Finite Element Analysis and Constitutive Modelling in Geomechanics
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Lecture - 40
Modified Cam Clay Models

Good morning students, very nice to meet all of you once again. The previous two classes we had looked at the elastic plastic constitutive modelling and we have seen how to derive the constitutive matrix during the plastic flow. The fundamental equation we have seen based on the requirement that the yield function value should remain constant or the change in the yield function value is 0, that we called as a consistency condition.

And the actual step-by-step derivation of that I have shown it with an example for the joint element and some other examples are going to be provided in your weekly tutorials and other handouts. You can go through them and the next topic that I am going to discuss is a brief introduction to analysis with pore pressures that is consolidation analysis and then the dynamic analysis, how to perform the analysis when we have the seismic loading or impact loading or some other such type of loading.

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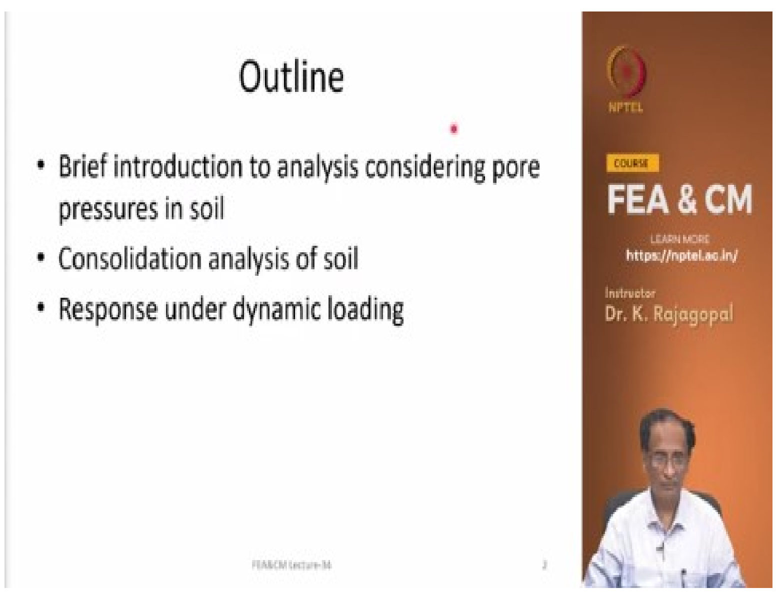

And this lecture will be only a brief introduction how to consider the pore pressures in the soil and then how to perform the consolidation analysis and then response under dynamic loading.

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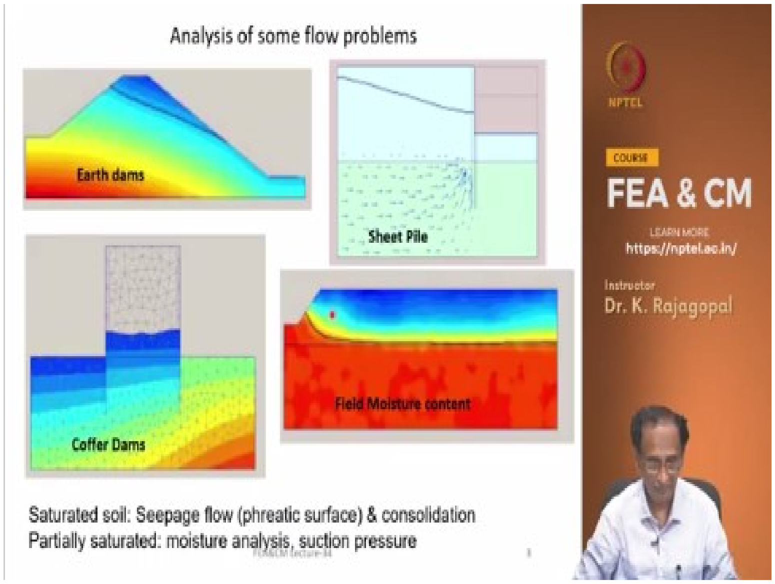
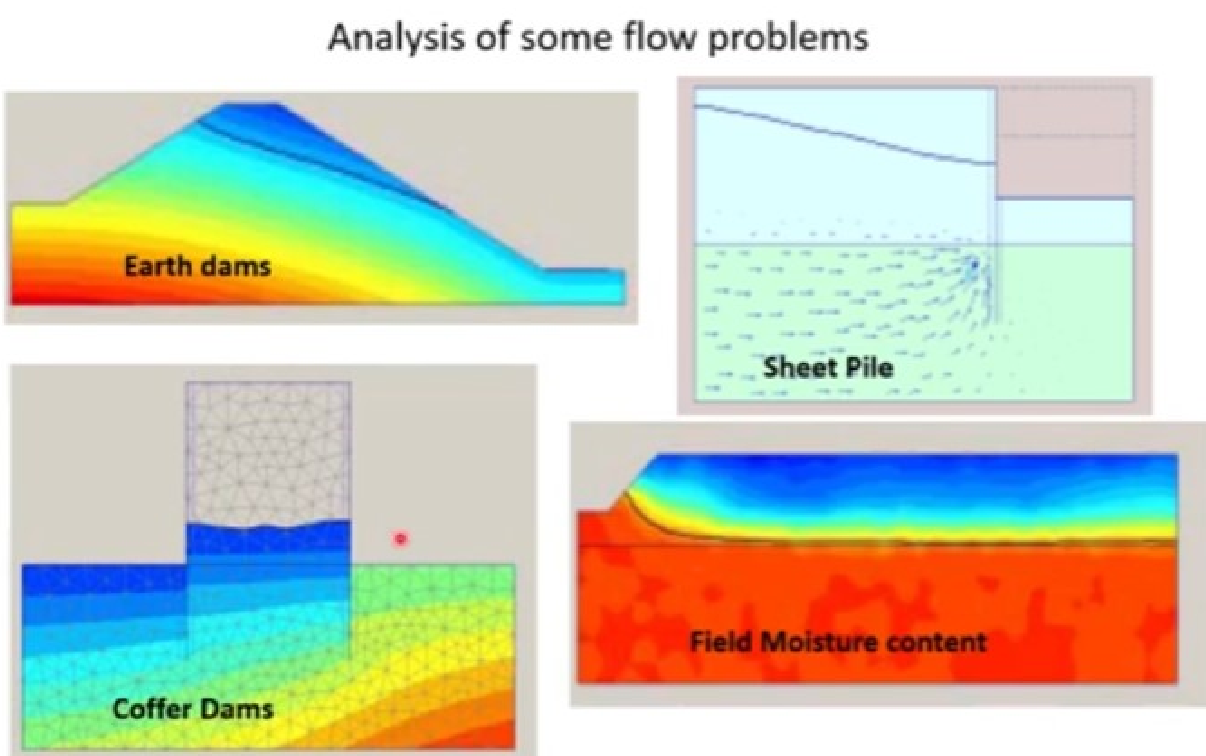


So, some of the flow problems are illustrated here say the flow through earth dams or flow around the sheet pile wall or a coffer dams or moisture content determination in the soil. Say the first three are for saturated soils the seepage flow and of course consolidation and in all these whenever we are doing the seepage, we are interested in knowing the top most flow line that is called as the phreatic surface phreatic line that is exposed to the atmospheric pressure.



And in the case of partially saturated soils, we would like to determine the moisture and also the suction pressure.

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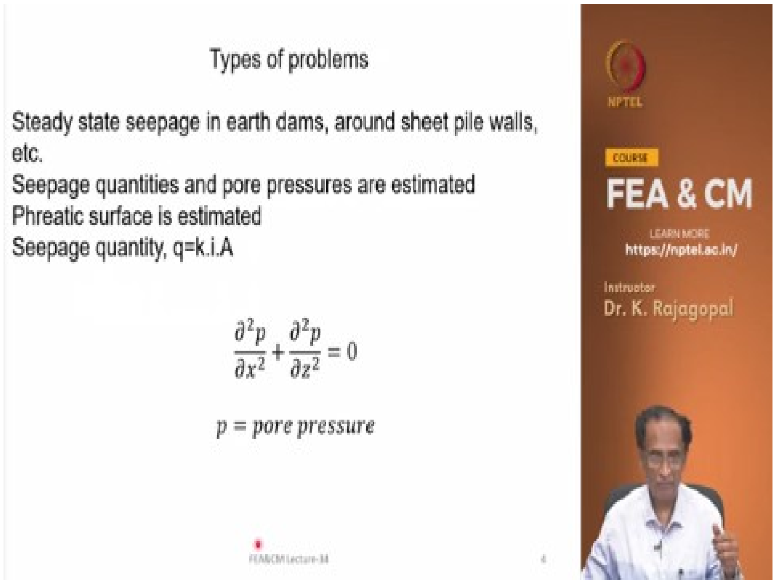

And the governing equations for different problems are like this for the steady state seepage for analysis of the flow through the soil medium is the LaPlace equation given like this. This is for a two dimensional case and for 3d also we have an additional coordinate y coordinate here the p is the pore pressure and x and z are the different Cartesian coordinates. And we by doing this analysis and by satisfying the boundary conditions we can determine the pore pressures at different places within the soil medium.

Seepage quantity, q=k.i.A

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial z^2}=0$$

$$p = pore\ pressure$$

And there are different methods for solving it both finite difference and the finite element and we can also incorporate the anisotropy. See what I have shown is only for isotropic soil medium but it is easy to modify this equation for anisotropic soil with the different coefficient of permeable in x and y directions. And we can also estimate at the seepage quantities as k times i A is very simple once we determine the pore pressures at different locations you draw a line.

And then we can get the gradient and then the i the head difference divided by the distance, k is the permeability and A is the cross-sectional area of flow.

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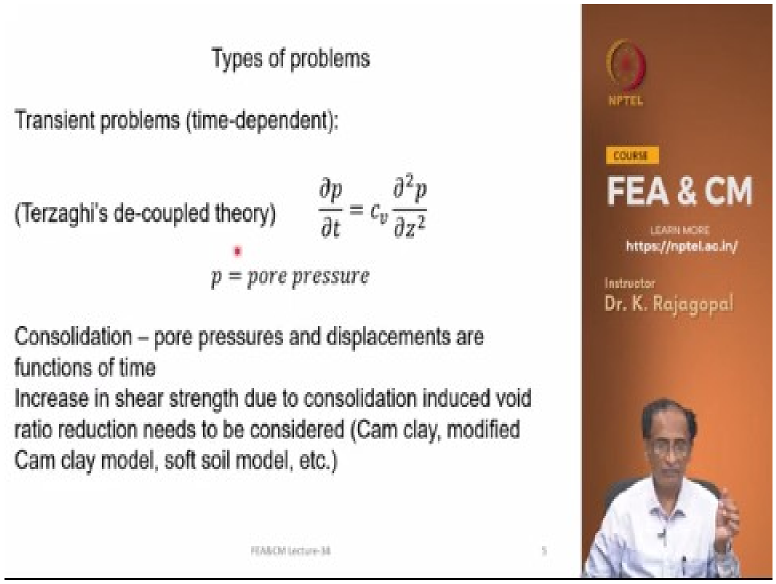
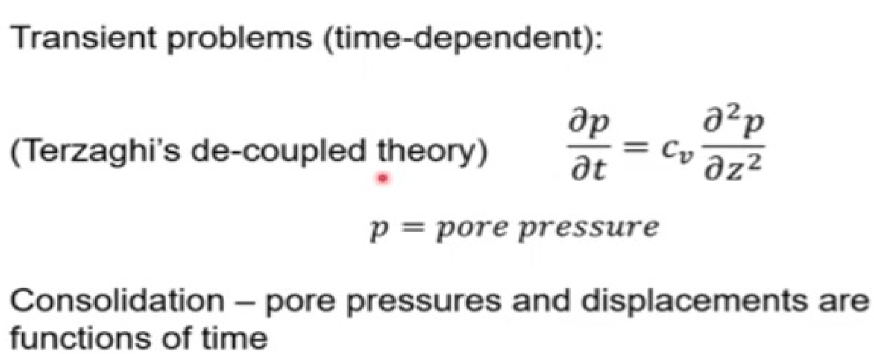


Then we can also do the transient type problems which are time dependent and typical of this is our Terzaghi's one dimensional consolidation equation dou p by dou t is c v dou square p by dou z square where p is the pore pressure and c v is our coefficient of consolidation. And in this analysis, we would like to determine the pore pressure and displacement. But then if we use the Terzaghi's decoupled theory there is only pore pressure as the nodal variable there is nothing else.



