FEM and Constitutive Modelling in Geomechanics

Prof. K. Rajagopal
Department of Civil Engineering
Indian Institute of Technology - Madras

Lecture: 5
Development of Equilibrium Equations for Beam Elements

Hello students and let us continue from the previous lecture where we looked at a 2-dimensional bar element. We had developed its equilibrium equations then we have also seen how to apply the constraints on the system to prevent the rigid body deformations. And let us continue for higher order elements in today's class.

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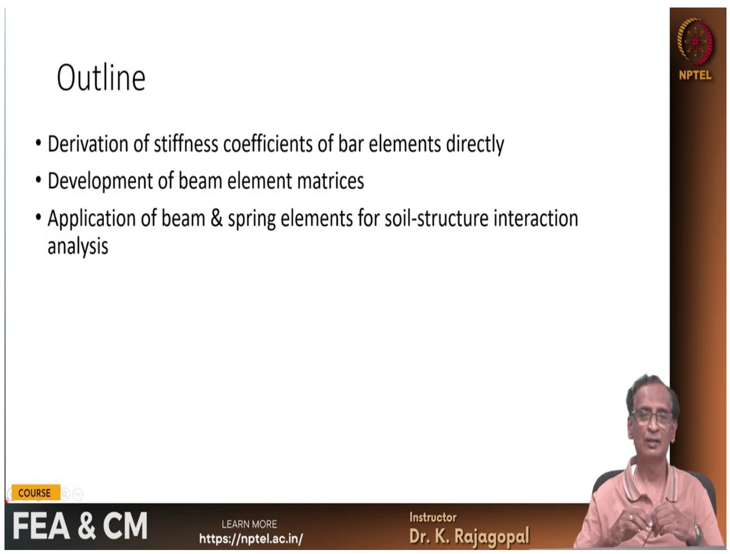

And before that let us do one more thing with the bar elements let us derive the stiffness coefficients directly by applying the unit displacements in the global directions directly. As previously we had looked at the local coordinate system where we defined 2 displacements one is axial and the other is shear.

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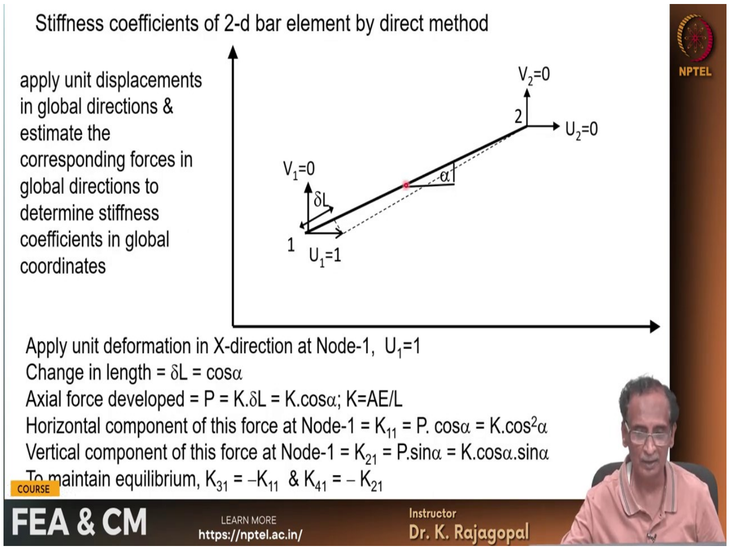
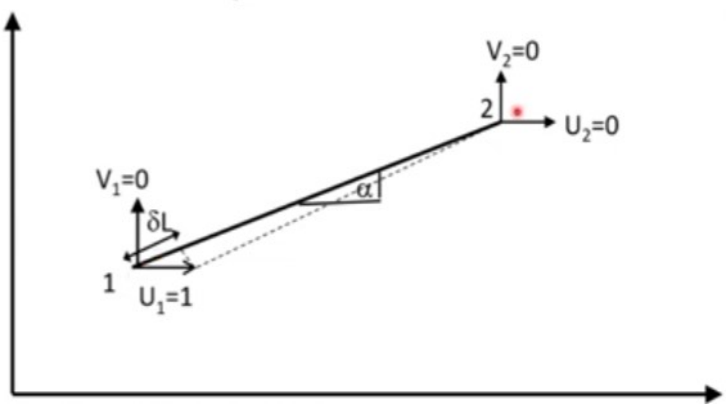


And let us look at another method for deriving the the stiffness coefficients let us take the same 2 node bar element and there are 4 displacements ux1 u y 1 u x 2 u y y 2 this is Node 1 and that is node 2 and let us say that there is an alpha here. And let us now apply the displacements directly in the global directions and then measure the forces developed in the respective directions. So, in that context let us apply unit displacement at node 1 in the x direction that is u 1 is 1 while all the other displacements are constrained.



So, we are setting v 1 to 0 v 2 0 u 2 to 0 and if you apply a unit deformation the x direction the change in the length Delta L is cosine Alpha and then the axial force developed is the stiffness multiplied by the change in the length K times Delta l and that is K times cosine Alpha and this K is AE by L and that is the axial force developed. And now we can resolve this this axial force into x direction and y direction.

Apply unit deformation in X-direction at Node-1, U_1=1
Change in length = $\delta L = \cos\alpha$
Axial force developed = $P = K.\delta L = K.\cos\alpha$; $K=AE/L$
Horizontal component of this force at Node-1 = $K_{11} = P.\cos\alpha = K.\cos^2\alpha$
Vertical component of this force at Node-1 = $K_{21} = P.\sin\alpha = K.\cos\alpha.\sin\alpha$
To maintain equilibrium, $K_{31} = -K_{11}$ & $K_{41} = -K_{21}$

So, the force in the x direction that is the force at Node 1 in the x direction is by definition K 11 because we applied a unit deformation in the degree of freedom 1 and then we are measuring the force in the same direction and that K times cosine Alpha result to x direction is K times cosine Square Alpha right. And then if you resolve this force in the vertical direction these axial force multiplied by sine Alpha and that will be K 21 because the force developed in direction 2 because of a unit displacement in direction one right.

So, P times sine Alpha that is K times cosine Alpha sine Alpha and then the force is developed at node 2 degrees of freedom 3 and 4 should be equal and opposite. So, we say that K 31 that is the force developed in degree of freedom 3 because of a unit displacement and degree of freedom 1 is - K 11 similarly K 41 is - K 21 right.

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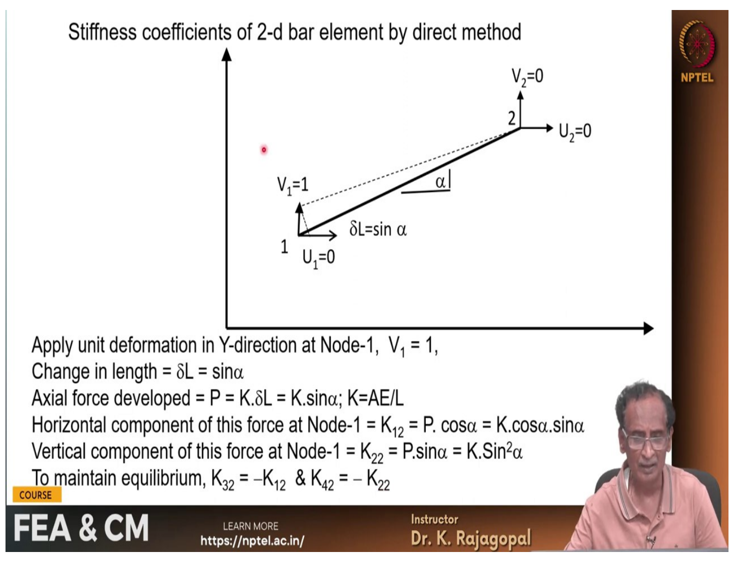
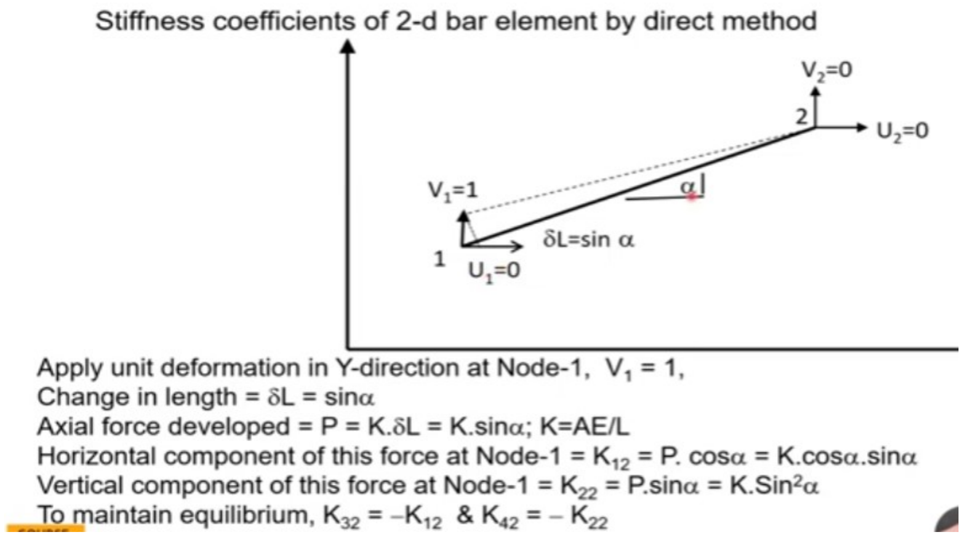


And similarly we can apply a unit displacement in the y direction at node 1 that is v 1 is 1 while we constrain all other displacements u 1 to 0 u 2 and v 2 to 0. And this unit deformation will change at the length of the element by sine Alpha right. So, the force developed the axial force developed is K times sine Alpha and if we resolve the force in the x direction is K 12 because the force developed in direction one because of a unit displacement in 2 that is p sine Alpha times cosine Alpha.



So, K times cosine Alpha sine Alpha is your now K 12 and the vertical component this vertical direction that is K 22 is a p times sine Alpha that is K times sine Square Alpha. Then the coefficients are the other node will be negative because we need to maintain the equilibrium they should be equal and opposite. So, K 32 is - K 12 and K 42 is - K 22 right and similarly we can apply unit deformations in the other degrees of freedom and measure the forces.

And we see that all these coefficients K cosine Square Alpha K sine Square Alpha K cosine Alpha sine Alpha and all these things they are exactly the same as what we had derived earlier and this is another way of deriving this stiffness coefficients directly by using the definition.

