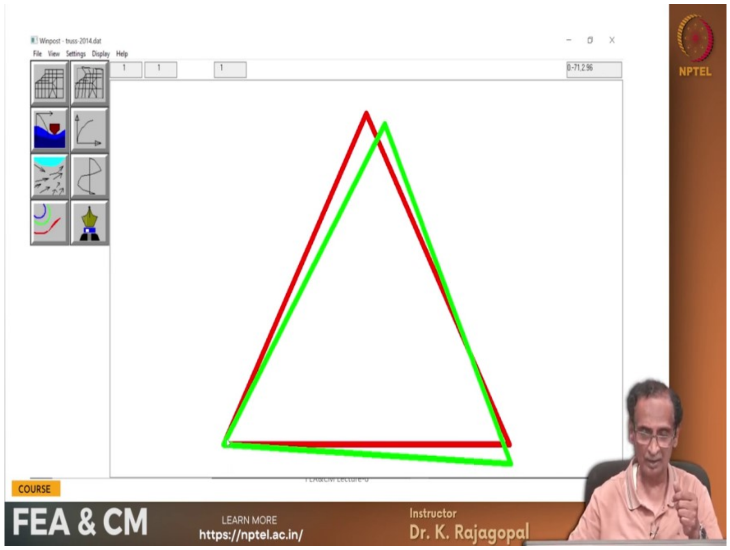

And if you graphically look at it; this red lines now the original mesh they represent the original motion then this green lines are they are the deformed mesh. And you see this actually it has undergone the structure as undergone a rigid body rotation that is because our node one is a fixed and hinged. So, you can actually rotate the whole thing and the right hand side is settled by 50 millimeter. So, the entire structure is rotated.

It is actually it has undergone a rigid body deformation. So, that means that the effect of settlements is not there. Then these elements will not develop any additional force because of the settlement of the right hand side support. So, whatever forces that we calculated earlier that is a tension 100 compression 100 tension 50 they will still be the same. That I am not doing here but if you use these displacements and calculate the element forces.

So, you will get the same forces that you got earlier. So, I think that is the last slide. So, I think that is the end of this lecture and if you have any questions please send an email to profkrg@gmail.com. And what I suggest is please take a pen and paper and go through this entire this differentiation because it is a bit messy and when you see it on the screen it might seem bewildering it might seem complicated.

But then it is so, simple it is a simple differentiation with respect to different quantities d 1 d 2 d 3. We do it by hand on pen and paper then you will easily understand. So, that is the thing and then we will continue further with other mathematical methods that are applicable for our equilibrium equations. So, thank you very much.