There is no displacement and what we do is we estimate the displacements based on the degree of consolidation at different times. But in finite element analysis we can consider both the displacements and pore pressures by considering other type of consolidation theory. And whenever we are doing the consolidation analysis, we should also look at the increase in the shear strength of the soil because of the consolidation induced the compression that leads to a decrease in the void ratio.

And some of the new models latest models like cam clay modified cam clay or soft soil models they account for this consolidation properties and you will see some you will hear some guest lectures on these topics later on.

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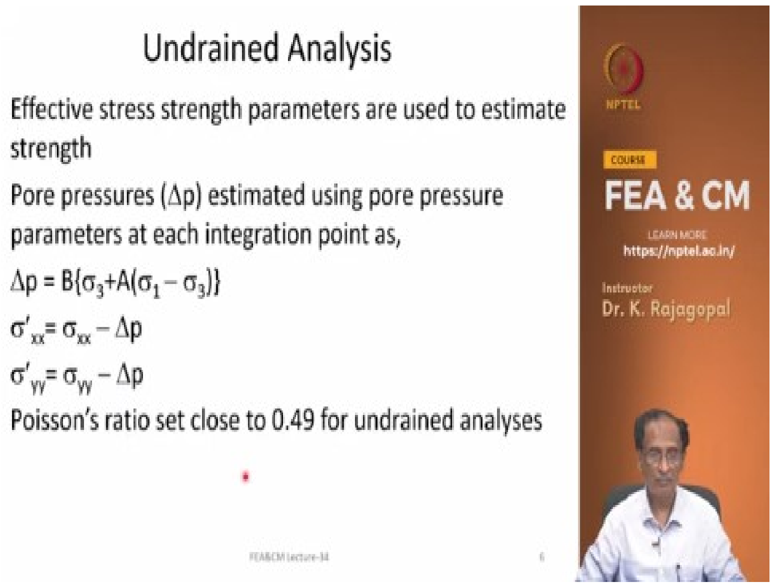

And the simplest analysis with pore pressures is the undrained analysis like assume that your soil or the boundary conditions are such that the pore pressures cannot escape from the soil. And that is an extreme case that we simulate in our consolidation tests either by UU tests or the CU test. And once you do this once you estimate the pore pressures by some other means either by using the Henkel's method or this Skempton's method we can estimate the effective stresses here.

You see the Skempton's equation for estimating the pore pressures in terms of the two principal stresses this is for 2d case B times sigma 3 + A times sigma 1 - sigma 3 where sigma 3 is our confining pressure or the minor principle stress and sigma 1 is the major principles stress sigma 1 - sigma 3 is the increase in the deviator stress and B is your increase in the confining pressure. And once you estimate the pore pressures the effective stresses are the total stress - the pore pressure.

Pore pressures (Δp) estimated using pore pressure parameters at each integration point as,

$\Delta p = B\{\sigma_3 + A(\sigma_1 - \sigma_3)\}$

$\sigma'_{xx} = \sigma_{xx} - \Delta p$

$\sigma'_{yy} = \sigma_{yy} - \Delta p$

Poisson's ratio set close to 0.49 for undrained analyses

And when we are doing the untrained analysis, we set the poisons ratio close to 0.49 maybe close to we can even set it to 0.499 depending on the precision that you use. And if we are doing the effective stress analysis, we need the effective stress shear strength parameters.

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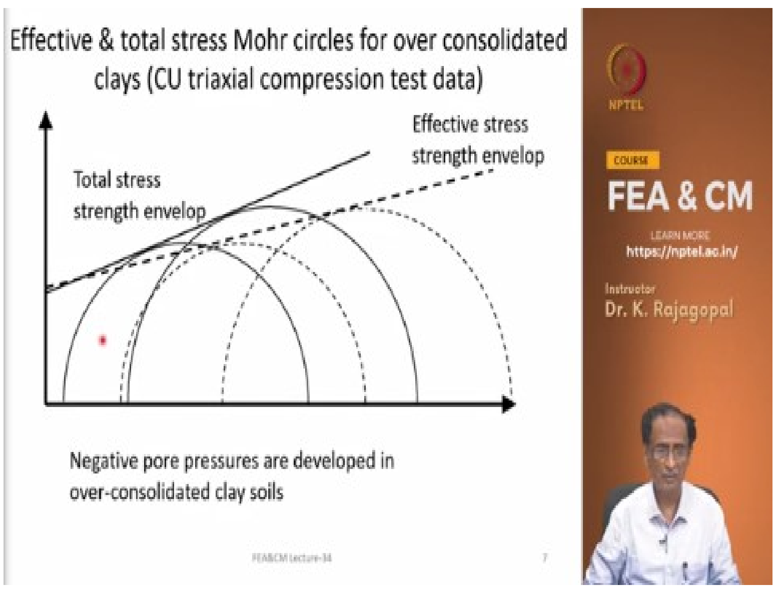

See here you see the strength envelopes for both total stress and effective stress Mohr circles. The solid lines the solid Mohr circles are the Mohr circles corresponding to our total stresses. And this particular data is for over consolidated clays where you develop negative pore pressure. So, there is an increase in the effective stresses under undrained conditions so, you see the Mohr circles moving to the right and this dotted line is your effective stress strength envelope.

And the intercept and the slope they give you the c and phi values and we can use these c prime phi prime along with the effective stresses that we determine through our finite element analysis and then we can perform all the calculations like our estimation of bearing capacity or the shear strength and so on.

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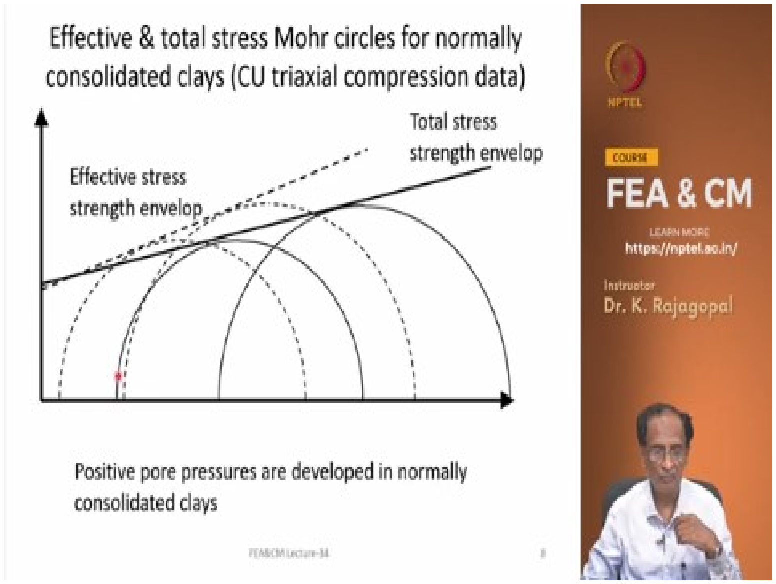

And the strength envelope for normally consolidated places like this is actually here we say we develop positive pore pressures under undrained loading and our effective Mohr circles will shift to the left because their effective stress reduces compared to the total stress. And will have a higher friction angle compared to that of the total stress strength parameters. By using one of these methods, it is easy to modify the previously discussed the methods.

For doing all these constitute modelling whether it is a bilinear elastic model or our hyperbolic model. Instead of using sigma 3 we will be using sigma 3 prime and then instead of sigma 1 we will be using sigma 1 prime by estimating the effective stresses. And we can perform the same analysis whatever we have done earlier in terms of the effective stresses.

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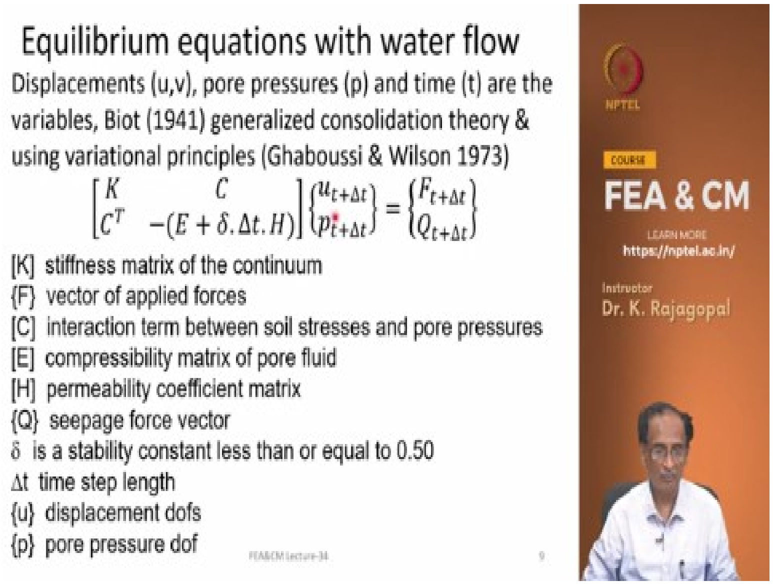

And the other thing that we normally do is the consolidation analysis. That is what happens if you have a drainage that is happening through the soil and the pore water pressure continues to dissipate with the time and then the effective stresses increase. And how do we do that analysis is actually the one approach that we have is by Terzaghi but then Biot in 1941 he proposed a generalized consolidation theory which we call as the couple theory.

Equilibrium equations with water flow

Displacements (u,v), pore pressures (p) and time (t) are the variables, Biot (1941) generalized consolidation theory & using variational principles (Ghaboussi & Wilson 1973)

$$\begin{bmatrix} K & C \\ C^T & -(E + \delta.\Delta t.H) \end{bmatrix} \begin{Bmatrix} u_{t+\Delta t} \\ p_{t+\Delta t} \end{Bmatrix} = \begin{Bmatrix} F_{t+\Delta t} \\ Q_{t+\Delta t} \end{Bmatrix}$$

[K] stiffness matrix of the continuum
{F} vector of applied forces
[C] interaction term between soil stresses and pore pressures
[E] compressibility matrix of pore fluid
[H] permeability coefficient matrix
{Q} seepage force vector
δ is a stability constant less than or equal to 0.50
Δt time step length
{u} displacement dofs
{p} pore pressure dof

Because it couples both the displacements and then the pore pressures together along with the compressibility of the soil. And by using that and by using the variational principles Ghaboussi and Wilson they adapted the Biot's theory for finite element analysis. It is actually the Biot's equation is for pure mathematical analysis but then when you have any sinusotropic soil or non-homogeneous soil or with complicated boundary conditions.

We have to use some numerical method for doing the analysis and Ghaboussi and Wilson they derived the finite element equations. Here I am not showing you the derivation but if you are interested you can refer to this paper and the details of this are given in the next slide. We can actually come out with the equilibrium equations and then the for the stresses are the forces and then based on the continuity equation.

And the flow into the volume is equal to the flow out of the volume are if there is a compressibility, we can include that in our continuity equation and we can derive this equation where u is the displacement vector p is the pore pressure and the t + delta t refers to the next time step. Because actually in all our problems with time we have a time also as a variable and these displacements they will go on changing with the time.

And on the right hand side F is the applied force vector the Q is the seepage force vector the Q is calculated based on the import pressures and here the K is our stiffness matrix of the continuum C is the interaction matrix between the pore pressures and then the soil skeleton and here we have the E is the compressibility matrix of the pore fluid and delta is the stability constant usually taken as about 0.5 and delta t is the time step length and H is the permeability coefficient matrix.

And this could be the isotropic or anisotropic it does not matter we can give any variation that you want. And by solving this we can get our displacements under the pore pressures. So, actually it is a very time-consuming process because the stability depends on the delta t and so we have to do as time stepping process to get the solution for the next time steps.