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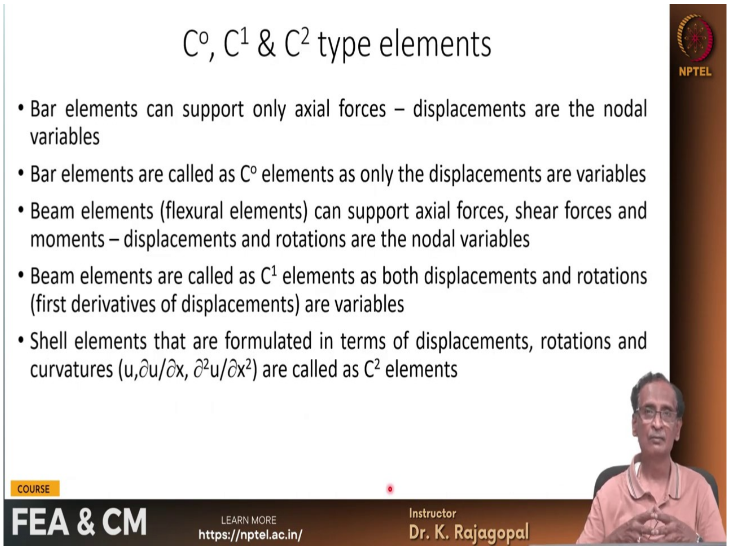

So, these bar elements we call them as a C naught type elements that they are formulated in terms of only the displacements at the nodal degrees of freedom and so, they are called as a C naught. But next we are going to look at the beam elements which are flexion flexural elements that have an axial force then Shear force. And then rotational degree of freedom and the corresponding moment capacity also it has down.

So, this beam element is called as a C 1 element because the nodal variables are both the displacements and then the rotation, rotation is the first order derivative of the displacements. So, it is; so, the beam element is called as C 1 and there are some higher order elements the shell elements where the nodal degrees of freedom are displacements first derivative of the displacements and then the second derivative of the displacement that is the curvature and these are called as C 2 elements.

Shell elements that are formulated in terms of displacements, rotations and curvatures ($u, \partial u/\partial x, \partial^2 u/\partial x^2$) are called as C^2 elements

But for geotechnical engineering we will be mostly concerned with C naught all our soils are formulated in terms of only the displacements. And then we do have beam elements because we need to use them for Sheet piles and other type of support structural elements they will not come across C 2 type elements.

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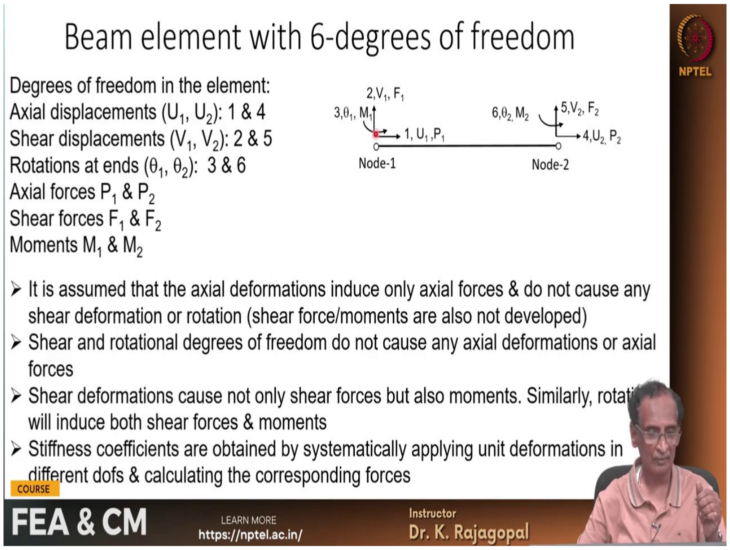
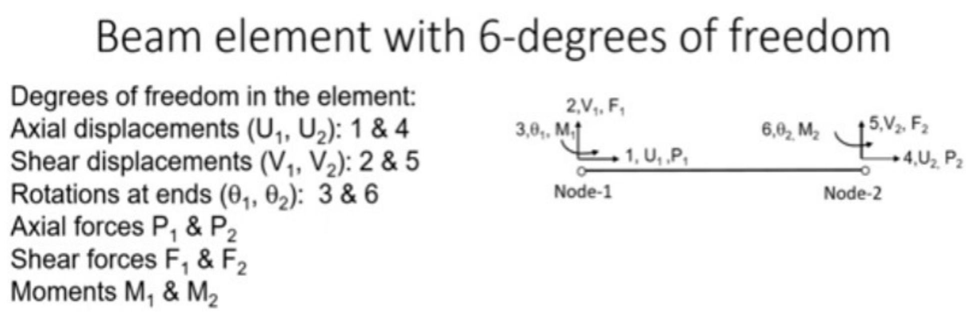


So, now let us move on to higher order elements that is the flexural element and the beam element with 6 degrees of freedom. Previously we had a bar element that has 4 degrees of freedom and this beam element has 6 degrees of freedom and letters initially consider only the element coordinate system that is our axial displacements under the shear displacements are defined with respect to the direction of the element.



So, we have at node 1, 3 degrees of freedom axial displacement u 1 Shear displacement v 1 and then a rotation theta 1 then the corresponding axial force P 1 Shear force F 1 and then a moment m 1. Similarly at node 2 we have u 2 v 2 and Theta 2 and the corresponding force is P 2 F 2 and m 2 right. And so, there are totally 6 degrees of freedom as shown here and the correspondingly 6 forces axial shear and moments and we assume that the axial.

And then the other forces the shear and moment are decoupled that is if you apply any axial deformation you will generate only the axial forces and they will not be any Shear force and the moment. And similarly if you apply any Shear deformation or rotation you will not produce any axial forces that is an assumption and it is a true if you are order of displacements are very very small.

Where you do not change the length by applying some Shear deformation and the shear deformations will cause only the shear forces but also the moments. Because now we are dealing with the flexural elements under any Shear deformation will be associated with some rotation then corresponding the moments of sort developed. And similarly if you apply any rotation it will not only develop some moment but also the shear force right.

And once again we can derive the stiffness coefficients by systematic application of the unit displacements in different directions. And initially we are going to define the stiffness coefficients in the local direction and then we will think of converting this into some Global coordinate system.

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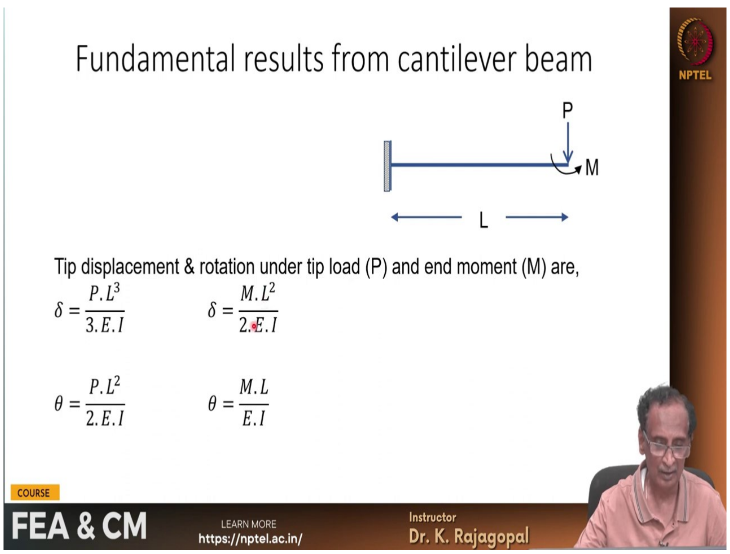
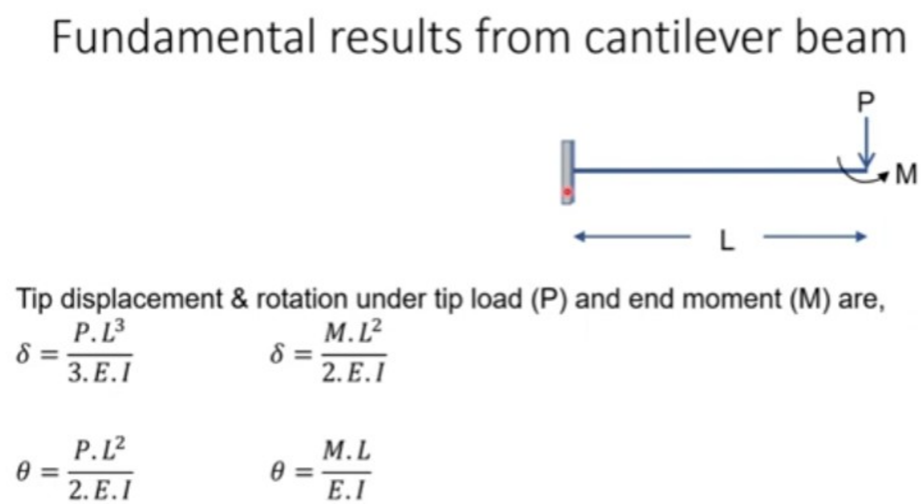


And to help us in developing our stiffness coefficients we need to know some fundamental results and I hope all of you know that if you have a cantilever beam subjected to some tip load and a tip moment. The resulting deformations under the tip load are P L 3 by 3E I and ml Square by 2 E I and the rotations are P L Square by 2 E I and Theta is ml by E I and we can use these equations to derive our stiffness coefficients.



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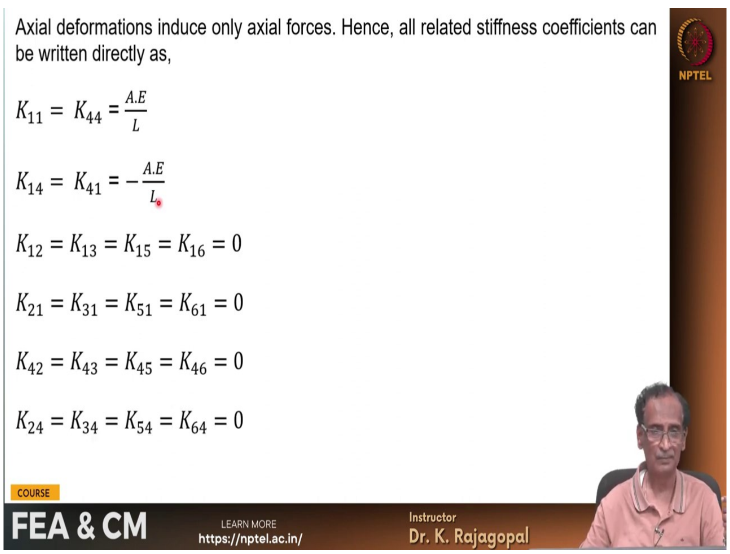

So, the application of axial deformations u 1 and u 2 and u 2 they will produce only the axial forces and all the other coefficients related to axial force the shear and rotational forces they are all zero K 11 and K 44 are AE by L K 14 and K 41 are - AE by L. The same thing as what we had seen with the bar elements.

