FEM and Constitutive Modelling in Geomechanics

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Lecture: 4
Development of Equilibrium Equations for 2-D Bar Elements and Truss Structures

And let us continue from our previous discussion on developing the system of equilibrium equations for axial elements and let us continue for 2 dimensional elements. And in this lecture let us look at only the 2 dimensional bar elements or 2 dimensional Prismatic elements.

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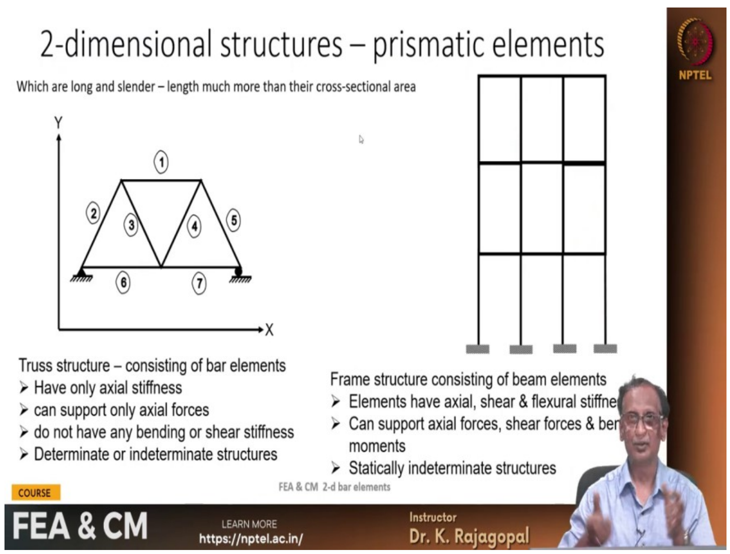
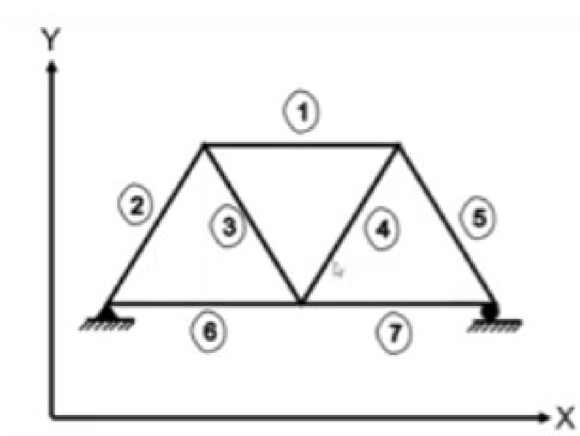
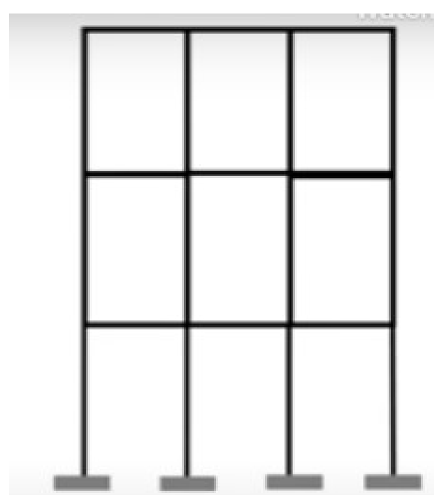


And there are different types of 2 dimensional structures that you can think of made of line elements like the trusses or the frames.





See these trusses they are made of only bar elements that can support only axial forces. And so, they have only axial stiffness and they do not have any bending or Shear stiffness and these structures could be either determinate or indeterminate and I hope that you already know what they are and then you can also have frames made of beam elements.

And these beam elements they have 3 forces the axial Force Shear force and then the bending bending force and the correspondingly they can have 3 stiffnesses axial stiffness, Shear stiffness and the flexural stiffness and these are basically statically indeterminate structures we require some other methods for solving them. Say the statically determinate structures can be solved by using our equilibrium equation Sigma of f is 0 and sigma of M is 0 net force and net moment is zero.

But then the statically indeterminate structures they require some other conditions and that some of those methods you would have already seen. But in this course we will see how they can be solved by using finite element techniques.