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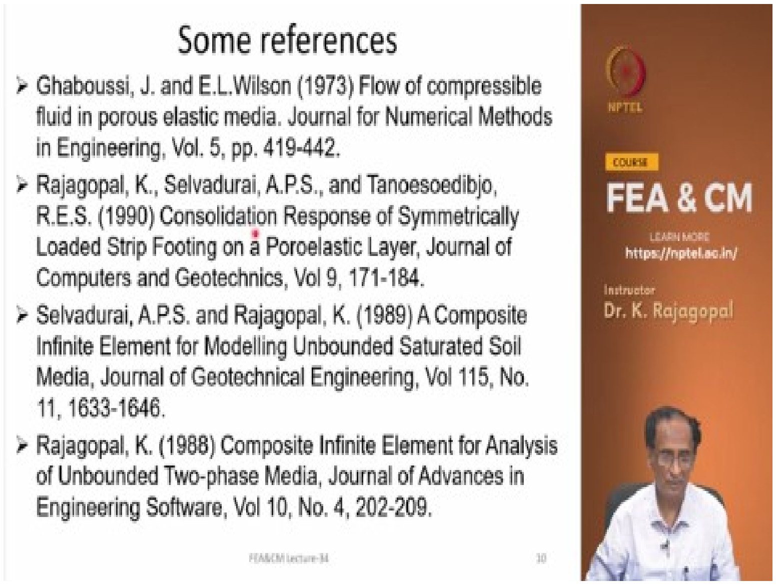

And some references are given here Ghaboussi Wilson in 1973 they developed that previously shown finite element equations that we can solve for getting the solutions.

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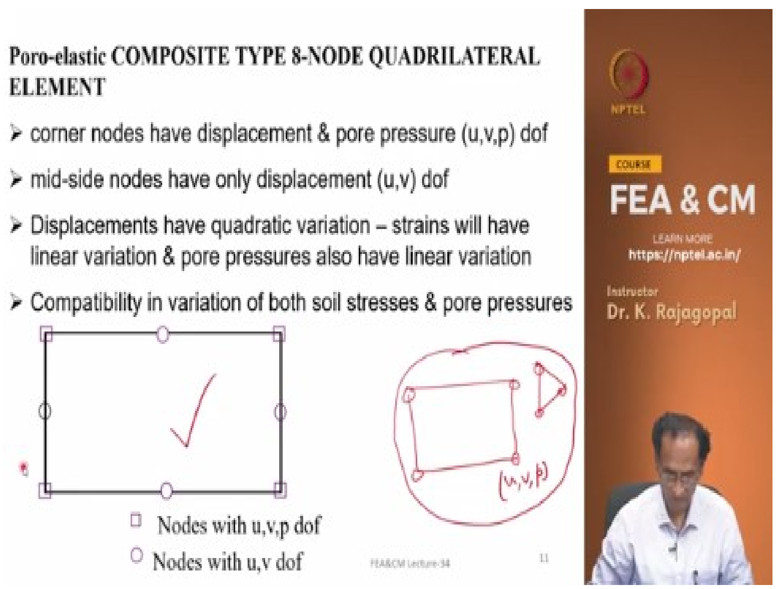

And for solving this type of problems that involve both the displacements and then the pore pressures we need some other type of continuum element, that is called as a pore elastic composite type element. And here you see one eight node quadrilateral with eight nodes four cardinal nodes and then the four mid side nodes. And the pore pressures are defined only at the corner nodes whereas the displacements are defined at all the eighth nodes.

So, the corner nodes they have both the displacements and then the pore pressures as the nodal variables whereas the mid side nodes they have only the displacements as the degrees of freedom. See the idea is that there should be a compatibility in the variation of strain and then the pore pressure. The strain means its stress it is the strain and stressor proportional to each other and it is a quadratic equation in terms of the displacements and the first derivative is your strain.

And that is also the order for the stress and your pore pressures because we have only two values one at each end there is a linear variation and then the pore pressures also will have the same linear variation. And this type of elements that ensure the compatibility in the variation between the between the soil stresses and then the pore pressures are more accurate compared to other elements where they define the pore pressures and displacements at all the nodes like.

For example, if you take a four node quadrilateral let me just see let us say you have a four node quadrilateral or a three node triangle. And by necessity we have to define the displacements and then the pore pressures at all the nodes and so that means that there is no compatibility between the between the pore pressures and then the stresses. So, the results that you get with this type of elements may not be as accurate as those we get with this type of elements.

So, for all these analyses we prefer using this type of composite type elements and when we are doing this consolidation analysis will have a problem. We see a negative term here and let us say you have an incompressible like E is the compressibility matrix and if you have an incompressible pore fluid that is the normal usual assumption E is going to be 0. And for time step or if you are interested in knowing the initial response at time t = 0 delta t is 0.

So, you have zero diagonal terms corresponding to the pore pressure. So, we know that we cannot have any zero diagonal terms because we will not be able to invert the matrix or use our Gauss elimination process.

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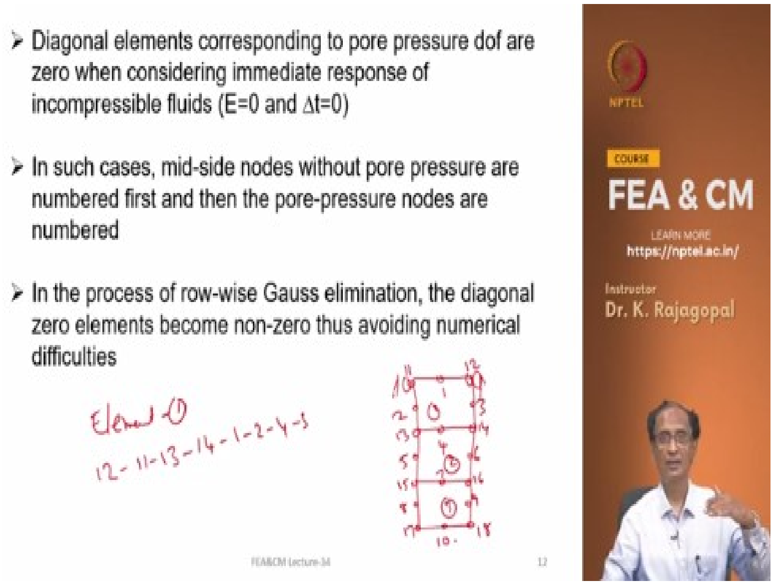
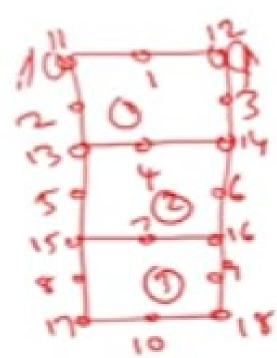


And for that we have a separate procedure what we do is we start numbering all the mid side nodes that do not have a pore pressure first. Say when you number a mesh you can arbitrarily give any node numbers like 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 let us say all these all the mid side nodes that do not have the pore pressures can be numbered first. And then the sorry let us say we are doing a one dimensional consolidation analysis.

Let us take a simple mesh like this, let us assume that all our elements are right node quadrilaterals with corner nodes having both the displacements and then the pore pressures are as nodal variables and only the mid side nodes they have the displacements. So, what we do is we can start numbering the mid side nodes first and then we can number these corner nodes. There are totally 18 elements in this mesh and we have three elements.



And so, the element one will have this it is defined between eight nodes say 12, 11, 13, 14, 1, 2, 4, 3 these are the eight notes of element one. Similarly for element 2 and element 3 we can define and since we have numbered only the on the nodes with the displacement first will have all non-zero diagonal terms say up to 10 rows means sorry 10 degrees 10 nodes means 20 degrees of freedom.

So, starting from 21st degree of freedom we may have some rows with some diagonal elements with the 0 and in the process of Gauss elimination these zeros will turn into to some non-zero numbers and then we can avoid the numerical difficulties. And this is one trick that is normally played out for getting the immediate solution. So, this and the consolidation analysis is actually it is usually done with this type of eight node the quadrilaterals.

And by using the Bios theory we get the direct solution of both the displacements and then the pore pressures at any time. And if we work with advanced constitutive models like the soft clay model the modified cam clay model, we can also incorporate the influence of the increase in the soil strength because of consolidation. And so that is an advancement so especially if you are doing analysis of any construction and a soft clay, we should go in for consolidation analysis.

And then go for models like modified cam clay or soft clay models that can include the influence of the void ratio and the strength of the soil. So, you will hear some lectures on these topics later on through some guest lectures through my colleagues in the department.

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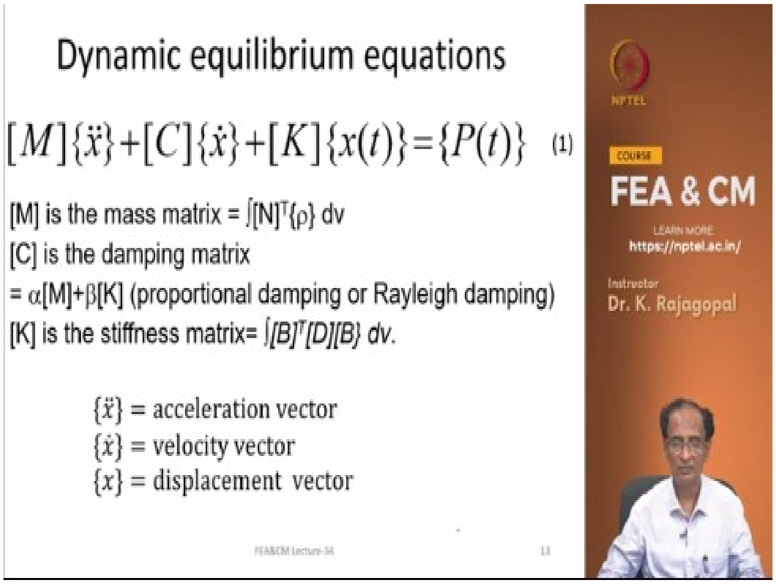

Now let us briefly see the dynamic analysis of the soil structures and the dynamic equilibrium equations they involve acceleration and then velocity and also the displacements like typical equation is a M x double dot + C x dot + K x = P where P is our load vector and that itself could be a function of time. It is not a steady state load it is not a constant load it could vary either in a sinusoidal form or arbitrary form.

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x(t)\}=\{P(t)\} \quad (1)$$

[M] is the mass matrix = $\int[N]^T\{\rho\}$ dv

[C] is the damping matrix

= α[M]+β[K] (proportional damping or Rayleigh damping)

[K] is the stiffness matrix= $\int[B]^T[D][B]$ *dv*.

$\{\ddot{x}\}$ = acceleration vector

$\{\dot{x}\}$ = velocity vector

$\{x\}$ = displacement vector

And in this M is the mass matrix that we can write as N transpose rho dv this is called as the consistent distribution of the mass and the C is the damping matrix. And could be written as alpha in terms of the mass and stiffness is alpha M + beta K this is called as the proportional damping or Rayleigh damping. And it is usually it is easy because once you know the stiffness matrix and mass matrix you can get your C.

And based on the values that you choose for alpha and M you can either have a mass proportional damping or stiffness proportional dump in which one is more predominant that you can decide. And the K is the stiffness matrix the usual stiffness matrix that is our B transpose B dv and in here x double dot is the acceleration x dot is the velocity and x is the displacement. And so how do we solve this and it is a time dependent problem.

And we have at each node we have acceleration and velocity and also displacement as the variables. So, are we going to consider all these three as nodal variables are only one of them that actually if we consider all the three it becomes the problem becomes very very complicated because you will have a very large number of degrees of freedom to solve for.

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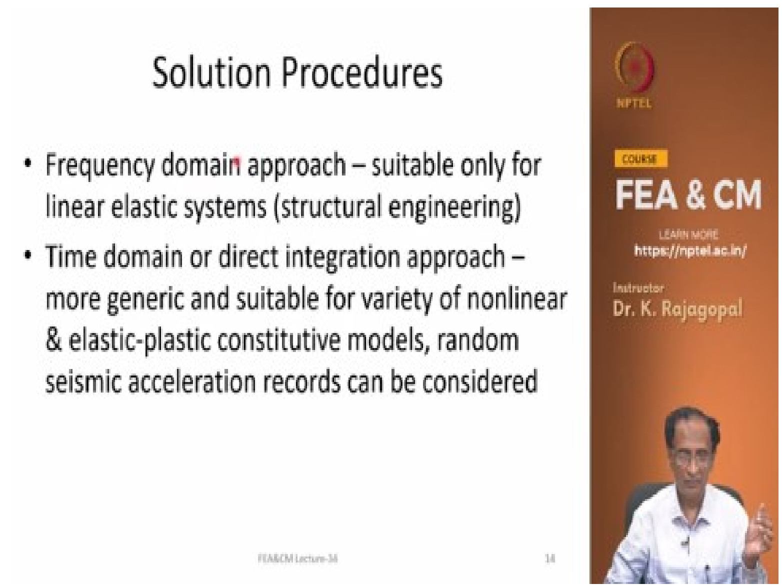

And so, what we do is we can we can approximate by different means the one method is the frequency domain approach. This is suitable for linear elastic systems where we get the solution separately for different nodes and then superimposed and that is typical for structural engineering problems. And there we cannot incorporate the effect of any non-linearity or the plastic deformation and so on.