Axial deformations induce only axial forces. Hence, all related stiffness coefficients can be written directly as,

$$K_{11} = K_{44} = \frac{A.E}{L}$$

$$K_{14} = K_{41} = -\frac{A.E}{L}$$

$$K_{12} = K_{13} = K_{15} = K_{16} = 0$$

$$K_{21} = K_{31} = K_{51} = K_{61} = 0$$

$$K_{42} = K_{43} = K_{45} = K_{46} = 0$$

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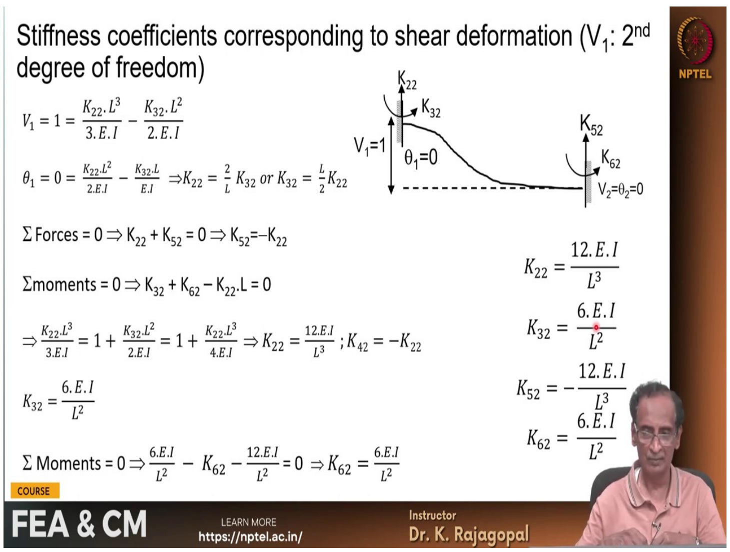
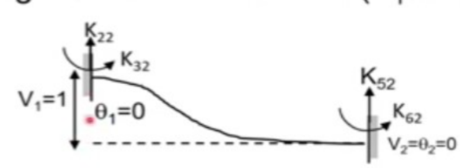


And now let us apply unit deformations in the shear and then the rotational degrees of freedom and see what happens. And let us take a node one and apply a unit deformation v 1 of 1 the shear definition of one and that will produce some force K 22 that is the shear force developed at node one and then the rotational force at the moment in a developed at node 1 that is K 32 because v 1 is actually the second degree of freedom.

Stiffness coefficients corresponding to shear deformation (V_1: 2nd degree of freedom)



And similarly there will be 2 more forces developed at the other end that is K 52 that is the force developed in degree of freedom 5 because of unit displacement and degree of freedom

2 and k62 will be moment developed at node 2 while we set Theta 1 to 0 and then v 2 and Theta 2 0 at the other end. So, now our Shear displacement at node 1 could be because of Shear force K 22 at the moment K 32 and v 1 is 1 and that we can equate to the displacements from both the shear force K 22 and then the Moment K 32 like this K 22 L Cube by 3 I – K 32 L Square by 2 E I got the flexural stiffness tries to resist the the deformation so, we have a - here.

$$V_1 = 1 = \frac{K_{22}.L^3}{3.E.I} - \frac{K_{32}.L^2}{2.E.I}$$

$$\theta_1 = 0 = \frac{K_{22}.L^2}{2.E.I} - \frac{K_{32}.L}{E.I} \Rightarrow K_{22} = \frac{2}{L} K_{32} \; or \; K_{32} = \frac{L}{2} K_{22}$$

$$\Sigma \text{Forces} = 0 \Rightarrow K_{22} + K_{52} = 0 \Rightarrow K_{52} = -K_{22}$$

$$\Sigma \text{moments} = 0 \Rightarrow K_{32} + K_{62} - K_{22}.L = 0$$

And then similarly our we have one more equation theta 1 is 0 that is K 22 L Square by 2 E I - K 3 2 L by E I and that is equal to 0. And so, we get a relation between K 32 and K 22 like this and we have 4 unknowns or 4 coefficients to be determined K 22 K 32 K 52 K 62 and v 1 of 1 and Theta 1 of 0 we got 2 equations and we need the 2 more equations and that we can get from our equilibrium equation the net force is zero net vertical force is zero and net moment is 0 around any point.

So, the sigma of vertical force is zero means that K 22 + K 52 is 0. So, K 52 is - of K 22 and similarly the moments about any point are 0 and let us consider the moment about this is 0.2 that will be K 32 + K 62 - K 22 times L where L is the length of the element that is equal to 0. And now we have 4 equations v 1 is 0 Theta 1 is 0. Sigma of vertical forces is 0 and then the net moment is zero.

$$\Sigma\, \text{Forces} = 0 \Rightarrow K_{22} + K_{52} = 0 \Rightarrow K_{52} = -K_{22}$$

$$\Sigma\, \text{moments} = 0 \Rightarrow K_{32} + K_{62} - K_{22}.L = 0$$

$$\Rightarrow \frac{K_{22}.L^3}{3.E.I} = 1 + \frac{K_{32}.L^2}{2.E.I} = 1 + \frac{K_{22}.L^3}{4.E.I} \Rightarrow K_{22} = \frac{12.E.I}{L^3}\ ; K_{42} = -K_{22}$$

$$K_{32} = \frac{6.E.I}{L^2}$$

$$\Sigma\, \text{Moments} = 0 \Rightarrow \frac{6.E.I}{L^2} - K_{62} - \frac{12.E.I}{L^2} = 0 \Rightarrow K_{62} = \frac{6.E.I}{L^2}$$

And by solving these 4 equations we get these 4 coefficients K 22 K 32 K Phi 2 K 62 right and actually you may know these terms 12 E I by L Cube 6 E I by L square and then we will also see Fourier by L later on these are all the same coefficients that you come across in our room moment distribution method and then the flexural sorry the the flexibility method and so on.

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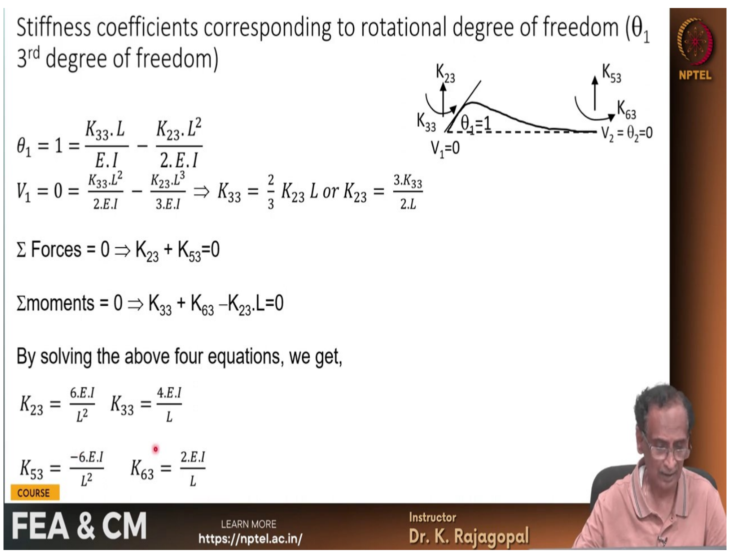
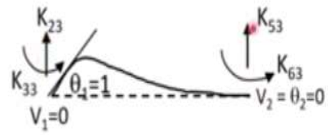


And now let us Supply a unit rotation at node one while we constrain all the other degrees of freedom so our Theta 1 is 1 and v 1 is 0 v 2 and Theta 2 are 0. and Theta 1 is the third degree of freedom. So, the moment developed at this node is K 33 and the shear first developed that this node is K 23 and the shear force developed at node 2 is K 53 and then the rotate in the moment is K 63. And so, we can do a similar exercise Theta 1 is 1 at node 1 and that is because of your moment and then the tip force K 33 L by E I - K 23 L Square by 2E I.

Stiffness coefficients corresponding to rotational degree of freedom (θ_1 3rd degree of freedom)



These are the equations that we had seen earlier for the cantilever beam and v 1 is zero. So, that is a K 33 l Square by 2 E I – K 23 L cube by 3I and that will give us some relation between K 33 and the K 23 and then we can find the other coefficients by using 2 more equations the net vertical force is zero. So, that is K 23 + K Phi 3 is 0. Then the net moment about any point like either this point at that point is zero.

K_{33}

$$\theta_1 = 1 = \frac{K_{33}.L}{E.I} - \frac{K_{23}.L^2}{2.E.I}$$

$$V_1 = 0 = \frac{K_{33}.L^2}{2.E.I} - \frac{K_{23}.L^3}{3.E.I} \Rightarrow K_{33} = \frac{2}{3}\, K_{23}\, L \; or \; K_{23} = \frac{3.K_{33}}{2.L}$$

Σ Forces = 0 $\Rightarrow K_{23} + K_{53} = 0$

Σmoments = 0 $\Rightarrow K_{33} + K_{63} - K_{23}.L = 0$

By solving the above four equations, we get,

$$K_{23} = \frac{6.E.I}{L^2} \quad K_{33} = \frac{4.E.I}{L}$$

$$K_{53} = \frac{-6.E.I}{L^2} \quad K_{63} = \frac{2.E.I}{L}$$

So, K 33 + K 63 - K 23 times L and by solving these 4 equations we get these 4 coefficients oh sorry yeah K 23 K 33 K 53 and K 63.

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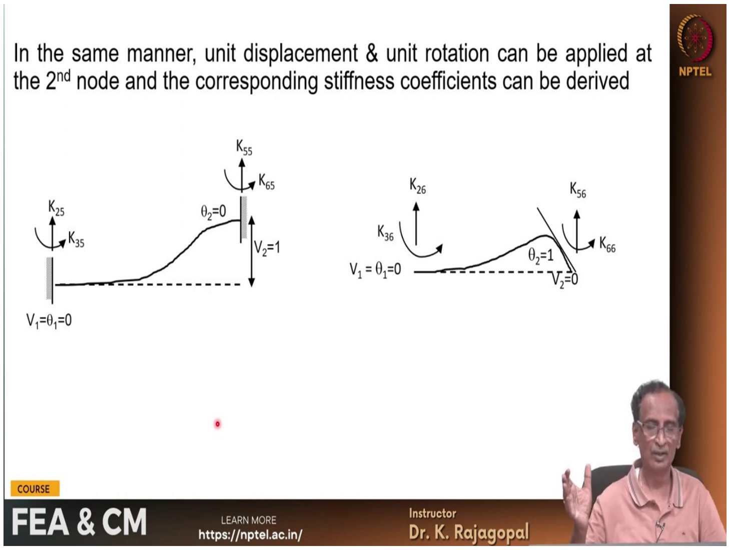

And similarly we can apply unit deformation and unit rotation at the other node at node 2 and then get these coefficients.