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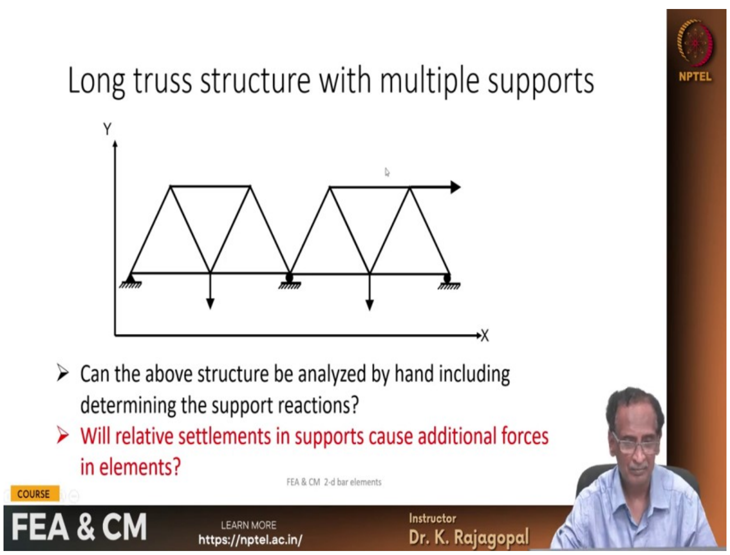
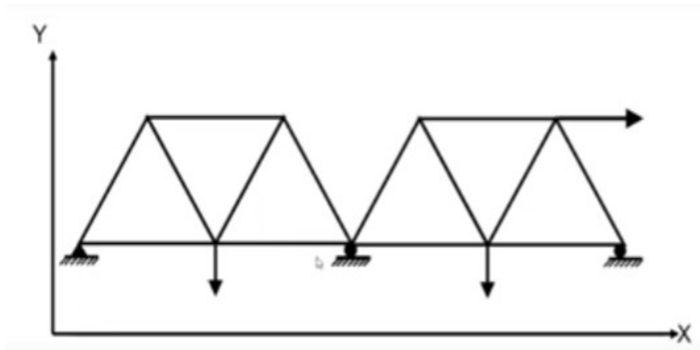


And we may have more complicated structures like this is a system with more than 2 supports. And so, these are all simple truss elements trust structures with the different supports and when we have very large number of elements and degrees of freedom can we solve them by hand? Well we can if you spend more time and more effort but then if you use a computer program becomes more easy.



And if you have multiple supports like this what will happen if one of the supports undergoes a relative deformation. Let us say the central support settles say by 20 millimeters in relation to the other supports what will happen to the element forces or the same question I could have asked even here. So, in this let us say that the right hand side support has settled by some amount what will happen to the forces in the elements.

So, that I will not give you the answer but you can think about and send me an answer then I will be able to comment.

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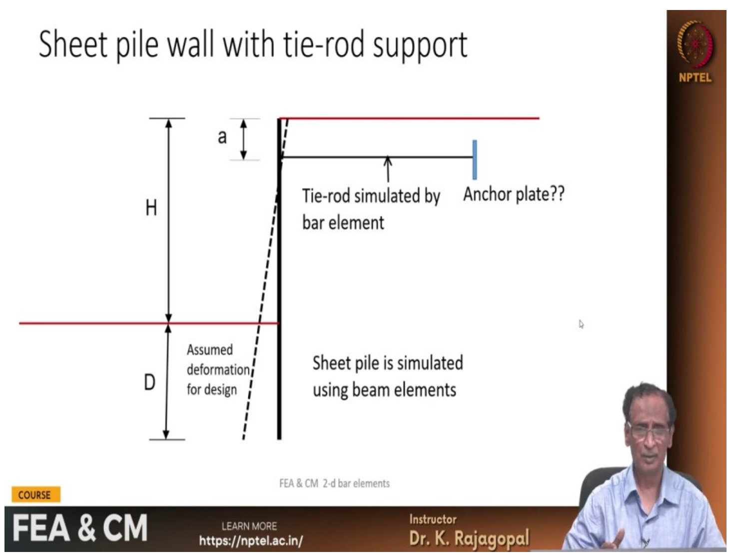
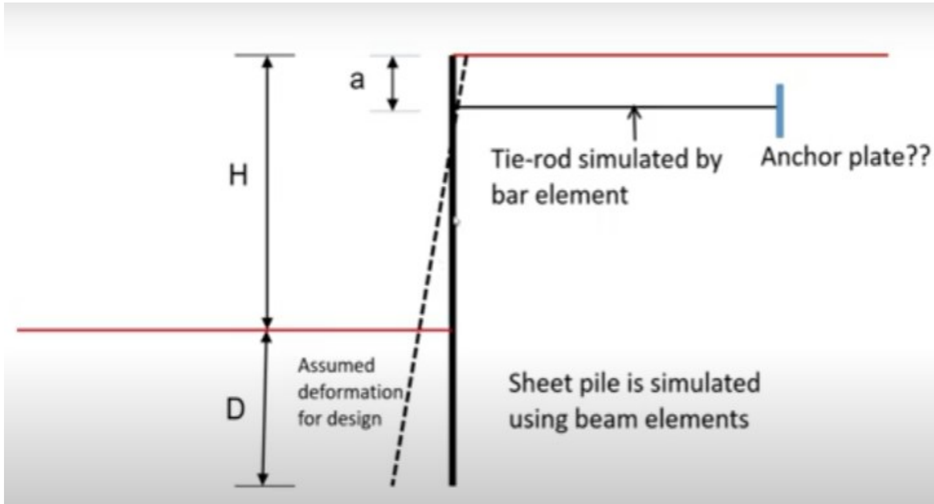