Then there is another method called as the time domain method or the direct integration approach and this is more generic and suitable for any type of non-linear problems or visco plastic viscoelastic or elastic plastic and so on. And we can also include the response under random seismic loading and so on. Whereas the frequency domain approach is good but it is only valid for some simple problems.

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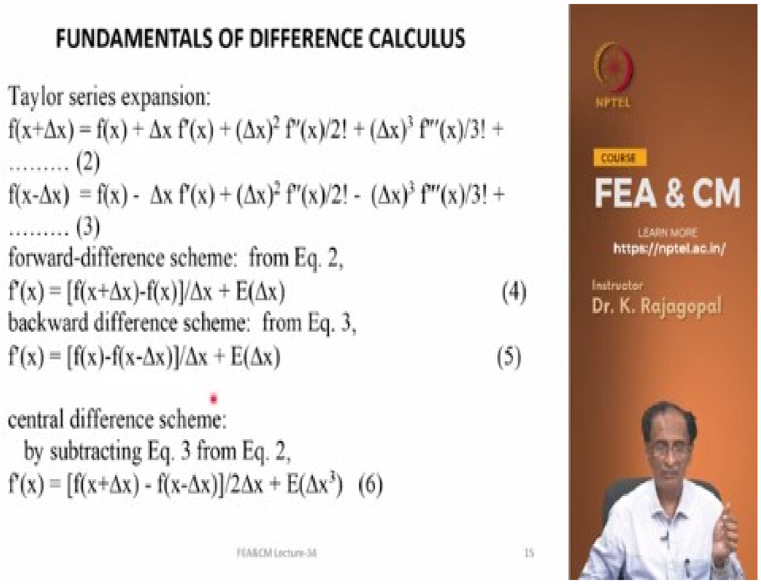

And we can convert our given equations into more simple form by using the difference equations here we see the Taylor series expansion of any function f of x around small variation delta x it is f of x + delta x is f of x + delta x times f prime x + delta x square f double prime under so on and f x - delta x like you can either add delta x or subtract delta x and then these are the Taylor series expansions.

Taylor series expansion:

$$f(x+\Delta x) = f(x) + \Delta x\, f'(x) + (\Delta x)^2 f''(x)/2! + (\Delta x)^3 f'''(x)/3! + \ldots\ldots\ldots \quad (2)$$

$$f(x-\Delta x) = f(x) - \Delta x\, f'(x) + (\Delta x)^2 f''(x)/2! - (\Delta x)^3 f'''(x)/3! + \ldots\ldots\ldots \quad (3)$$

forward-difference scheme: from Eq. 2,

$$f'(x) = [f(x+\Delta x)-f(x)]/\Delta x + E(\Delta x) \quad (4)$$

backward difference scheme: from Eq. 3,

$$f'(x) = [f(x)-f(x-\Delta x)]/\Delta x + E(\Delta x) \quad (5)$$

central difference scheme:

by subtracting Eq. 3 from Eq. 2,

$$f'(x) = [f(x+\Delta x) - f(x-\Delta x)]/2\Delta x + E(\Delta x^3) \quad (6)$$

And by manipulating this we can develop the forward different schemes backward different schemes central different schemes and so on. And the central difference scheme is the most more preferred one because you can get convergence faster as we see here so, if you want to get the f prime x by forward or backward different scheme the we can neglect all the higher order terms from delta x square.

And then we can write f prime of x is f of x + delta x - f of x by delta x and the error is of the order delta x because all the terms above that are eliminated. Whereas if you look at the backward different scheme is also the same by looking at this, we can write f prime of x is f of x - f x - delta x by x delta x and by central different scheme if you add up these two equations our sorry if you subtract f prime of x can be written as f of x + delta x - f of del x - delta x by 2 delta x.

And then this term gets eliminated and you have the order of delta x to the power 3 the error is of this order and if delta x itself is small delta x cube will be even smaller.

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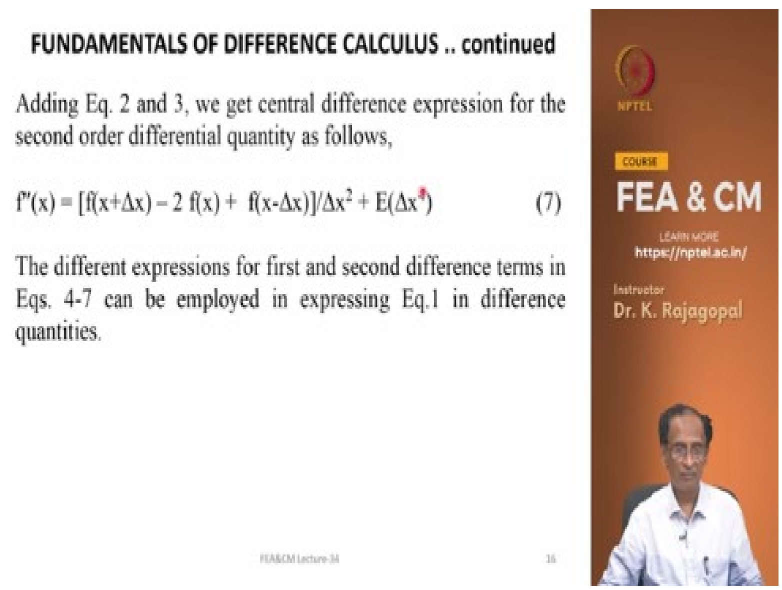

And we can also determine the double prime f double prime by adding these two equations two and three and we get slightly different equation and here the error is of the order delta x to the power 4 because we are neglecting the terms starting from delta x to the power 4. And this equation for f prime x is actually our x dot this is related to your first derivative that is a velocity term and this is our second derivative related to our acceleration term.

$$f''(x) = [f(x+\Delta x) - 2 f(x) + f(x-\Delta x)]/\Delta x^2 + E(\Delta x^4) \qquad (7)$$

And we can substitute this in our governing equation M x double dot + C x dot + K x = P and convert the entire equation into only in terms of displacements or different time steps.

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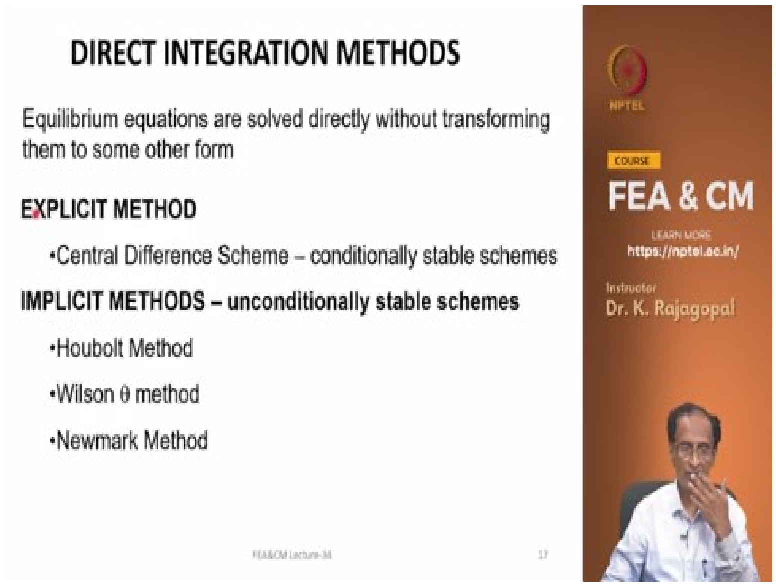

And when we do that, we can do in two different ways one is an explicit method central different scheme conditional stable algorithm is actually these explicit methods. They are only conditionally stable whereas implicit method is actually we have the unknown terms both on the left hand side and the right hand side. Whereas in the explicit method all the unknown terms are on the left hand side on the right hand side the we know all the quantities.

They refer to the previous time step whereas in the implicit method the terms unknown terms are both there on the left hand side and the right hand side.

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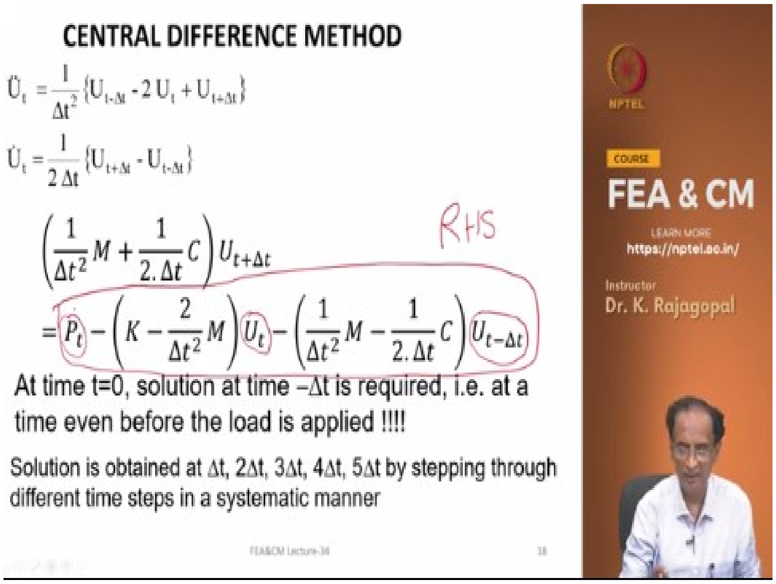

The central difference scheme is like this our acceleration at time t we can get by looking at the displacements at two different time steps U t + delta t and U t - delta t and then the time the displacement at this time step U t like this and the velocity at time t is the difference in the velocities at time t + delta t and U t - delta t by 2 delta t and we can substitute these two expressions in our governing equation and then we can convert our equation.

And we get a matrix like this sorry server the U double dot and U dot once you place and then our displacement at time t is a known quantity and we take to the right hand side. Whereas the displacements at time t + delta t are unknowns and so we retain them on the left hand side and so we have on the right hand side the load at time t - K - 2 by 2 delta square M times U t and so on. And then on the left hand side we have some effective stiffness matrix M by delta t square + C by 2 delta t times displacement at the next time step.

CENTRAL DIFFERENCE METHOD

$$\ddot{U}_t = \frac{1}{\Delta t^2}\{U_{t-\Delta t} - 2\,U_t + U_{t+\Delta t}\}$$

$$\dot{U}_t = \frac{1}{2\,\Delta t}\{U_{t+\Delta t} - U_{t-\Delta t}\}$$

$$\left(\frac{1}{\Delta t^2}M + \frac{1}{2.\Delta t}C\right)U_{t+\Delta t}$$
$$= P_t - \left(K - \frac{2}{\Delta t^2}M\right)U_t - \left(\frac{1}{\Delta t^2}M - \frac{1}{2.\Delta t}C\right)U_{t-\Delta t}$$

At time t=0, solution at time $-\Delta t$ is required, i.e. at a time even before the load is applied !!!!

Solution is obtained at Δt, $2\Delta t$, $3\Delta t$, $4\Delta t$, $5\Delta t$ by stepping through different time steps in a systematic manner

And so, we can repeatedly solve this equation and then get our solution. But let us say we are interested in first knowing the response at time t = 0. So, if you substitute t = 0 that is the response at delta t we have this U 0 that is our initial conditions but then if you look at this the displacement at the previous time step U at - delta t, that is even before we start applying the load, we should know what happened before that.

And but there is a method for getting that but once you get it then it is a simple recursive equation, we go on getting the solution at time delta t, 2 delta t, 3 delta t and so on.