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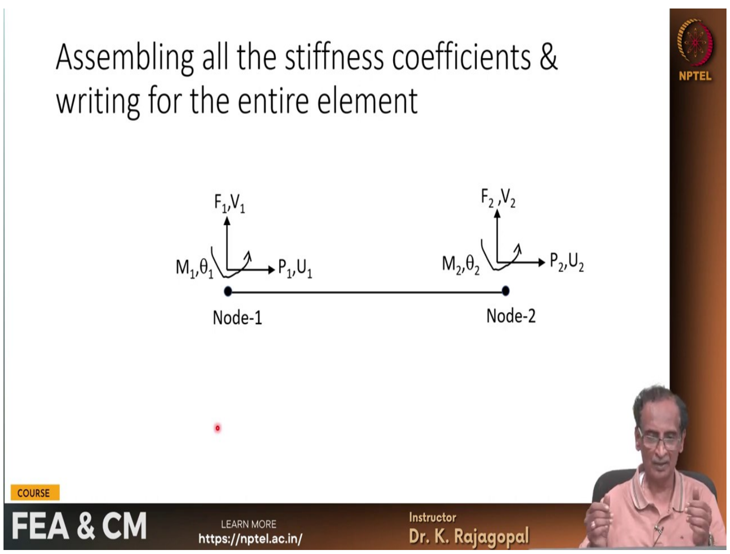

And now we can solve or we can assemble all the equations and write equilibrium equations at the at the element level in terms of the forces P 1 v 1 M 1 P 2 v 2 m 2 and then the reaction forces right.

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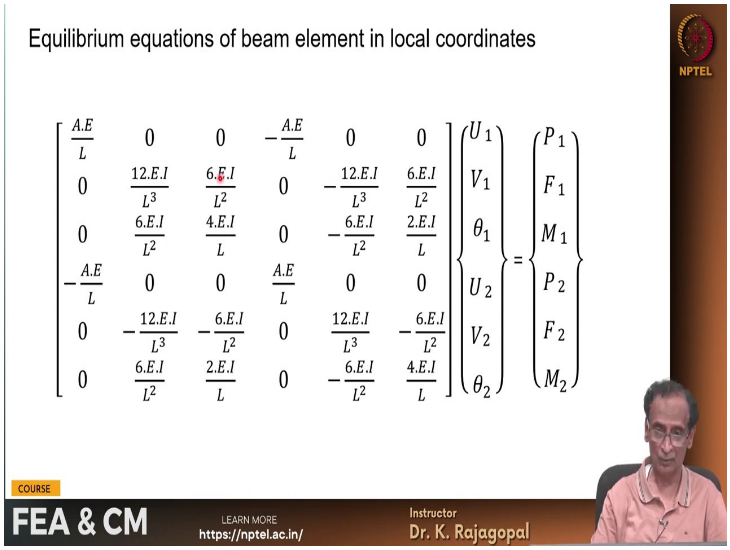

So, that will be like this now the axial forces they are not affected by deformation or the rotation. And similarly if you apply any rotation you will not develop any axial force. So, this is the set of equilibrium equations for the beam element in its own local coordinates.

Equilibrium equations of beam element in local coordinates

$$\begin{bmatrix} \frac{A.E}{L} & 0 & 0 & -\frac{A.E}{L} & 0 & 0 \\ 0 & \frac{12.E.I}{L^3} & \frac{6.E.I}{L^2} & 0 & -\frac{12.E.I}{L^3} & \frac{6.E.I}{L^2} \\ 0 & \frac{6.E.I}{L^2} & \frac{4.E.I}{L} & 0 & -\frac{6.E.I}{L^2} & \frac{2.E.I}{L} \\ -\frac{A.E}{L} & 0 & 0 & \frac{A.E}{L} & 0 & 0 \\ 0 & -\frac{12.E.I}{L^3} & -\frac{6.E.I}{L^2} & 0 & \frac{12.E.I}{L^3} & -\frac{6.E.I}{L^2} \\ 0 & \frac{6.E.I}{L^2} & \frac{2.E.I}{L} & 0 & -\frac{6.E.I}{L^2} & \frac{4.E.I}{L} \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ \theta_1 \\ U_2 \\ V_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ F_1 \\ M_1 \\ P_2 \\ F_2 \\ M_2 \end{Bmatrix}$$

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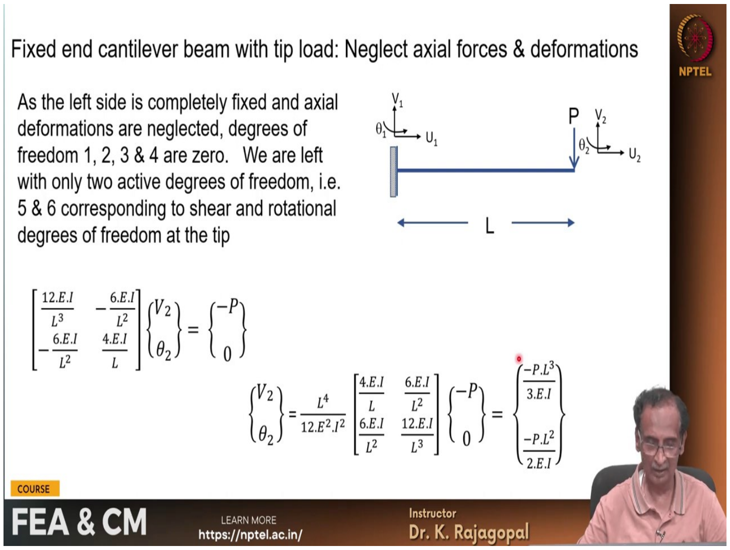
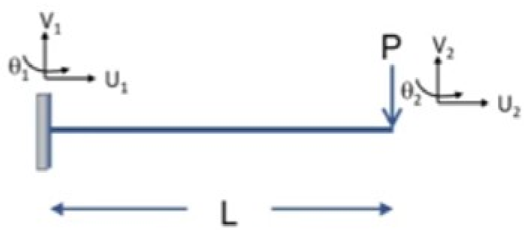


And let us do a small problem let us take a fixed and cantilever beam with the tip load and let us neglect axial deformation axial forces and axial deformations u 1 and u 2 we are not considering P 1 and P 2 also we are not considering. So, as the left side is completely fixed and axial deformations are neglected the degrees of freedom corresponding to these fixed degrees of freedom can be eliminated.



So, this one 2 3 4 equations can be can be deleted then we are only left with only 2 active degrees of freedom that is v 2 and Theta 2 that is a degrees of freedom 5 and 6 right. So, this is our equilibrium equation that we have and we can solve this by inversion. And so, v 2 is – P L cube by 3E I and Theta 2 is - PL Square by 2 E I these are our familiar equation that we already know and we just got back the same thing whatever we already know by applying the equilibrium equations that we have derived.

$$\begin{bmatrix} \frac{12.E.I}{L^3} & -\frac{6.E.I}{L^2} \\ -\frac{6.E.I}{L^2} & \frac{4.E.I}{L} \end{bmatrix} \begin{Bmatrix} V_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} V_2 \\ \theta_2 \end{Bmatrix} = \frac{L^4}{12.E^2.I^2} \begin{bmatrix} \frac{4.E.I}{L} & \frac{6.E.I}{L^2} \\ \frac{6.E.I}{L^2} & \frac{12.E.I}{L^3} \end{bmatrix} \begin{Bmatrix} -P \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{-P.L^3}{3.E.I} \\ \frac{-P.L^2}{2.E.I} \end{Bmatrix}$$

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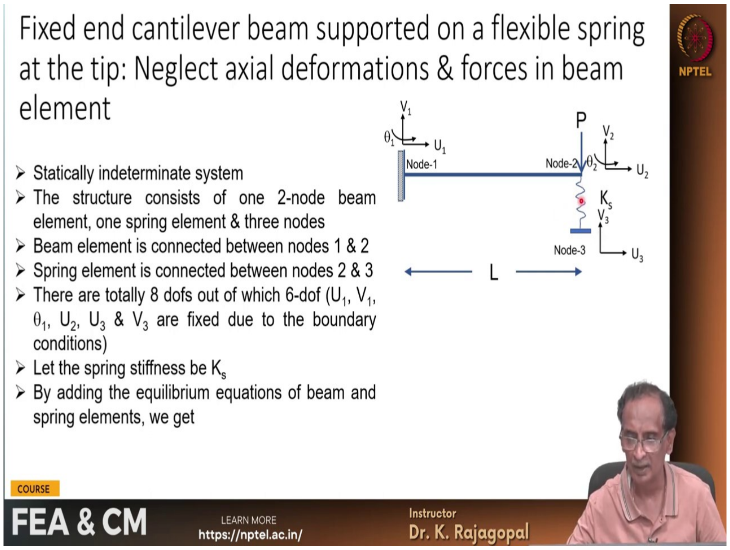
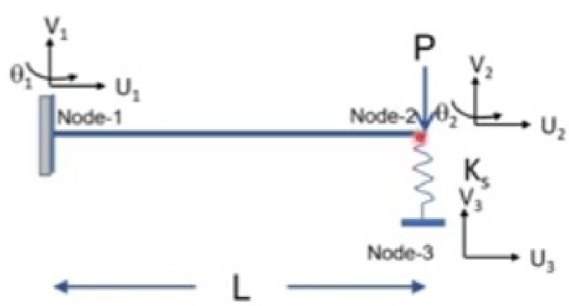


So, now this problem a little bit and let us assume that now we have supported the node 2 on a spring maybe that could correspond to the support that we get from the soil from some other member and now we want to solve this problem. So, it is actually it is a statical indeterminate problem right because we do not know what is the compression and if you know the compression you can find the force right.



It is actually it is a interaction problem it is not straightforward that you can solve but say if your spring is rigid that it will not deform then we can then we can use that as an additional equation and then find the solution that we have. It is basically it is a statical indeterminate problem it is a typical soil structure interaction problem. Structure is consisting of 3 nodes node 1 node 2 and node 3 and it consists of one beam element between Node 1 and node 2.

Then one spring element node 3 under node 2 and our spring element is like our uniaxial bar element right and the beam element is connected between Node 1 and node 2 and the string element between nodes 2 and 3. And we have the corresponding the degrees of freedom at node 1 u 1 v 1 Theta 1 at node 2 u 2 v 2 Theta 2 node 3 u 3 and v 3 out of this node 1 is fixed. So, u 1 v 1 Theta 1 of 0 then we are neglecting the axial deformation.

So, u 2 is 0 then node 3 is fixed u 3 and v 3 are 0 right. So, let us say that our spring stiffness is KS and since we are working with the local coordinate system we can directly write the equations for the beam element then the spring is in the direction of the shear force at node 2. So, we can directly add the contribution of the spring to the to the shear force component at node 2 right.