So, what is the practical exam practical application for all these elements that we have been talking about the axial elements and then the beam elements and so on. One typical application for that could be your sheet pile walls with the tie rod and maybe possibly with an anchor plate. And all of you would have analyzed this type of structures in your Foundation Engineering courses.



One of the methods is to assume the sheet file wall as a rigid element and bend about this point and then we can assume that the soil in the front is under passive conditions whereas the soil behind is an active conditions and then by equilibrium we determine the depth of embedment and then we can also determine the force developed in the tie rod. But then we can do the same thing more elegantly by using our finite element methods.

And while doing so, our sheet pile can be simulated by using our flexural elements like beam elements and the tie rod can be simulated by a bar element and then what about the anchor plate? For that there are different methods that we will we will see later.

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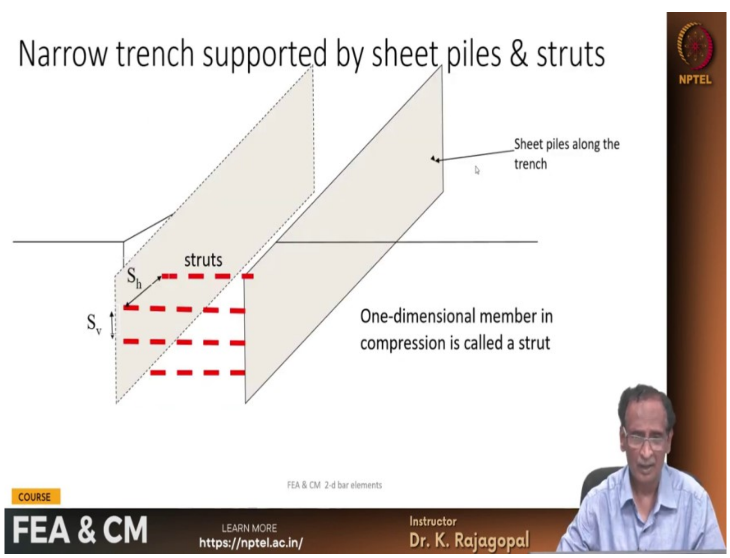

Or another example is an arrow excavation supported by sheet by walls and then number of struts. See the tie rod element is in tension it is a one dimensional element in tension whereas distract is also a one-dimensional element but in compression the compression elements we call them as struts and then the tension elements we call them as tie elements.

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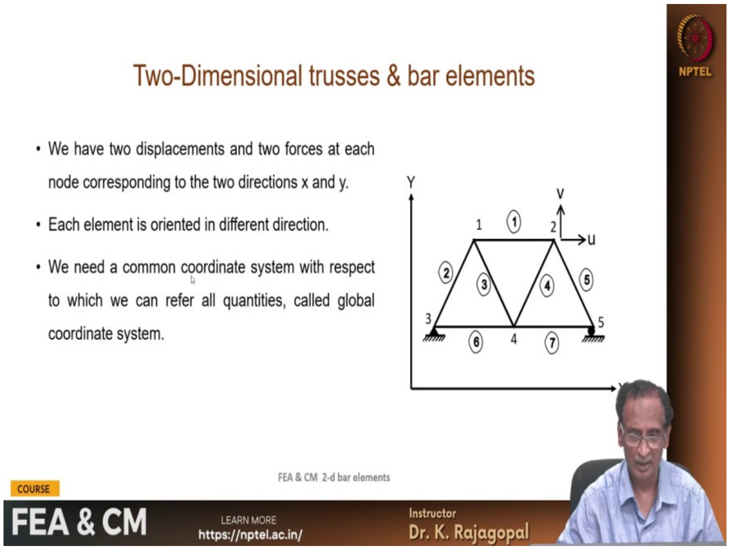
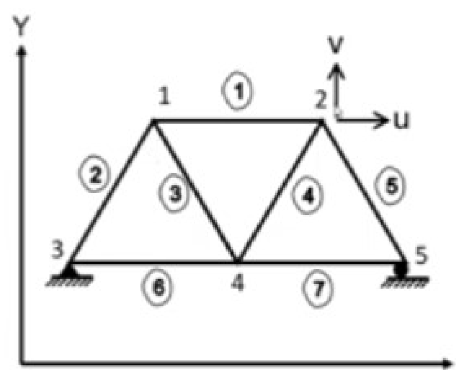