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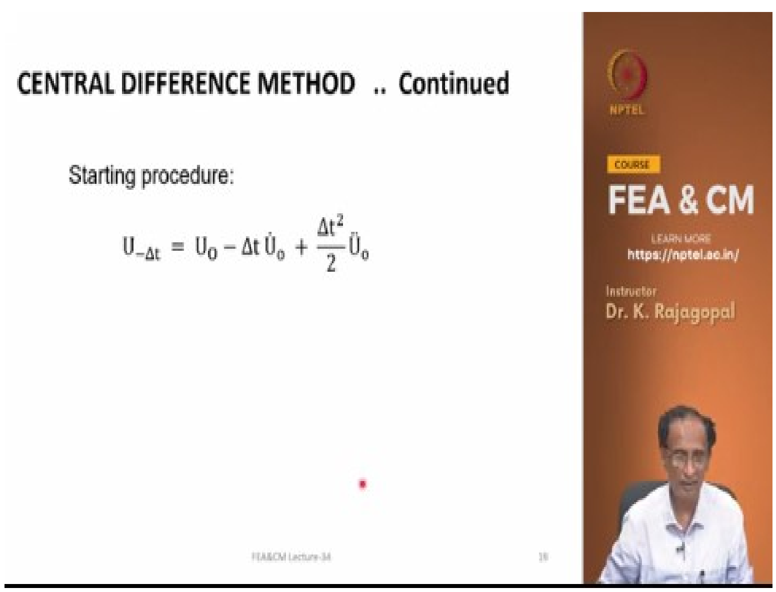

By time stepping schema by the time marching scheme and the starting procedure the displacement at time - delta t we can get by setting this to the time t to 0. And then we can get the displacement at the previous time step as U 0 - delta t U dot 0 + delta t square by 2 accelerations at initial time step. And once you have this, we can do the time marching scheme.

Starting procedure:

$$U_{-\Delta t} = U_o - \Delta t\,\dot{U}_o + \frac{\Delta t^2}{2}\ddot{U}_o$$

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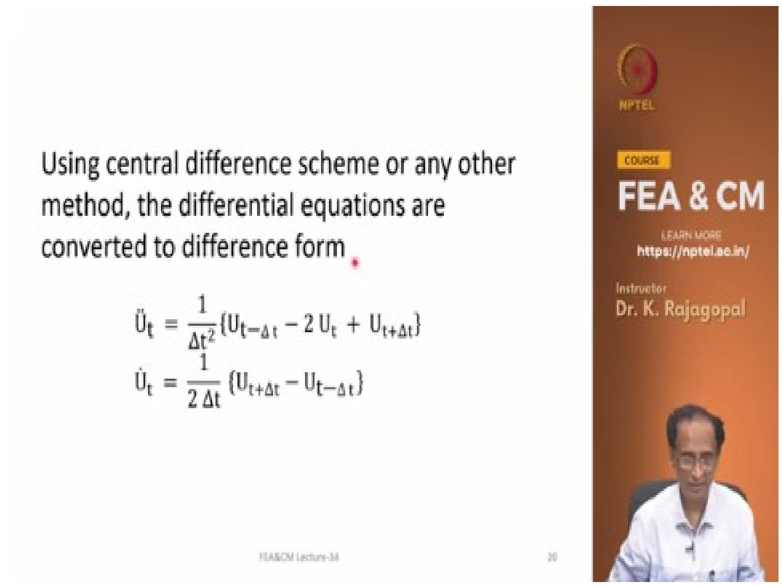

And so, this is what we can do for converting our differential equation in terms of the displacements. Displacements are different time steps t + delta t t - delta t and so on and this is the explicit method where we have all the known quantities on the right hand side. So, if you look at this all the right hand side terms they are either corresponding to current time t or the previous time t - delta t.

$$\ddot{U}_t = \frac{1}{\Delta t^2}\{U_{t-\Delta t} - 2\,U_t + U_{t+\Delta t}\}$$
$$\dot{U}_t = \frac{1}{2\,\Delta t}\{U_{t+\Delta t} - U_{t-\Delta t}\}$$

And so, we can and of course the load at this current time t and we can easily form the right hand side vector and then the left hand side is the unknown the displacement at U t + delta t and this is our explicit method. Our explicit equation for solving and it is a good procedure but only thing is it is conditionally stable that we will see with the numerical example for some delta t's you might not get any convergence.

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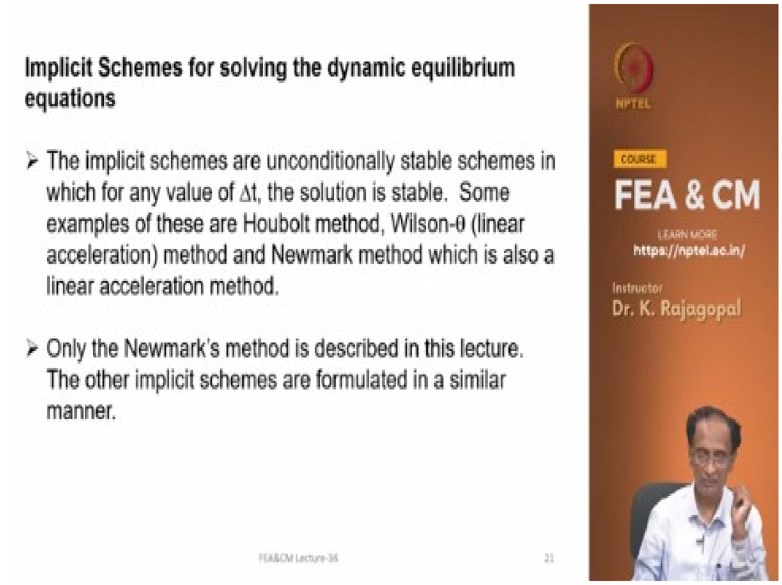

And there are other methods called as implicit methods my laser pointer and these implicit schemes are unconditionally stable whatever may be the delta t you get a stable solution. And some methods are by Houbolt Wilson theta and then Newmark method. In this lecture I am explaining only the Newmark method the other methods are very similar like very similar to Newmark method and can refer to any good textbook for getting the details of the other methods.

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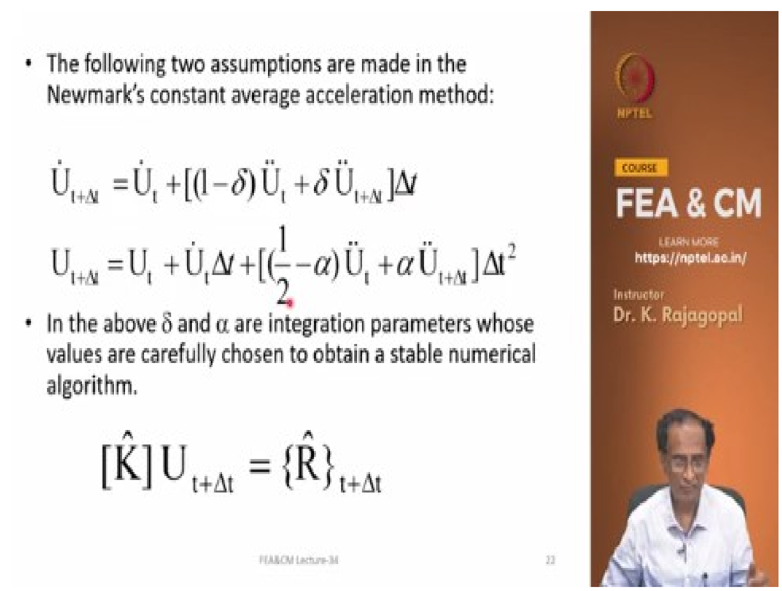

And in the Newmark method is called as the average acceleration method the velocity at time step the next time step t + delta t we can write in terms of the velocity at time t + 1 - delta acceleration at time t and then delta times acceleration at time t + delta t multiplied by delta t. See here is the velocity + acceleration times delta t is your velocity. So, you see here you see the implicit nature of this equation because you have the velocity at time t + delta t written in terms of acceleration at t + delta t.

$$\dot{U}_{t+\Delta t} = \dot{U}_t + [(1-\delta)\ddot{U}_t + \delta\ddot{U}_{t+\Delta t}]\Delta t$$

$$U_{t+\Delta t} = U_t + \dot{U}_t\Delta t + [(\frac{1}{2}-\alpha)\ddot{U}_t + \alpha\ddot{U}_{t+\Delta t}]\Delta t^2$$

And t + delta t refers to the next time step for which we need to know the solution we need to determine the solution. And similarly, our displacement at time t + delta t we can write like this the displacement at time t + the velocity at time t multiplied by delta t + the average acceleration multiplied by delta t square. And this delta and alpha are called as integration parameters we have to choose them so that we get stable algorithm.

$$[\hat{K}]\, U_{t+\Delta t} = \{\hat{R}\}_{t+\Delta t}$$

And if you substitute these equations in our governing equation M x double dot + C x dot + K x = P of t. We can convert them into some equivalent stiffness matrix K hat U t + delta t is the force vector at t + delta t and the force vector is a known quantity and then our displacement at the next time step t + delta t is the unknown quantity.

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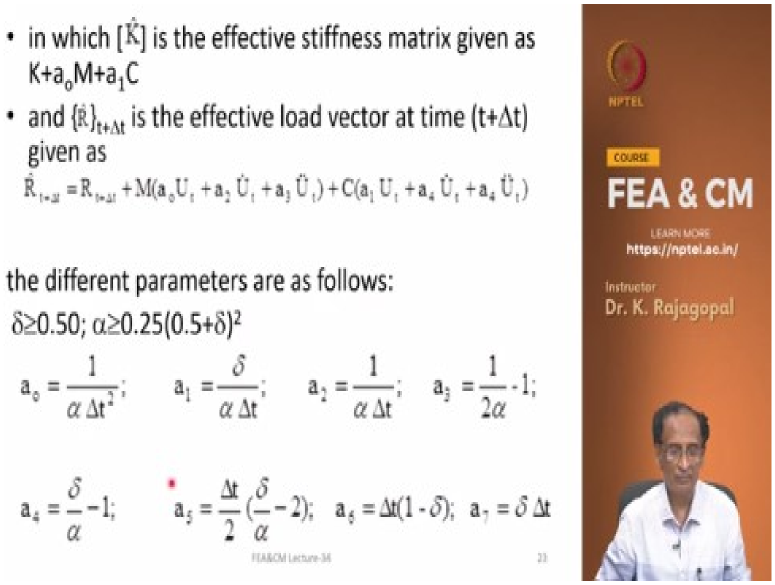
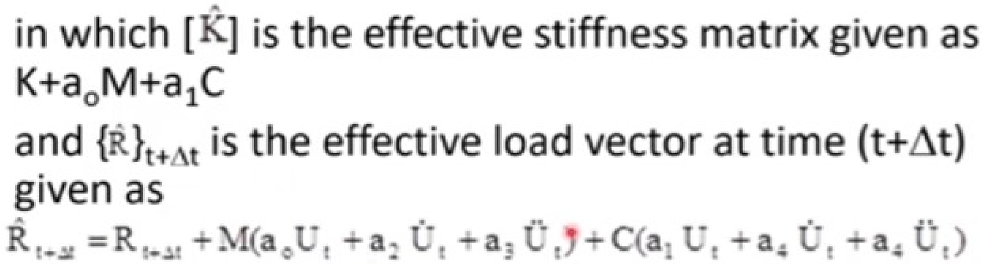


And the K hat is written as K + a 0 M + a 1 C where a 0 and a 1 these are some constants that we will see later and R hat is the effective load vector at time t + delta t that is sum total of the applied loading plus mass times all this quantities and the C times all these quantities which are all dependent on the values at the previous time the displacement at time t velocity at time t acceleration at time t and so on.



And so, the load vector we can easily determine because R t + delta t is the applied load and so we know what is that applied load. And our delta and alpha are chosen like this delta should be greater than or equal to 0.5 usually we take it as 0.5 and alpha should be greater than 0.25 times 0.5 + delta whole square. And if delta is taken as 0.5 alpha comes out as 0.25 and a 0, a 1, a 2, a 3 and all these parameters they are given here.

the different parameters are as follows:

$\delta \geq 0.50$; $\alpha \geq 0.25(0.5+\delta)^2$

$$a_0 = \frac{1}{\alpha\,\Delta t^2}; \quad a_1 = \frac{\delta}{\alpha\,\Delta t}; \quad a_2 = \frac{1}{\alpha\,\Delta t}; \quad a_3 = \frac{1}{2\alpha} - 1;$$

$$a_4 = \frac{\delta}{\alpha} - 1; \quad a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right); \quad a_6 = \Delta t(1 - \delta); \quad a_7 = \delta\,\Delta t$$

Basically, we get them by solving by substituting these terms in the governing equation and the simplifying we get these terms and the delta t is the time step length that we can fix.