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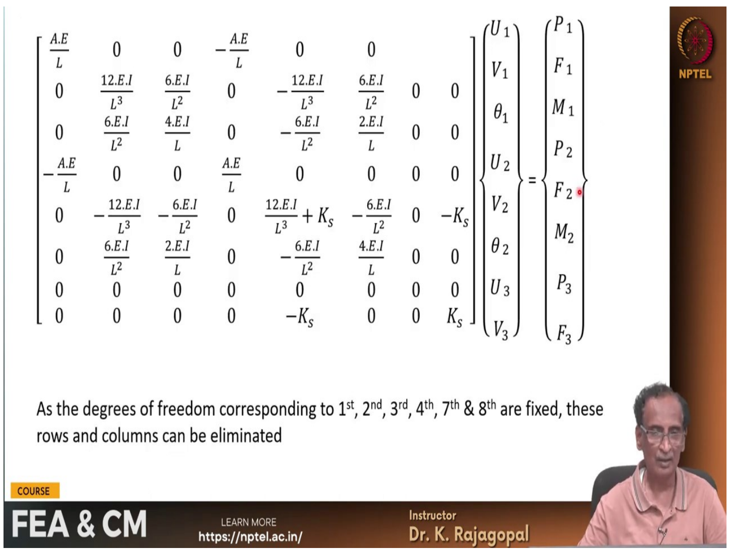

So, we can directly add this K s to v 2 or F 2 and then we have F 3 is K s and then the interaction terms are – K s and – K s. So, actually basically this is our equilibrium equation the stiffness coefficient that we derived earlier for the beam element then we have added the contribution from the spring right and this is the combined equilibrium equation for the entire structure consisting of a beam and then a spring.

$$
\begin{bmatrix}
\frac{A.E}{L} & 0 & 0 & -\frac{A.E}{L} & 0 & 0 & & \\
0 & \frac{12.E.I}{L^3} & \frac{6.E.I}{L^2} & 0 & -\frac{12.E.I}{L^3} & \frac{6.E.I}{L^2} & 0 & 0 \\
0 & \frac{6.E.I}{L^2} & \frac{4.E.I}{L} & 0 & -\frac{6.E.I}{L^2} & \frac{2.E.I}{L} & 0 & 0 \\
-\frac{A.E}{L} & 0 & 0 & \frac{A.E}{L} & 0 & 0 & 0 & 0 \\
0 & -\frac{12.E.I}{L^3} & -\frac{6.E.I}{L^2} & 0 & \frac{12.E.I}{L^3}+K_s & -\frac{6.E.I}{L^2} & 0 & -K_s \\
0 & \frac{6.E.I}{L^2} & \frac{2.E.I}{L} & 0 & -\frac{6.E.I}{L^2} & \frac{4.E.I}{L} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -K_s & 0 & 0 & K_s
\end{bmatrix}
\begin{Bmatrix} U_1 \\ V_1 \\ \theta_1 \\ U_2 \\ V_2 \\ \theta_2 \\ U_3 \\ V_3 \end{Bmatrix}
=
\begin{Bmatrix} P_1 \\ F_1 \\ M_1 \\ P_2 \\ F_2 \\ M_2 \\ P_3 \\ F_3 \end{Bmatrix}
$$

As the degrees of freedom corresponding to 1st, 2nd, 3rd, 4th, 7th & 8th are fixed, these rows and columns can be eliminated

And in this the degrees of freedom 1 2 3 4 and then and then 5 6 sorry the 7 and 8 are fixed. So, these rows and columns can be deleted.

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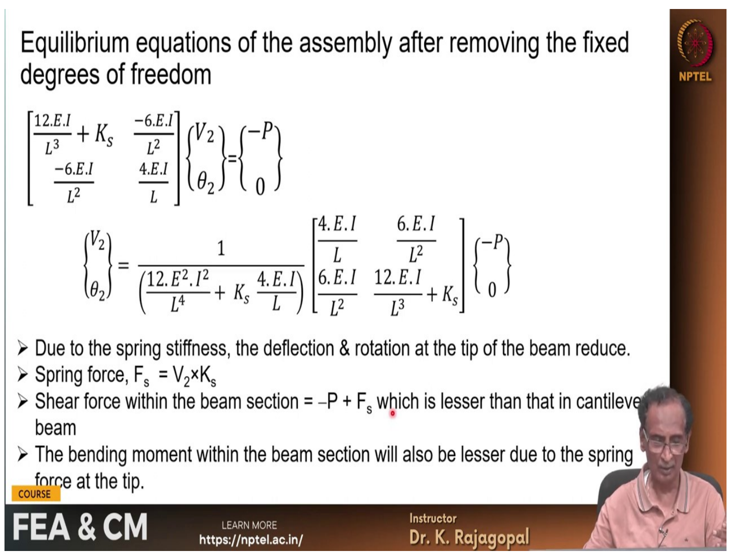

And we are left with only 2 degrees of freedom v 2 and Theta 2. here this is our stiffness Matrix and by inverting it we can get our v 2 and Theta 2 like this it is basically it is the same equation that we had earlier except our K s is there and then the denominator we have some other this function. Where K s is your spring stiffness and because of the spring stiffness we can expect a smaller deflection at the tip.

Equilibrium equations of the assembly after removing the fixed degrees of freedom

$$\begin{bmatrix} \frac{12.E.I}{L^3}+K_s & \frac{-6.E.I}{L^2} \\ \frac{-6.E.I}{L^2} & \frac{4.E.I}{L} \end{bmatrix} \begin{Bmatrix} V_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} V_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{\left(\frac{12.E^2.I^2}{L^4}+K_s\frac{4.E.I}{L}\right)} \begin{bmatrix} \frac{4.E.I}{L} & \frac{6.E.I}{L^2} \\ \frac{6.E.I}{L^2} & \frac{12.E.I}{L^3}+K_s \end{bmatrix} \begin{Bmatrix} -P \\ 0 \end{Bmatrix}$$

And then a smaller rotation and then the even the shear force in the beam section will be smaller and then the end moment also will be smaller. And the force developed in the spring is v 2 times K s and then the shear force in the beam previously it was just simply - P because there is only some tip load but now because we have some reaction force in the spring that is F s.

So, the shear force is - P + F s which is obviously lesser than the shear force in the cantilever beam right let us give some numerical values to appreciate the effect of spring stiffness. Then bending moment correspondingly bending moment in the in the beam section also will reduce.

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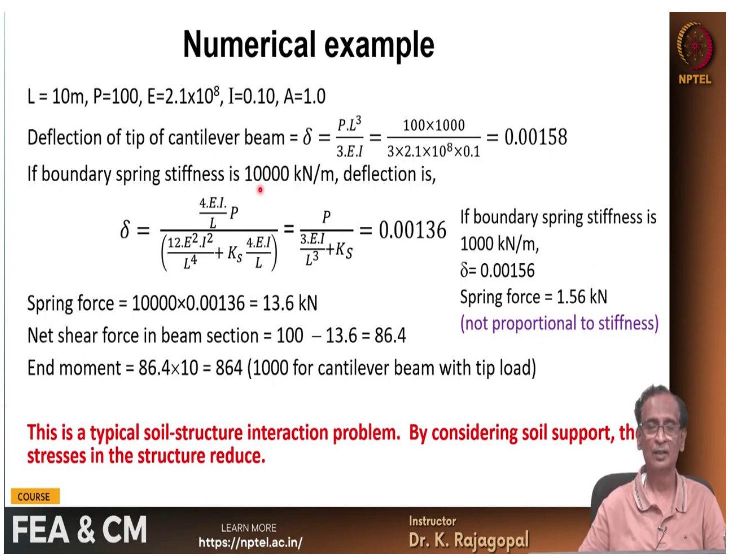

So, let us look at the numerical example let us take a beam of length 10 meters and let the loading at the tip be 100 and the Young's modulus and then I the section moment of inertia of this section and then the cross sectional area I have just arbitrarily given some values and the deflection of the cantilever beam is just simply PL cube by 3E I. So, if you substitute all the numbers you will get 0.00158 and in some units and the same units as your material properties.

Numerical example

L = 10m, P=100, E=2.1x10^8, I=0.10, A=1.0

Deflection of tip of cantilever beam = $\delta = \frac{P.L^3}{3.E.I} = \frac{100\times1000}{3\times2.1\times10^8\times0.1} = 0.00158$

If boundary spring stiffness is 10000 kN/m, deflection is,

$$\delta = \frac{\frac{4.E.I.}{L}P}{\left(\frac{12.E^2.I^2}{L^4} + K_S\frac{4.E.I}{L}\right)} = \frac{P}{\frac{3.E.I}{L^3}+K_S} = 0.00136$$

Spring force = 10000×0.00136 = 13.6 kN

Net shear force in beam section = 100 − 13.6 = 86.4

End moment = 86.4×10 = 864 (1000 for cantilever beam with tip load)

And now let us add a boundary spring having a stiffness of 10000 kilo Newton per meter right some number that that I have just given and now let us see what is the effect of this boundary spring. And our Delta we have derived this equation earlier. So, if you substitute all the members you will get .00136. Previously our deflection was 0.10058 and because of the spring stiffness it is slightly reduced 0.00136.

And then the spring force is a 10000 times 0.00136 that is 13.6 and then the net Shear force is 86.4 actually I'm not looking at the sign conversion of just I have calculated only the numerical value see previously our Shear force and the beam section was 100 that is corresponding to the tip load but now because of this reaction force from the spring it is only 86.4 and then the end moment is going to be 86.4 times 10 that is 864.

And the cantilever beam with the tip load the end moment was 1000. So, there is a significant reduction in the in the moment. And now you might ask what is the effect of this boundary spring whether the spring stiffness and then the spring force they are directly proportional to each other. They may not be because it is actually it is an interaction between the soil and then the structure right.

So, we cannot separately look at the contribution of either the spring or the beam. Now we are combining both of these and let us say let us write this by taking some other boundary spring stiffness. Let us take now a stiffness of 1000 previously it was 10000 now it is only one thousand and our trip displacement is 0.00156 very close to 158 and then the spring force is 1.56. So, it is not proportional.

So, you see when our spring stiffness was ten thousand the spring force was 13.6 and I reduced the the boundary spring stiffness by 10 times and if there is a proportionality and the spring force should have been 1.36 but now we get a spring force of 1.56. Actually that depends on also this the structure like let us say you are Young's modulus of the beam is very low then the spring has to has to take a higher and larger part.

Then obviously the compression in the spring will be more and then your corresponding force will be more will be higher and that is why we call this type of problems as soil structure interaction problems because the response depends not only on the soil but also on the structure. So, this is just a small illustration of our beam and then the spring element that we derived earlier.

And we also see that by considering the support from the soil our forces that are developed in the in the beam section are reduced not only the shear but also the the bending moment.

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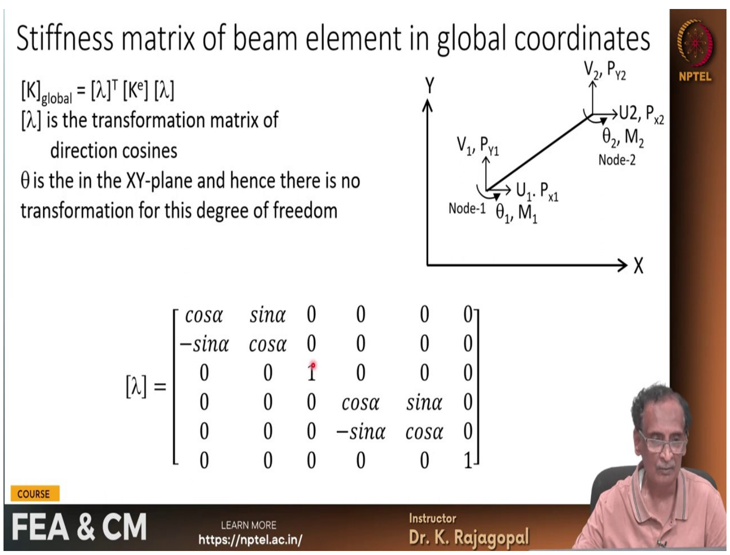


And so, now let us look at the beam element equilibrium equations in global coordinates. So, we can use the same transformation matrix Lambda transpose K Lambda that we had derived earlier for the bar element because there we had the 2 displacements u and v 1 at each node.