So, let us look at a 2-dimensional truss consisting of bar elements all axial elements and each node will have 2 degrees of freedom because previously we had only one degree of freedom because we had only axial. Now we have we have to think of a Cartesian coordinate system x and y. Then we have the structure here consisting of 5 notes one 2 3 4 5 and between each of these nodes we have one element.



Element one connected between nodes One and Two element 2 between one and 3 and then 3 4 5 6 7. Then at each node we have 2 displacements one in the x Direction and one in the y direction u and v. And this Cartesian coordinate system we also call it as a as a global coordinate system because it is common for the entire structure.

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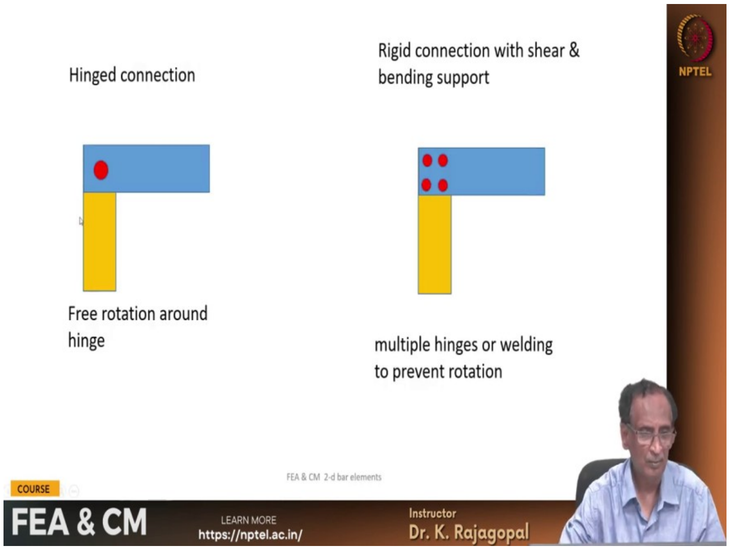
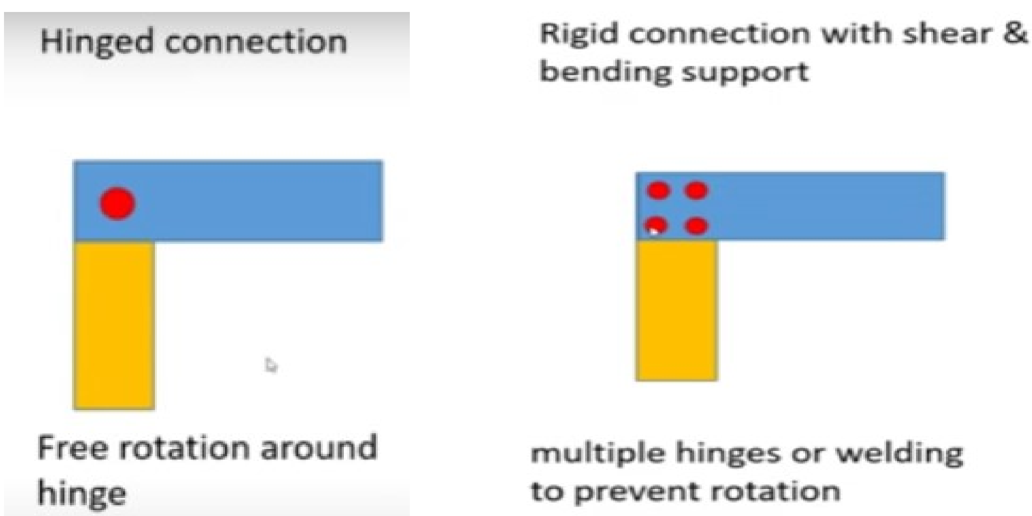


And what do we mean by hinge and then the rigid connections. So, all these bar elements that are connected with a hinge element with a hinge connection like say just imagine this as a bar element under the 2 ends we have the hinges. So, that this element can be rotated freely around this connection point and that can imagine by joining 2 plates with one single hinge like this and they can be rotated around each other.