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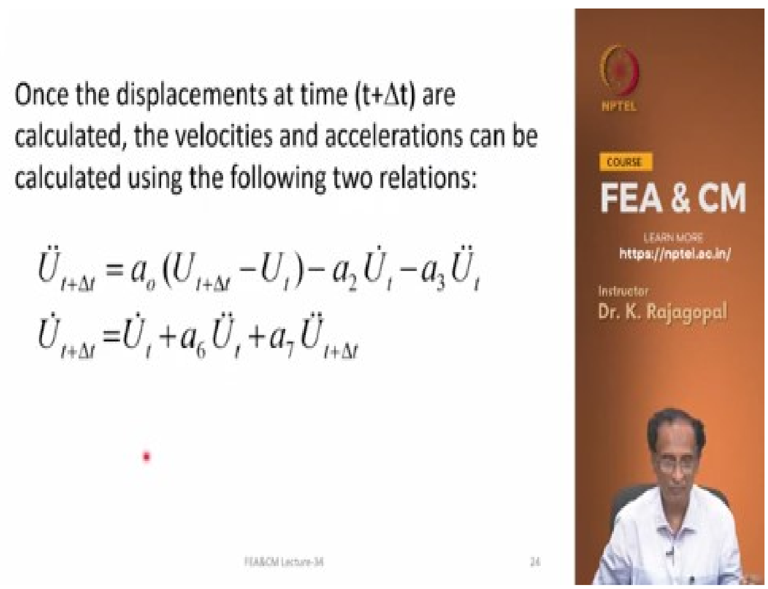

And once the displacements at time t + delta t are determined then the acceleration at the next time step is a 0 times U t + delta t - U t – a 2 times U dot t – a 3 U double dot t and then the velocity at time t + delta t is obtained like this is. Actually, basically we can use these equations to get these quantities because we know what is acceleration at a time t + delta t and then we substitute that and then get our velocity at the next time step.

Once the displacements at time (t+Δt) are calculated, the velocities and accelerations can be calculated using the following two relations:

$$\ddot{U}_{t+\Delta t} = a_o (U_{t+\Delta t} - U_t) - a_2 \dot{U}_t - a_3 \ddot{U}_t$$
$$\dot{U}_{t+\Delta t} = \dot{U}_t + a_6 \ddot{U}_t + a_7 \ddot{U}_{t+\Delta t}$$

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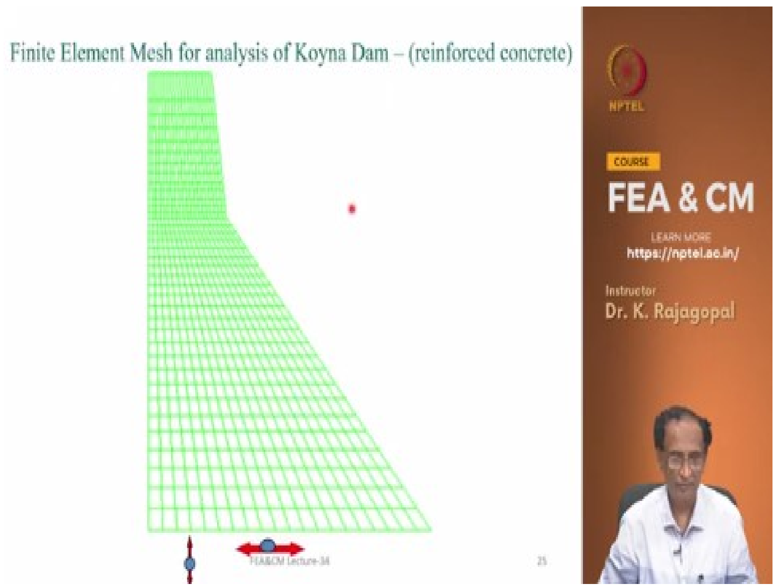

So, let me show you one example this particular one is from this particular one is the finite element mesh for Koyna Dam. And the Koyna dam is a reinforced concrete dam and this particular result that I am going to show is from Abacus program. And it was subjected to some earthquake acceleration because of the hydrostatic pressures and then because of the water penetrating into the ground.

And that has caused the some earthquake and basically it is not an earthquake prone zone but then it is a subjected earthquake and the geologists think that because of the voids in the ground this earthquakes happen.

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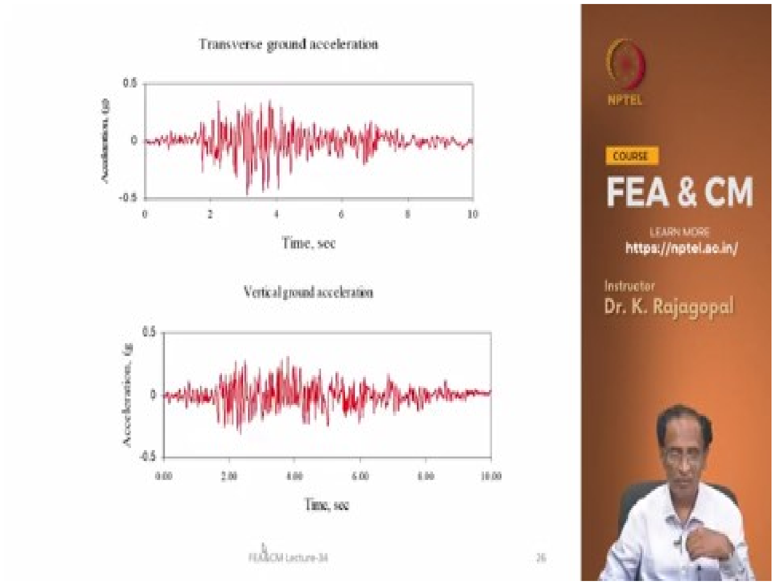

And the recorded that quick accelerations are given here. This is the transverse ground motion and then this is the vertical ground acceleration. The acceleration with the time and the record is given for 10 seconds and this was applied at the ground level like the we take all the nodes corresponding to the base and apply the acceleration. Both acceleration the horizontal direction and then the vertical direction and we can get our displacement.

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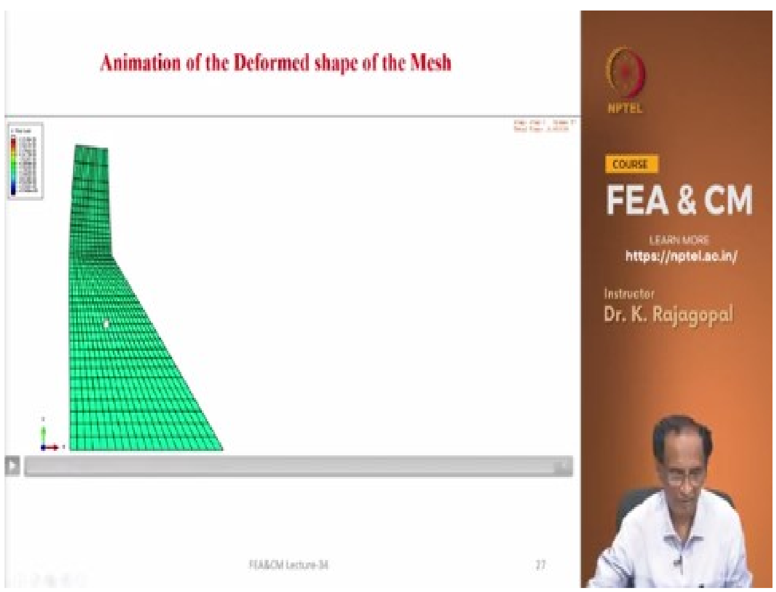

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And here is an animation for the displacement.

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See basically what the program has done is it has performed the analysis saved the displacements at different time steps and then plotted. The displacement shape are different time steps continuously it has started plotting and it comes out as in an animation form like this.

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And we are interested in knowing these stresses and because of this the change in the cross section here this is the most critical area where the concrete was subjected to a lot of tensile stresses that we can see through this animation.

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We can see the red dots that indicate the tensile stress the all these red dots they indicate the locations where the tensile stresses are more than the capacity of the concrete.

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And we can assume that all these zones have failed.

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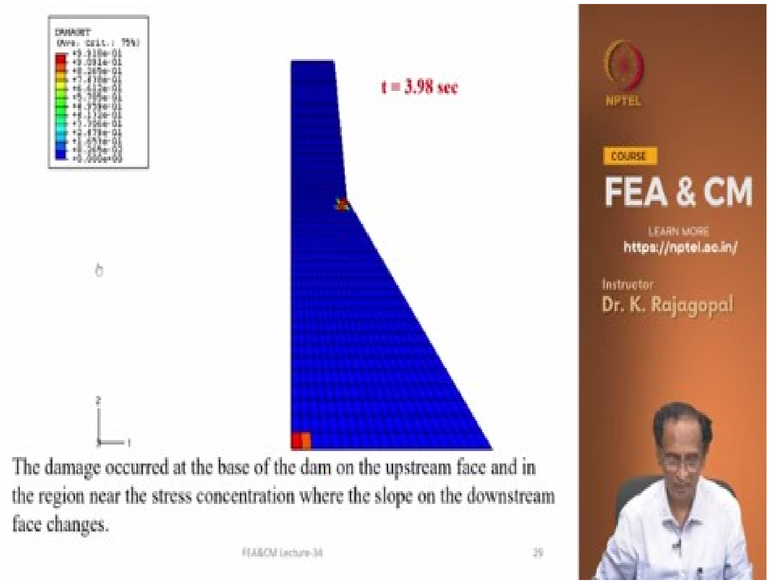

In fact, when you compare the damage, they in that actually happened in the dam or to the finite element predictions they both matched very nicely.

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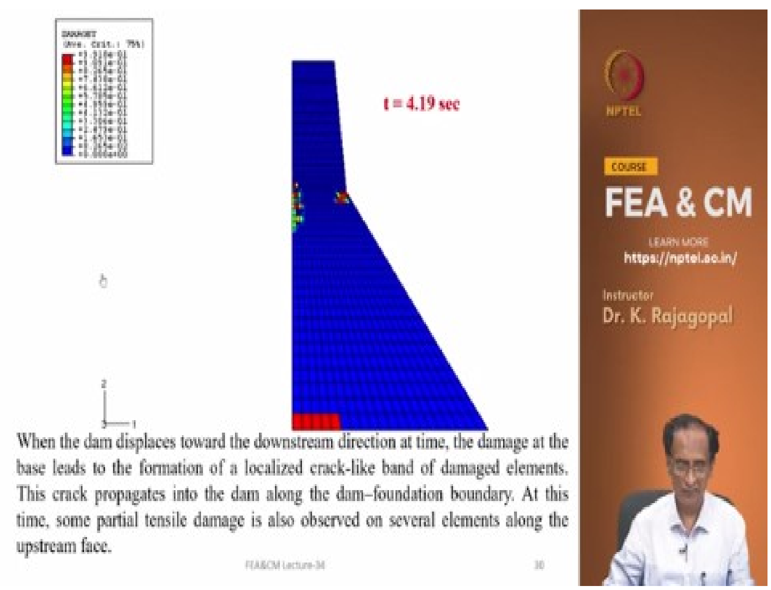

And different times there is a different damage.

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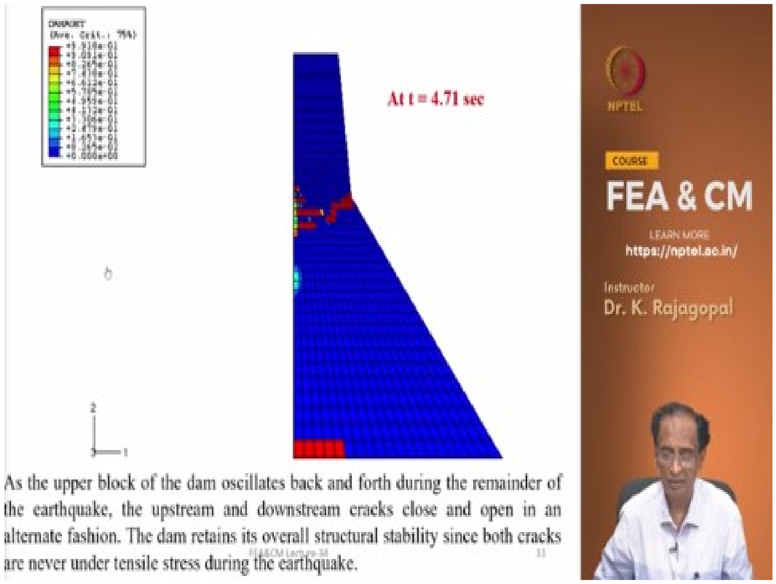

And that these are all the red zones these are the damaged portions.