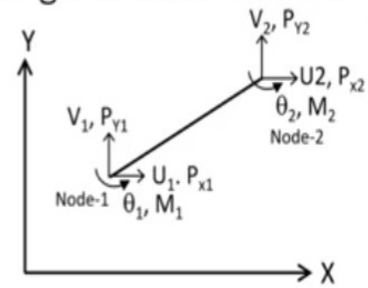
Now we have one rotational degree of freedom Theta one and this Theta 1 is actually in the x y coordinate in the in the plane of the x y x y plane.

Stiffness matrix of beam element in global coordinates

$[K]_{global} = [\lambda]^T [K^e] [\lambda]$

$[\lambda]$ is the transformation matrix of direction cosines

θ is the in the XY-plane and hence there is no transformation for this degree of freedom



So, even if you rotate to x Prime y Prime is Theta will remain the same because it is about a point about an axis that is a perpendicular to this plane; plane of analysis. So, our transformation matrix will have only one for the Theta. So, the Lambda for this beam element will be a 6 by 6 Matrix the transmission Matrix or the of the direction cosine Matrix because the cosine Alpha sine Alpha terms then for the rotation we have unit value 1.

$$[\lambda] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So, by going through this orthogonal transformation we can get the stiffness Matrix of the beam element and the global coordinates.

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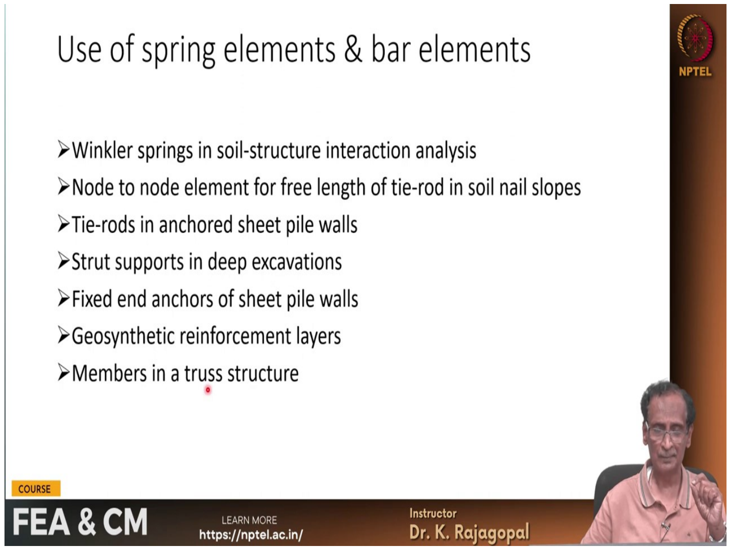

So, what are the or where what are the applications for the spring element and then the bar elements that we have developed. So, if the spring elements typically we can use them in the soil structure interaction analysis as Winkler Springs. And then node to node element for the free length portion of the of the tie rod in soil nail walls or the pre-stressed grouted anchors and then thyroids and anchored sheet piles then strut supports in deep excavation are the fixed and anchors of the sheet pile walls.

Or we can use these bar elements as geosynthetic reinforcement layers or members in the in a trust structure that we had seen the example earlier in the lecture 4. So, these are some of the applications for bar and spring element. Spring element is actually it is a it is a one-dimensional element whereas bar is a is a 2 dimensional element both are essentially the same both have both can support only the axial forces.

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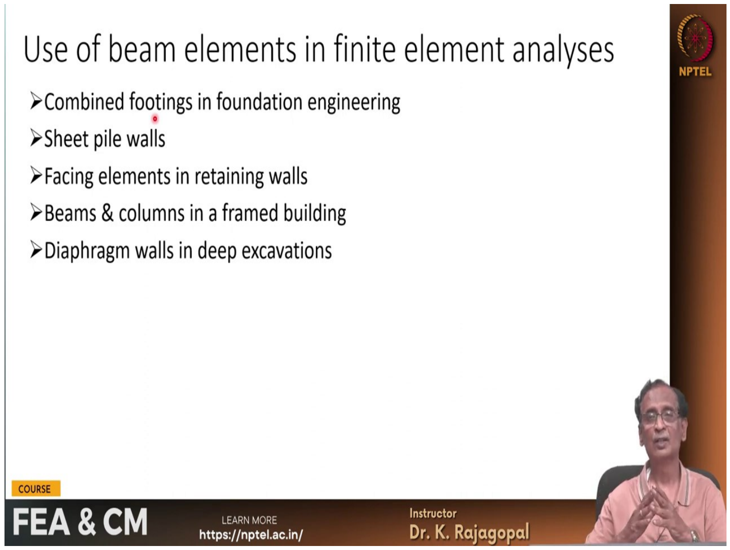

And what are the applications for the beam elements? See when we are dealing with any combined footing in Foundation Engineering we can represent the combined footing using a beam element right then we can represent the sheet pile walls with the beam element. The facing elements in retaining walls is actually any element that has some flexural stiffness can be represented by the beam elements.

Beams and columns in a frame it building and the dark from walls in our deep x equations and so, these we will see lot of applications for spring elements bar elements and the beam elements when we go for analysis of any soil structure like say any retaining wall that is supported by some anchors or some crops. And that type of structures requires apart from the modelling of the soil we require some extra elements for sheet pile walls or for thyroids and so on.

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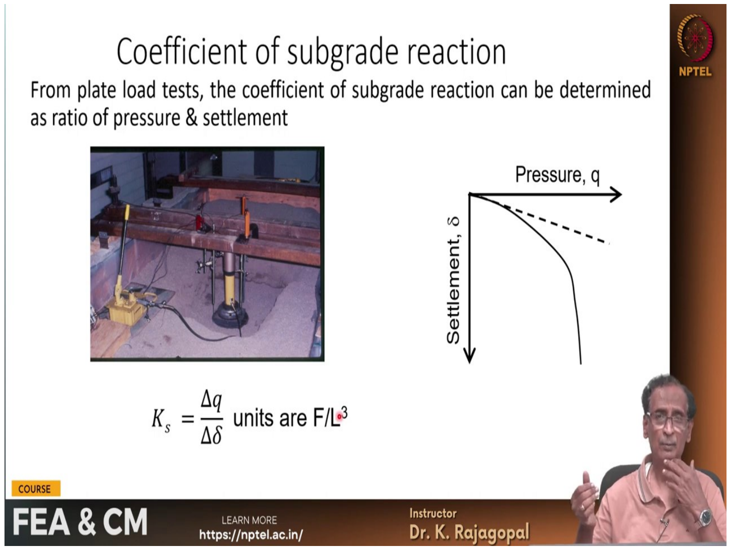

We will see some examples later on. Now let us do one thing slightly different see previously I had referred to the soil structure interaction and the one property that we require for this soil structure interaction analysis is the Winkler spring Winkler spring modulus. And that string modulus can be obtained from our plate load test plate load test is a very common test that we perform in geotechnical engineering.

So, here we have a photograph here is a plate and some pressure is applied then we measure the deformation and we plot a graph between the pressure and then the settlement of the plate and we get a graph like this and then the initial slope is taken as this the coefficient of subgrade reaction of the soil and this K s is the slope that is the Delta q by Delta the pressure divided by the by the deformation.

$$K_s = \frac{\Delta q}{\Delta \delta} \text{ units are F/L}^3$$

And the units for the K s that is the coefficient of subgrade reaction or the force per L Cube units that is kilo Newton per cubic meter units and this is a simple test that we perform for getting our coefficient of subgrade reaction. And so, if we do not perform this test there are other methods for getting it from our allowable bearing pressure and then the allowable settlement also we can estimate this coefficient of subgrade reaction.

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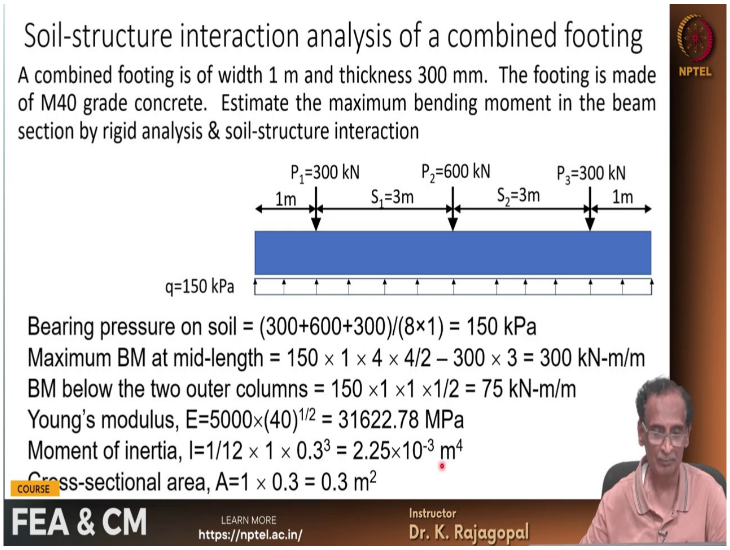
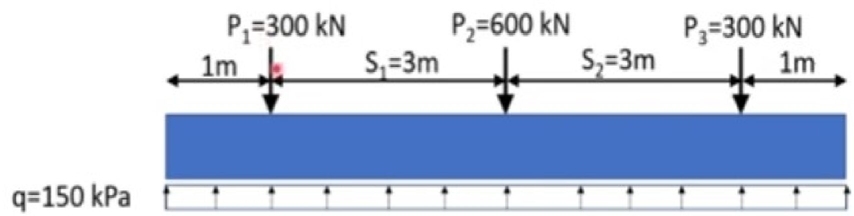


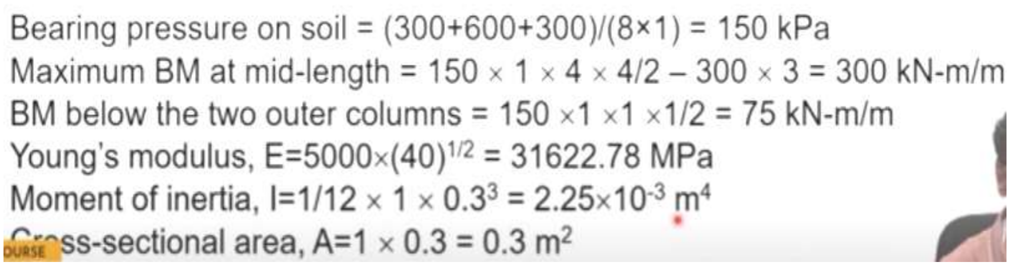
And let us do a small problem. Let us take a combined footing of 8 meters length and then there are 3 column loads column one column 2 column 3; 300 600 and 300 kilonewtons and then it is the combined footing is of thickness 300 millimeters. And then the width in the outer plane direction is one meter and let us say that the footing is made of M40 grade concrete. And we are asked to estimate the maximum bending moment in the beam section both by the rigid analysis and soil structure interaction.