Then if you give a multiple hinges let us say 4 hinges more than one hinge like 2 or 3 4 and so on or with a weld then you will not be able to freely rotate when we rotate we develop some forces some Shear force and then some moments and so on. And the trust structures all the elements are connected with one single hinge whereas a name in a building frame you can imagine either a welded connection or more rigid connection with the more than one hinge at each joint like this.

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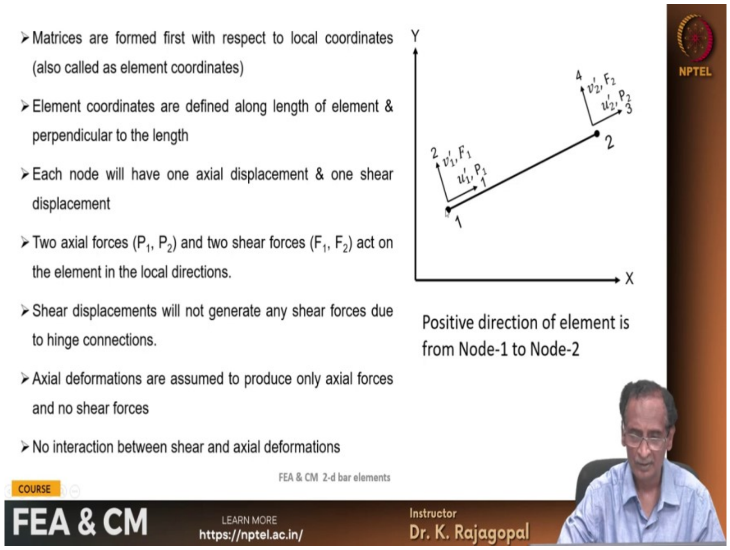

And how do we form the equilibrium equations for these systems. So, once again we think of farming the elements for one single element and then and then joining them together for the entire structure and these element equilibrium equations and it is more easy. If we form them in the local coordinates are with respect to the element coordinates and the element coordinates are defined along the length of the element and perpendicular to the element.

And each node will have one axial displacement and one Shear displacement that is if you think in terms of the element coordinates then we have 2 axial forces P 1 and P 2 and 2 Shear forces F 1 and F 2 and the shear displacements will not generate any Shear forces due to the hinged connections server F 1 and F 2 they are actually zero. Although we we can consider them but for this bar elements are they are zero.

And the axial deformations will produce only the axial forces and there will not be any Shear forces. Basically we are saying that all our deformations are so small that the axial deformations will only produce axial forces but they will not produce any Shear forces. And there is no interaction between the shear and axial deformations that is if you apply some Shear deformation you will develop only Shear forces but no axial forces.

Similarly when you apply an axial deformation we will develop only the axial forces but no Shear forces. So, here we see this element having 2 nodes one and 2 and the x and y these are the Cartesian coordinates x is the along the defined along the horizontal axis y along the vertical axis and u 1 Prime is the axial deformation at Node 1 and u 2 Prime is the axial deformation at node 2 v 1 Prime and v 2 Prime are the shear deformations perpendicular to the length of the element P 1 F 1 P 2 F 2 are the axial and Shear forces.

And the positive direction of this element is defined from Node 1 to node 2 that is any Vector drawn from 1 to 2 it is a it is called as a positive vector and that is the positive direction of the element. Because actually we have to have some consistency in how we Define each element so, that we can combine all the contributions together.