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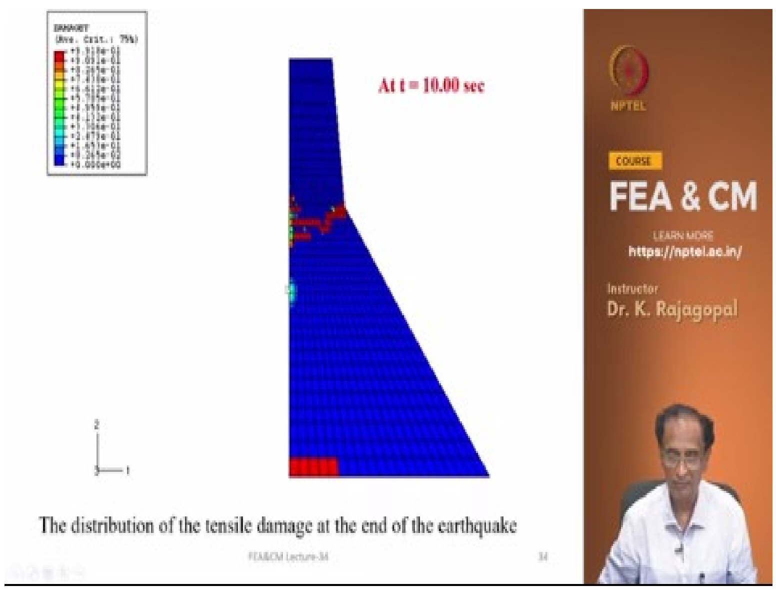

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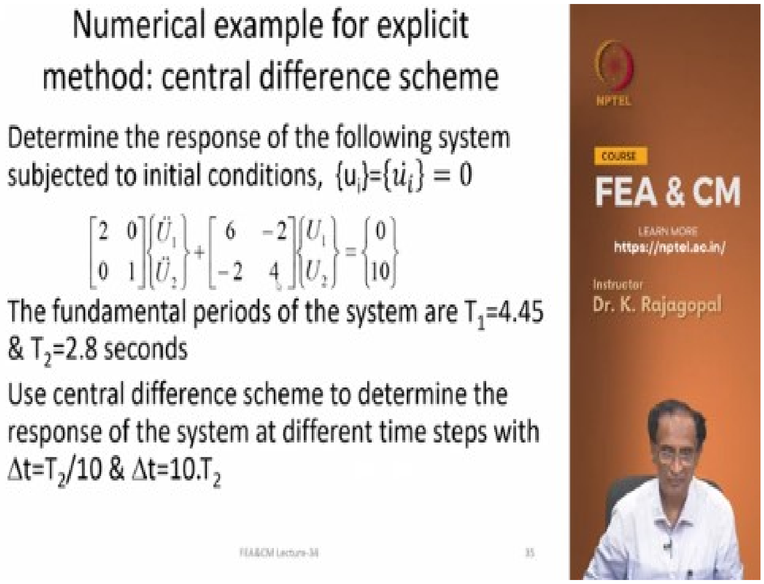
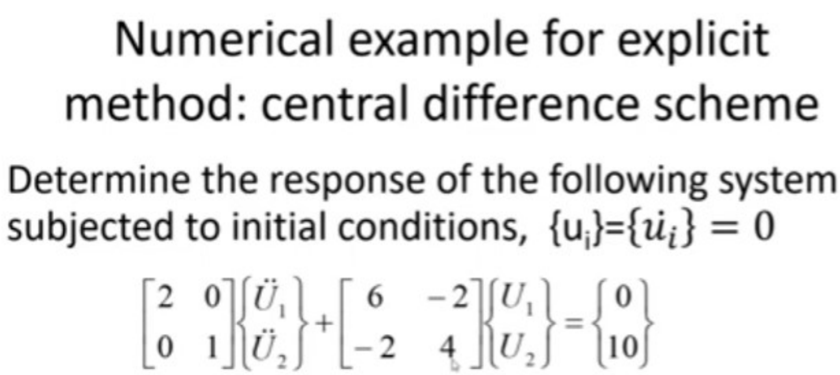


And is actually this particular example that I showed was solved by this Newmark method by the implicit method. And let me demonstrate one simple numerical example this I am showing only for understanding purpose and the similar procedure is also followed in finite elemental analysis. But then you will be dealing with the much larger problems. Let us consider a system a two degree of freedom system without any damping.



And this is our mass matrix and there are only two degrees of freedom, the acceleration at degree of freedom 1 acceleration and degree of freedom 2. And then the stiffness matrix the displacements U 1 and U 2 is subjected to loading the instantaneous loading. And then we want to know what happens with the time what are the displacements and what are the other quantities. And there are two degrees of freedom.

So, we will have two fundamental periods the T 1 is 4.45 and T 2 is 2.8 this is a corresponding to mode 1 and this is corresponding to mode 2. And usually, the fundamental the first vibration mode will have a higher time period compared to the higher orders of vibrations. So, T 2 is a 2.8 T 1 is 4.45 and let us solve this system by using the explicit method the central difference method with the delta t of T 2 by 10 and delta T of 10 times T 2.

As the initial displacements & velocities at t=0 are given as 0, the initial accelerations can be obtained directly from,

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{Bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

Initial accelerations are {0, 10}. If time increment, $\Delta t=0.28$, displacements at time step, $t=-\Delta t$ can be obtained from

$$U_{-\Delta t} = U_o - \Delta t\, \dot{U}_o + \frac{\Delta t^2}{2} \ddot{U}_o$$

$$\begin{Bmatrix} U^1_{-\Delta t} \\ U^2_{-\Delta t} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - 0.28 \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \frac{0.28^2}{2} \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.392 \end{Bmatrix}$$

That is 0.28 and 28 seconds and delta t how you choose not only based on the stability but if you choose a very large time step you will be missing out the initial response. So, even 0.28 may not be sufficient if you are interested in knowing what happens at a short time intervals because it is actually this loading is applied instantaneously. And we are interested in knowing shortly after the load is applied like milliseconds.

In that case the delta t maybe even smaller and as the time progresses you can increase the delta t so that you can accelerate much faster.

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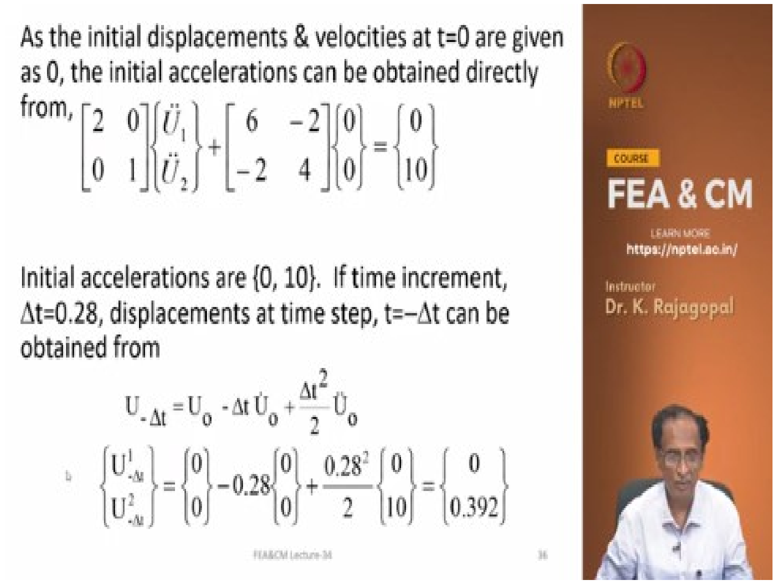

Let us take to start with a delta t of 0.28 and we need the response at the previous time step t of minus delta t and the equation for the starting the analysis is given here and the U 1 and U 2 at time step - 0.28 or 0 and 0.392 and our initial velocity and acceleration are 0 and 0 sorry I think there is the initial displacement and velocities are given as 0. And we can get the initial acceleration by putting in the initial displacements in our governing equation.

And we get the U 1 and U 2 acceleration as 0 and 10 and we use that value here and we get the displacement a degree of freedom 1 and 2 at the time step - delta t like this.

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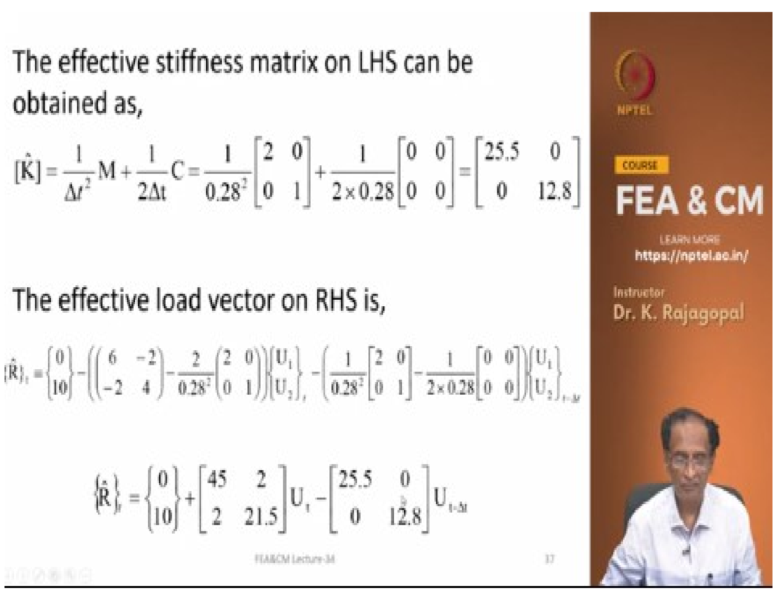

The effective stiffness matrix on LHS can be obtained as,

$$[\hat{K}] = \frac{1}{\Delta t^2}M + \frac{1}{2\Delta t}C = \frac{1}{0.28^2}\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix} + \frac{1}{2\times 0.28}\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}25.5 & 0\\0 & 12.8\end{bmatrix}$$

The effective load vector on RHS is,

$$\hat{R}\}_t = \begin{Bmatrix}0\\10\end{Bmatrix} - \left(\begin{pmatrix}6 & -2\\-2 & 4\end{pmatrix} - \frac{2}{0.28^2}\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}\right)\begin{Bmatrix}U_1\\U_2\end{Bmatrix}_t - \left(\frac{1}{0.28^2}\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix} - \frac{1}{2\times 0.28}\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\right)\begin{Bmatrix}U_1\\U_2\end{Bmatrix}_{t-\Delta t}$$

$$\{\hat{R}\}_t = \begin{Bmatrix}0\\10\end{Bmatrix} + \begin{bmatrix}45 & 2\\2 & 21.5\end{bmatrix}U_t - \begin{bmatrix}25.5 & 0\\0 & 12.8\end{bmatrix}U_{t-\Delta t}$$

FEA&CM Lecture-34 37

And then this is our K hat and the effective stiffness matrix is M by delta t square and the C by 2 delta t and this is our it is a constant value in the explicit method. And then the effective load vector the right hand side is given like this and it has the displacements are changing the right hand side will go on changing.

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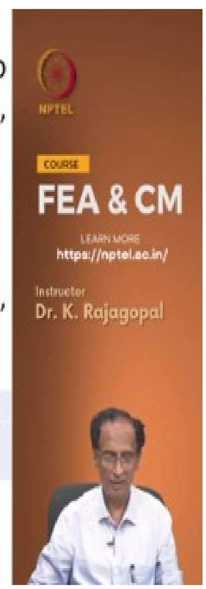
Hence, we need to solve the recursive equation to find the solution at different time steps, Δt, 2 Δt, 3 Δt, 4 Δt, etc.

$$\begin{bmatrix}25.5 & 0\\0 & 12.8\end{bmatrix}U_{t+\Delta t} = \hat{R}_t$$

Solving the above equation at different time steps, solution can be obtained as,

Time	Δt	$2\Delta t$	$3\Delta t$	$4\Delta t$	$5\Delta t$	$6\Delta t$	$7\Delta t$	$8\Delta t$	$9\Delta t$	$10\Delta t$	$11\Delta t$	$12\Delta t$
U_t	0.0 0.392	0.0307 1.45	0.168 2.83	0.487 4.14	1.02 5.02	1.70 5.26	2.40 4.90	2.91 4.17	3.07 3.37	2.77 2.78	2.04 2.54	1.02 2.60

FEA&CM Lecture-34 38



And we can solve this recursive equation and we get the solutions at different time steps delta t 2 delta t 3 delta t 4, 5, 6 and so on I have solved up to 12 delta t and these are actually manual calculations you can just simply take a calculator or write an Excel program and solve them it is very simple. And we see that as the time step is increasing the displacement is fluctuating and at 12 times the delta t 12 times 0.28 the displacements are 1 and 2.6.