A combined footing is of width 1 m and thickness 300 mm. The footing is made of M40 grade concrete. Estimate the maximum bending moment in the beam section by rigid analysis & soil-structure interaction

By rigid analysis what I mean is the analysis not considering the soil or in other words we consider the footing as an extremely rigid object that whatever may be the soil that you have the soil becomes flexible. So, the unit pressure at the foundation level is the total load divided by divided by the by the plan area and I have not considered the self weight of the footing because it gets compensated when we calculate our Shear force and bending moment.

So, for soil structure interaction analysis we do not consider any sulfate only when it comes to the settlements we require the self rate. So, let us calculate the bending moment in this section without considering the soil structure interaction. So, the bearing pressure on the soil is 150 kPa and then the maximum bending moment at the mid length is 150 times 4 times 4 by 2 times 11 is the unit width in the perpendicular direction that is - 300 times 3 that comes to 300 kilo Newton meter per meter.



Per meter means in the perpendicular direction we are considering the unit length and then the bending moment below the 2 extreme column 2 outer columns is 75 kilo Newton meter and the Young's modulus is approximately 5000 times square root of fck and our that is the compressive strength of the concrete that is a that is a 40. So, that is 31622.78 mPa right and the moment of inertia for this section is 112 BD Cube that is a 2.25 times 10 to power of -3 meters to the power 4 right. And the cross-sectional area is a 0.3 meter Square.

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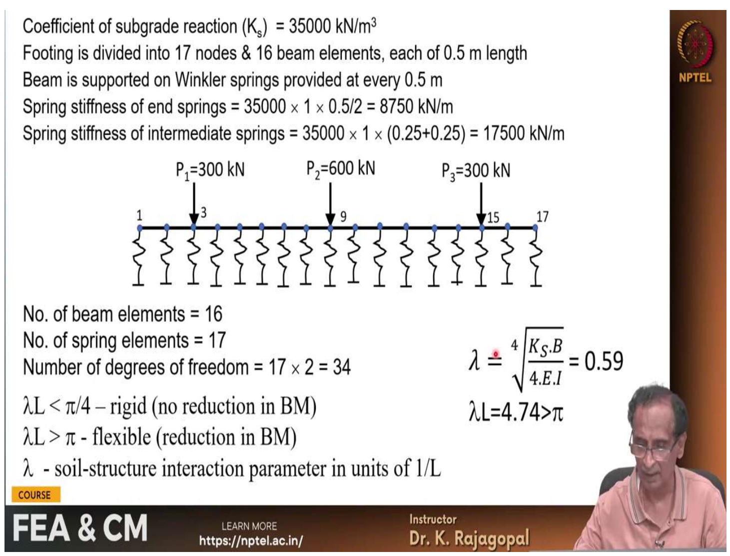
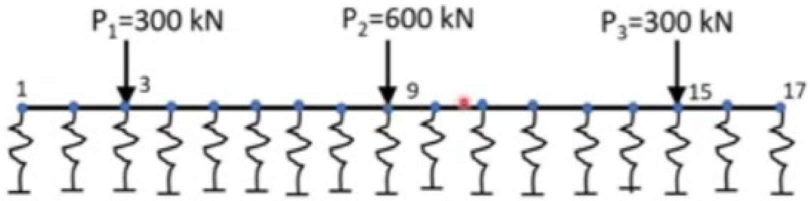


And we can perform a typical soil structure interaction analysis by considering the combined footing as a beam elements and in this particular case I have divided this beam element into 16 Elements by considering each beam element of length 0.5. So, we have totally 70 nodes 1

2 3 4 5 6 7 8 9 10 1112 13 14 15 16 17 notes then there are 17 notes corresponding to the to the soil or the below the Stream.



And these are all the Winkler strings having some stiffness that we will see and then the column loads 300, 600 and 300. And let us assume that our coefficient of subgrade reaction case is 35 000 kilo Newton per cubic meter and this is obtained as I mentioned from the plate flow test right but then our spring stiffness is in the units of kilo Newton per meter units not in this kilo Newton per cubic meter units.

Coefficient of subgrade reaction (K_s) = 35000 kN/m^3
Footing is divided into 17 nodes & 16 beam elements, each of 0.5 m length
Beam is supported on Winkler springs provided at every 0.5 m
Spring stiffness of end springs = 35000 × 1 × 0.5/2 = 8750 kN/m
Spring stiffness of intermediate springs = 35000 × 1 × (0.25+0.25) = 17500 kN/m

And so, we need to convert this modulus into some stiffness that is in units of kilo Newton per meter and there are 17 nodes 16 beam elements each of 0.5 meters length and then the beam is supported and Winkler Springs provided at every 0.5 meters. Actually my task y only 0.5 meters because in the theoretical solution the Winkler Springs are provided continuously. So, when we consider the integral equation we have a continuous support.

But because we are dealing with finite element analysis we put these only at some discrete points and the solution that we get may be only approximate it may not be exact. And by considering more number of Winkler Springs we may be able to approach the theoretical solution. And how we convert this coefficient of subgrade reaction to the spring stiffness is very simple you take this value 35000 multiplied by area the area of the footing that is controlled by each of these Springs.

So, if you look at the 2 end Springs we can assume that they support the footing up to Mid length up to half the length of this first beam element that is the length of the beam element

0.5 and half the length is 0.25 and in the outer plane direction we have a width of 1. So, that is spring stiffness for the 2 end springs is 35000 times 0.5 by 2 times 1 that is 8750 and the spring stiffness for the intermediate Springs are for all the interior ones are 35000 multiplied by one without a plane direction multiplied by 0.25 to the left and 0.25 to the right that is 17500.

And then in the soil structure interaction analysis we have one parameter that is the 4th root of K s B by 4 E I that is the Lambda that is a soil structure interaction parameter. And if your Lambda L where L is the length of the footing if Lambda L is greater than Pi we consider the footing as flexible. And if you perform any soil structure interaction analysis say we will see some reduction in the bending moment and if your Lambda L is less than Pi by 4 we can consider that as rigid and there will be only marginal reduction.

No. of beam elements = 16
No. of spring elements = 17
Number of degrees of freedom = 17 × 2 = 34

$\lambda L < \pi/4$ – rigid (no reduction in BM)
$\lambda L > \pi$ - flexible (reduction in BM)
λ - soil-structure interaction parameter in units of 1/L

$$\lambda = \sqrt[4]{\frac{K_S.B}{4.E.I}} = 0.59$$

$\lambda L = 4.74 > \pi$

I should have said marginal reduction but it is no reduction because if Lambda is very very small. And for this particular problem that we have my Lambda is 0.59; 0.59 times 8 Lambda L is 4.74 which is greater than Pi. So, we can expect significant reduction in the bending moment and that is what actually.

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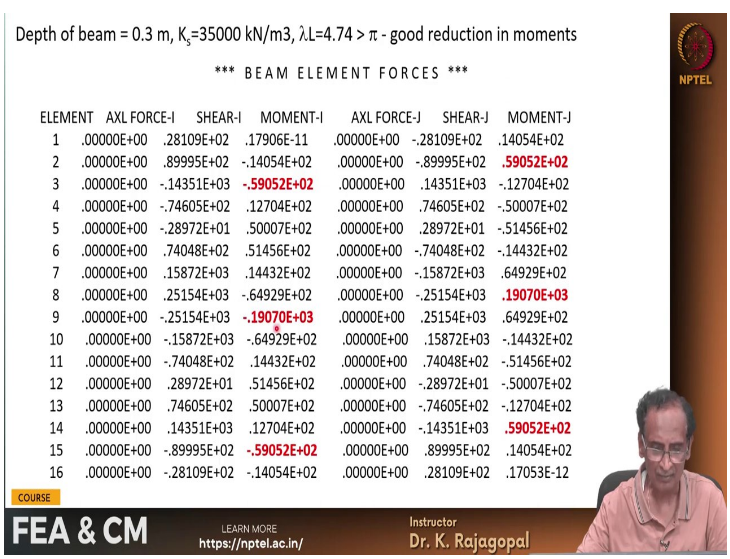

Depth of beam = 0.3 m, K_s=35000 kN/m3, λL=4.74 > π - good reduction in moments

*** BEAM ELEMENT FORCES ***

ELEMENT	AXL FORCE-I	SHEAR-I	MOMENT-I	AXL FORCE-J	SHEAR-J	MOMENT-J
1	.00000E+00	.28109E+02	.17906E-11	.00000E+00	-.28109E+02	.14054E+02
2	.00000E+00	.89995E+02	-.14054E+02	.00000E+00	-.89995E+02	**.59052E+02**
3	.00000E+00	-.14351E+03	**-.59052E+02**	.00000E+00	.14351E+03	-.12704E+02
4	.00000E+00	-.74605E+02	.12704E+02	.00000E+00	.74605E+02	-.50007E+02
5	.00000E+00	-.28972E+01	.50007E+02	.00000E+00	.28972E+01	-.51456E+02
6	.00000E+00	.74048E+02	.51456E+02	.00000E+00	-.74048E+02	-.14432E+02
7	.00000E+00	.15872E+03	.14432E+02	.00000E+00	-.15872E+03	.64929E+02
8	.00000E+00	.25154E+03	-.64929E+02	.00000E+00	-.25154E+03	**.19070E+03**
9	.00000E+00	-.25154E+03	**-.19070E+03**	.00000E+00	.25154E+03	.64929E+02
10	.00000E+00	-.15872E+03	-.64929E+02	.00000E+00	.15872E+03	-.14432E+02
11	.00000E+00	-.74048E+02	.14432E+02	.00000E+00	.74048E+02	-.51456E+02
12	.00000E+00	.28972E+01	.51456E+02	.00000E+00	-.28972E+01	-.50007E+02
13	.00000E+00	.74605E+02	.50007E+02	.00000E+00	-.74605E+02	-.12704E+02
14	.00000E+00	.14351E+03	.12704E+02	.00000E+00	-.14351E+03	**.59052E+02**
15	.00000E+00	-.89995E+02	**-.59052E+02**	.00000E+00	.89995E+02	.14054E+02
16	.00000E+00	-.28109E+02	-.14054E+02	.00000E+00	.28109E+02	.17053E-12

This is the result that that you can get by using a fine intelligent program and this particular one is from my own program I will be giving this program for you, you can and along with instructions on how to use the program and it is a very simple program and the axial forces are neglected. So, all the axial forces are zero there are total is 16 beam elements then the shear force and then the bending moment.

You see that the bending moment at the 2 ends is 0 10 to the power of - 1 that is practical zero but then the shear force is not zero. So, actually technically the shear force should be zero at the 2 ends but because of the of the spring elements that we have the shear force need not be equal to zero and the maximum bending moment that that we get in the beam section is 190. See if you neglect the soil structure interaction analysis it is 300 and if you consider the soil structure interaction there is a significant reduction almost a 30 percent reduction. Then of the 2 outer columns the bending moment is 59 whereas with rigid analysis you get 75.

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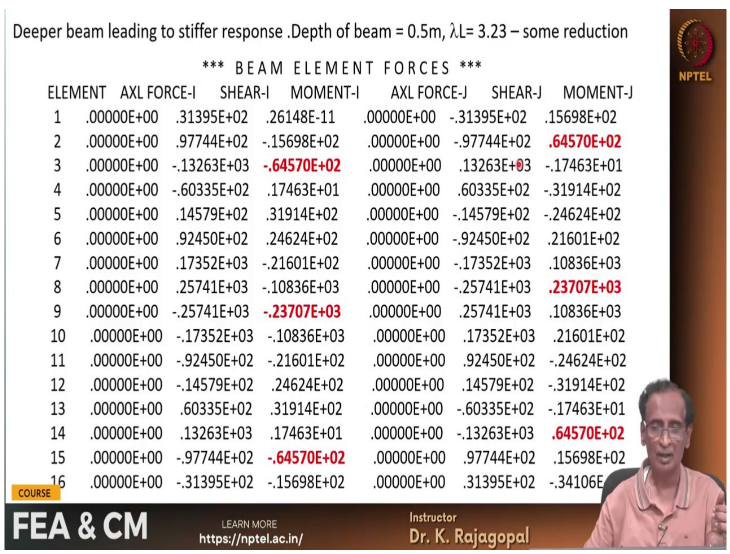
Deeper beam leading to stiffer response .Depth of beam = 0.5m, λL= 3.23 – some reduction

*** BEAM ELEMENT FORCES ***

ELEMENT	AXL FORCE-I	SHEAR-I	MOMENT-I	AXL FORCE-J	SHEAR-J	MOMENT-J
1	.00000E+00	.31395E+02	.26148E-11	.00000E+00	-.31395E+02	.15698E+02
2	.00000E+00	.97744E+02	-.15698E+02	.00000E+00	-.97744E+02	.64570E+02
3	.00000E+00	-.13263E+03	-.64570E+02	.00000E+00	.13263E+03	-.17463E+01
4	.00000E+00	-.60335E+02	.17463E+01	.00000E+00	.60335E+02	-.31914E+02
5	.00000E+00	.14579E+02	.31914E+02	.00000E+00	-.14579E+02	-.24624E+02
6	.00000E+00	.92450E+02	.24624E+02	.00000E+00	-.92450E+02	.21601E+02
7	.00000E+00	.17352E+03	-.21601E+02	.00000E+00	-.17352E+03	.10836E+03
8	.00000E+00	.25741E+03	-.10836E+03	.00000E+00	-.25741E+03	.23707E+03
9	.00000E+00	-.25741E+03	-.23707E+03	.00000E+00	.25741E+03	.10836E+03
10	.00000E+00	-.17352E+03	-.10836E+03	.00000E+00	.17352E+03	.21601E+02
11	.00000E+00	-.92450E+02	-.21601E+02	.00000E+00	.92450E+02	-.24624E+02
12	.00000E+00	-.14579E+02	.24624E+02	.00000E+00	.14579E+02	-.31914E+02
13	.00000E+00	.60335E+02	.31914E+02	.00000E+00	-.60335E+02	-.17463E+01
14	.00000E+00	.13263E+03	.17463E+01	.00000E+00	-.13263E+03	.64570E+02
15	.00000E+00	-.97744E+02	-.64570E+02	.00000E+00	.97744E+02	.15698E+02
16	.00000E+00	-.31395E+02	-.15698E+02	.00000E+00	.31395E+02	-.34106E

And now let us consider a slightly deeper beam of instead of 300 millimeters I am considering half a meter thick beam and Lambda L is a 3.23. So, we should still get some deduction because it is it is greater than greater than Pi but not as much as what we had in the previous case. So, now we get 237 slightly more than what we had earlier and then 64.5 previously we had 59 and 190.

Now we have 64.5 and 237 and let us consider very very soft soil K s is a 2 kilo Newton per cubic meter and let us say the beam depth is 300 millimeters. So, your Lambda L is 0.41

which is very less than 5 by 4. So, in this case the results should be very close to the rigid case. So, we see our maximum bending moment is 299.9 then at below the 2 columns outer columns it is 74.98 that is very close to 75 right.

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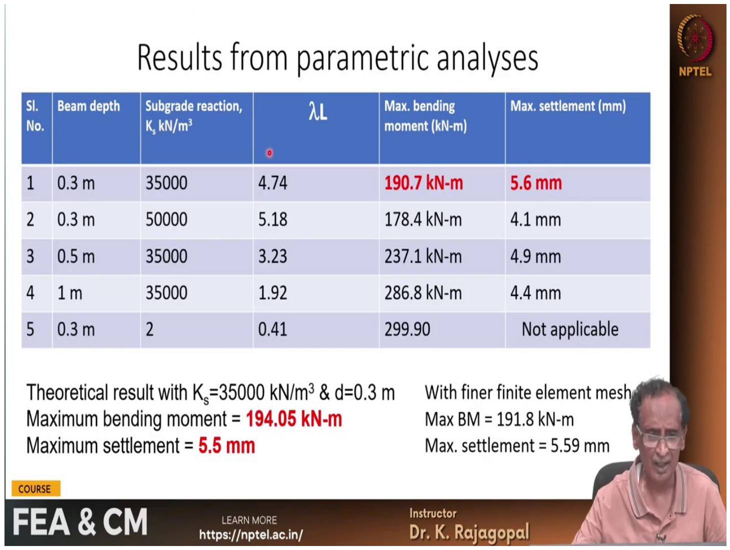
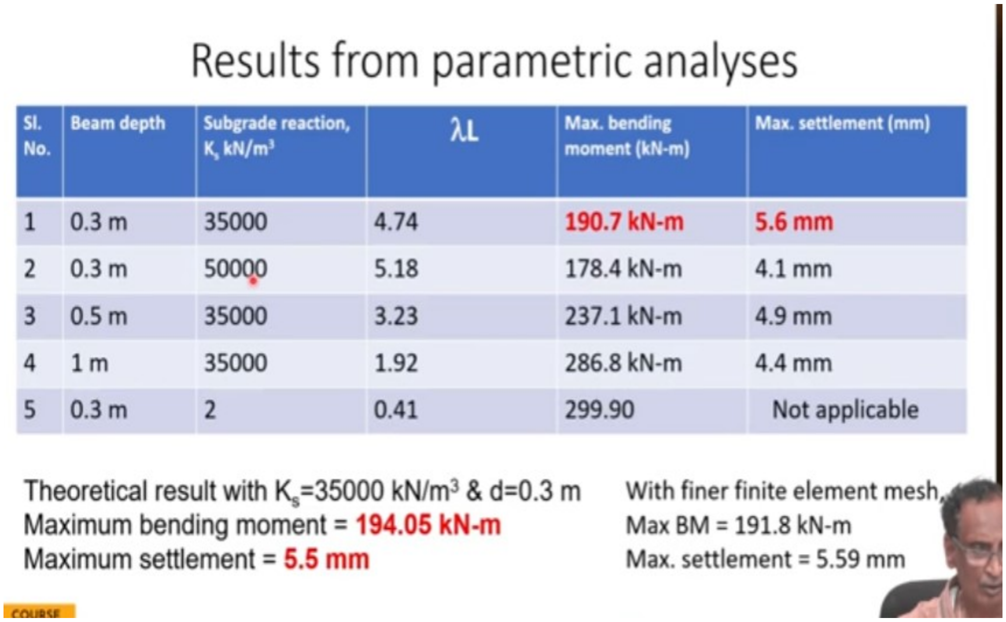


And so, here I have some results from different parametric studies and for this problem the theoretical solution with the K s of 35000 and the depth of 0.3 meters maximum bending moment is 194 kilo Newton meter then the maximum settlement is 5.5 millimeters. And through finite element analysis with this mesh discretization we get 190.7 and so, if I take a stronger soil with a subgrade reaction of 50000.

Your bending moment is a lower and then the maximum settlement is also smaller and if I take a deeper beam with the same subgrade stiffness we get some increase in the bending moment and then some reduction in the settlement. Then I am taking one meter thick beam just to see what happens and your subgrade reaction is 35000 Lambda L is pi by 2 is 1.92. And so, the effect of soil structure interaction may not be significant it is.



Results from parametric analyses

Sl. No.	Beam depth	Subgrade reaction, K_s kN/m³	λL	Max. bending moment (kN-m)	Max. settlement (mm)
1	0.3 m	35000	4.74	190.7 kN-m	5.6 mm
2	0.3 m	50000	5.18	178.4 kN-m	4.1 mm
3	0.5 m	35000	3.23	237.1 kN-m	4.9 mm
4	1 m	35000	1.92	286.8 kN-m	4.4 mm
5	0.3 m	2	0.41	299.90	Not applicable

Theoretical result with K_s=35000 kN/m³ & d=0.3 m
Maximum bending moment = 194.05 kN-m
Maximum settlement = 5.5 mm

With finer finite element mesh,
Max BM = 191.8 kN-m
Max. settlement = 5.59 mm

So, you see this is 286.8 and the settlement is 4.4 then if we take very very low subgrade reaction just to make the beam as re as rigid as possible the Lambda is 0.41 and you are maximum bending moment is 300 very close to 300 299.9. And it is actually if you take a finer mesh see in this case I have taken 16 beam elements and if we perform with 32 number of beam elements bending moment is slightly increased like 191.8.

So, next you increase it to 64 elements you may be moving the closer to this theoretical result and the settlement is also moving the closet to this. So, this is how we can use our beam element and then the spring element and then the bar element for practical applications and we will see the applications more applications in later classes. So, I think that is the end of this lecture.

So, in this just to summarize we have seen how to derive the stiffness coefficients of 2-dimensional bar element by applying displacements in the global directions. And then we have seen the the stiffness Matrix the development for a beam element initially we had the considered only the element coordinates and then we had seen how to develop now the stiffness Matrix of the beam element and global coordinates by Lambda transpose K e times Lambda.

Then we had seen some examples of the soil structure interaction by using our beam element and then the then the spring stiffness for Winkler springs. So, I will give you a computer program that you can utilize for doing this analysis and you can explore yourself what will happen. If you take a as a very coarse mesh with not considering say 16 beam elements but only consider only say 6 or 8 beam elements.

What happens or with a very large number of beam elements what happens. So, all these things you can you can explore later. So, if you have any questions please do contact me at this email address profkrg@gmail.com. So, thank you very much we will meet next time.