And we can continue like and because there is no damping motion will continue forever because we have, we did not consider or we did not include the damping.

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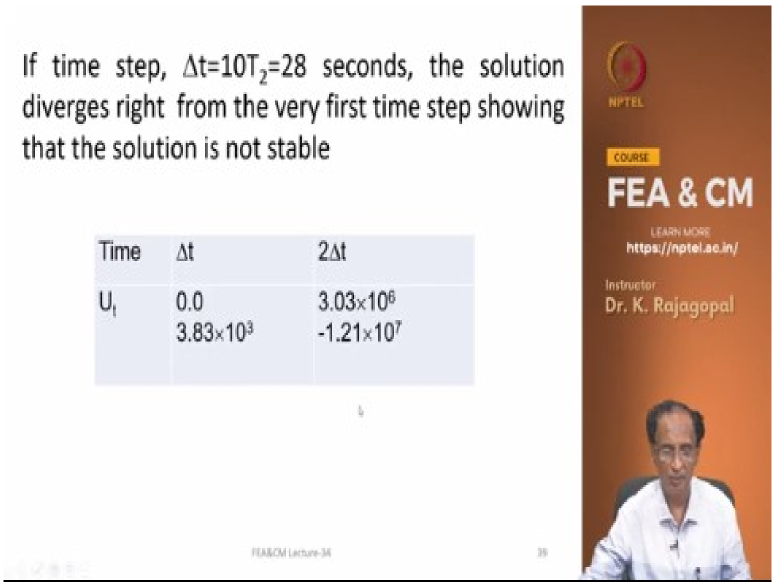

And let us now see what happens with a delta t of 10 times the t 2 28 seconds. Why do we want to use 28 seconds? That is if you want to know what happens at a large time interval fuse a larger delta t you need to spend a lesser computational effort and we can go through the time much faster. But then if you use a delta t of 28 seconds this is what we get. At the starting time step delta t our displacement is some 300830.

Time	Δt	$2\Delta t$
U_t	0.0 3.83×10^3	3.03×10^6 -1.21×10^7

And that 2 times the delta t the displacements are even larger. So, we can say that it has started diverging and so it is an unstable solution like although it is a numerical value it is giving some displacement but it is actually it is an unreasonable value because with delta t of 0.28 even after 12 times or 12 iterations, we get a reasonable displacement. And we see that with a larger time step the there is no convergence the solution is diverging.

So, that is the disadvantage with the central different scheme although it is a very simple scheme but the stability depends on the delta t.

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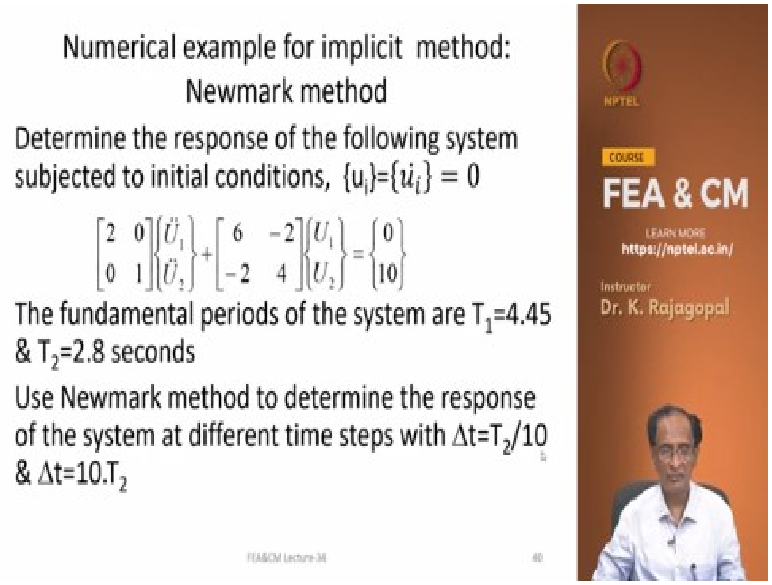

And now let us look at the Newmark's method and let us solve with delta t of T 2 by 10 that is 0.28 and then 10 times the T 2 that is the 28 seconds.

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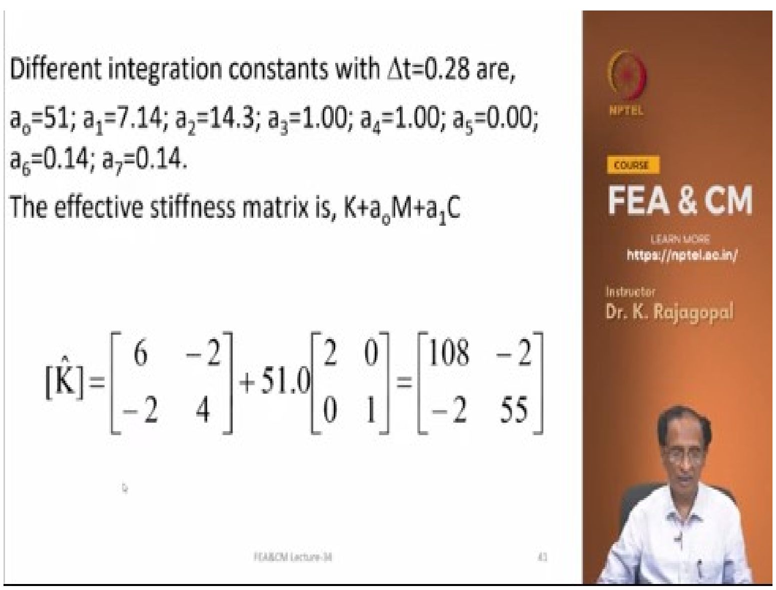

And these are all the constants a 0, a 1 and so on and the effective stiffness matrix K hat is this and then the right hand side load vector is like this and the displacement at any time t + delta t is K hat multiplied by U t + delta t is R hat at t + delta t and then we can continue the solution.

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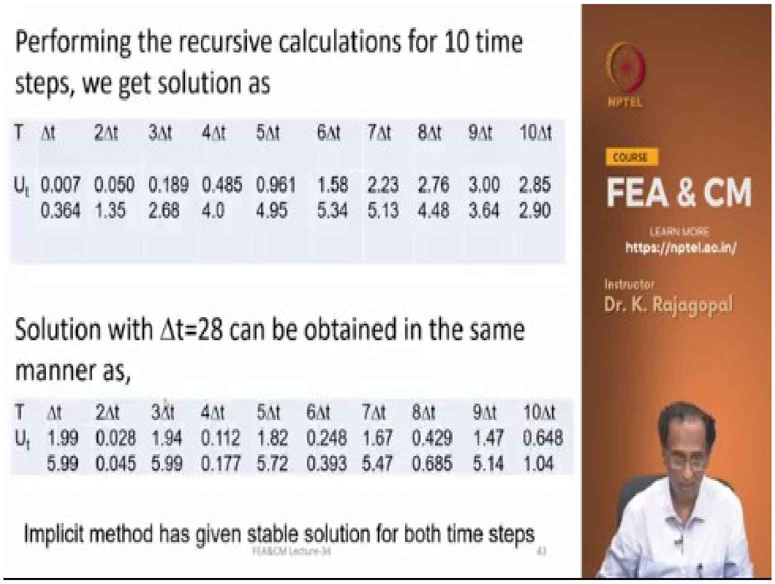

And so, for 10 iterations like this is the solution with 0.28 and it is once again fluctuating the solution is not the same. And you see that this solution is different from what we get from the central difference scheme and actually this the Newmark's solution is a better more accurate compared to the central difference scheme. And if you compare this and that there is some difference and with the delta t of 28 this is the response that we get.

And in fact, the once again we are getting up and down because there is no damping and after 10 times the delta t that is the 280 seconds our response is still within the finite values like 0.648 and 1.04 is going up and down. Whereas with central different scheme the whole thing has blown up. So, this shows the advantage of the implicit methods this is particularly the Newmark's method and similar solutions you can get the even with the Wilson theta method or the Houlbert method.

And both of them are very similar to this Newmark's method so before I close let me just show you an Excel spreadsheet program the annotation.

(Video Starts: 57:16)

So, you have a Excel spreadsheet program for Newmark's method and this particular one is for a single degree of freedom. This particular one is for impulse loading and you can actually these are all the delta is 0.5 and alpha is 0.25 as I as we discussed and then the mass is given stiffness is given and you can calculate the critical damping of the system. And then the loading is given here the peak load is 500 seconds and then the duration of the impulse is given here 0.05.

And then we can get the response by a Newmark's method it is a very simple one like it is even simpler than the two degree of freedom example that we considered earlier. And let us calculate the response without any damping let us say there is no damping. And you see here the response is given here like it is actually it is a beautiful sinusoidal response there is no reduction in the displacement because there is no damping.

And now let us consider some damping let us say some 0.01 very small damping and then we see gradually the displacements are reducing with the time. And if you use a larger value like let us, say 0.1 this damping factor in here, we are using damping factor the damping is damping factor multiplied by the critical damping critical damping factor. And here you see a beautiful reduction in the displacements for higher damping ratio.

And if you use a very large damping ratio of one if the damping is more than the critical damping you get you do not get any oscillations. You directly get your the constant displacement this is what you get here there is there is too much of damping and within a small time the displacement has become 0 and you can actually it is a very simple spreadsheet program that lets you examine the influence of a different time steps like here the delta t is suggested.

I think based on based on the H6 is what it is based on the impulse loading and the delta t is what you can suggest you can include let me just use a very small very large value delta t sorry I think your delta t is too large because actually it is larger than the impact duration impulse duration so you will not get anything. Because your impulse duration is 0.05 and if you use a delta t of greater than the impulse time you will not get anything.

With the and we can try even with the 0.05 let us try with 0.01 and you will get some response and say if you use a 0.05 within one time step you are at the end of your impulse. So, you will not see any response it is not the numerical instability but it is you are beyond your impact loading. So, your delta t should be proportional to your impact duration. So, let us say we use a larger value of one second and even with 0.1 you will get a reasonable response.

You will get is actually you will get a different shape response because now you are it is not really impulse loading it so it is tested on the system for a longer time. So, you will get this type of confusing response so like I am applying for a smaller time step and you can see is actually depending on the way you apply the loading and depending on the duration of the loading you get a different response.

And you can look at the influence of your damping ratio and then the impulse duration and then the stiffness coefficient and other things like here. The stiffness is given as 2000 and let me just give this as a real impulse 0.05 and here what happened. We get a displacement of about the 0.0 maybe 0.065 and let us increase the stiffness let us say we are increasing to 5000. And you will see the displacement has reduced point 0.04.

And then you get a different response because now our critical time period has changed and then this the way the system response is different. So, you can examine all these factors using this simple Excel spreadsheet program and this particular one is based on the Newmark's direct integration methods.

(Video Ends: 1:04:42)

So, that is a brief introduction to analysis of the systems with pore pressures and then the dynamic analysis I try to give you a simple examples numerical examples with hand calculations. So, that you understand the procedures and a similar procedure is followed even in the finite element analysis. So, if you have any questions, please send an email to this address profkrgmail.com and this will be my last lecture in the course.

But this will be followed by some guest lectures by my colleagues who will be talking to you on other constitute models our constitute to models like cam clay models modified cam clay and then other critical state models. Then soft clay modelling and then some practical analysis using plexus program. So, I have about five guest lectures that also they will be uploaded on the course website you can go through them.

And there will be assignments and other things that you need to solve for your grading purposes. So, thank you very much and look forward to interacting with you further on any of these topics thank you very much.