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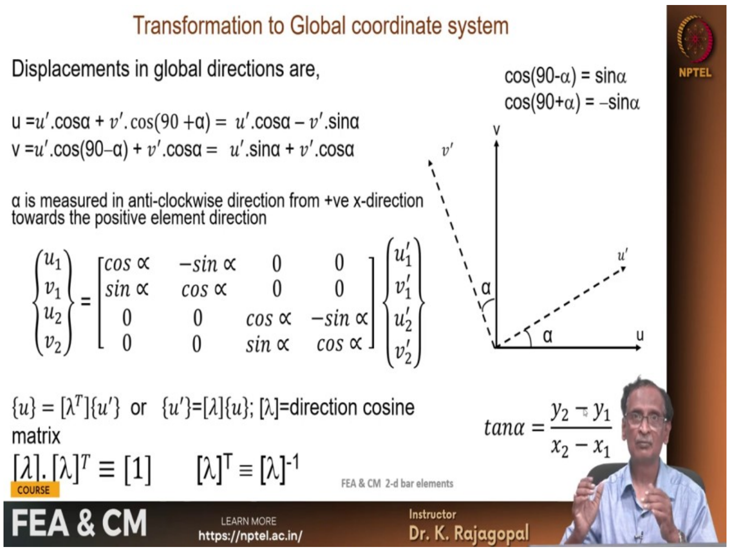


And let us form these equations at the element level and for that we require element stiffness Matrix. And let us say that our stiffness coefficient is AE by L because now we are saying that this element can only support axial forces. And so, AE by L is the axial stiffness and using our fundamental definitions of K ij we can directly write the stiffness Matrix in the element coordinates K Prime as AE by L 1 0 -1 0 and so on. See these zeros they correspond to shear deformations.

Because we are saying that if you apply initial deformation this element will not produce any Shear forces because of the hinged connections and if we are playing any axial deformation they will produce only the axial forces and no Shear forces. So, because of that we have too many zeros here and this element's difference Matrix is oriented in the element coordinates. So, basically each element may have different directions.

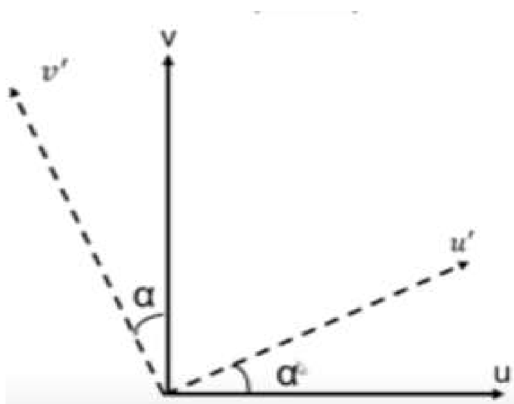
And so, the contribution of each element we cannot directly add them together because for example if you see the structure that we had seen earlier. See the element one here is oriented along the horizontal Direction whereas element 2 is maybe in this direction element 3 in this direction 4 in this direction 5 in this direction and so on. So, you cannot directly add the contribution of all these elements together we need to convert them to some common system like the common coordinate system like our Cartesian system x and y.

So, that we can add them together. So, for that we need some understanding and let us define the relation between our local displacements u Prime v Prime and then the global displacements u and v, u and v are the displacements in the 2 Cartesian directions x and y u Prime is the axial deformation v Prime is the shear deformation with respect to the element and if Alpha is the angle of this element orientation of this element.

We can define or we can relate u Prime v Prime to u and v through this

$$u' = u\cos\alpha + v\sin\alpha$$
$$v' = -u\sin\alpha + v\cos\alpha$$

and this u Prime is u times cosine Alpha + v Prime sorry v v times sine Alpha and v Prime is -u times cosine of sine Alpha + v times cosine Alpha right.



And so, we can actually transform them or in other words we can define the global displacements the Cartesian displacements u and v in terms of u Prime v Prime like this u is u Prime of cosine Alpha and + v Prime Times cosine of 90 + Alpha.

$$u = u'.\cos\alpha + v'.\cos(90+\alpha) = u'.\cos\alpha - v'.\sin\alpha$$
$$v = u'.\cos(90-\alpha) + v'.\cos\alpha = u'.\sin\alpha + v'.\cos\alpha$$

Is actually in terms of the cosines is more easy and our from trigonometry we know that cosine of 90 - Alpha is sine Alpha cosine of 90 + Alpha is - sine Alpha. So, if you apply that u is u Prime cosine Alpha - v Prime sine Alpha and v is u Prime times cosine 90 - Alpha + v Prime times cosine Alpha right is basically the angle between that particular direction and then this u Prime and v Prime axis.

And Alpha is measured in the anticlockwise direction from the positive x direction towards the positive element direction.

So, for example if this is your positive coordinate the element direction the alpha is measured starting from positive x axis in the anti-clockwise direction this is your Alpha. And so, our global displacements in terms of the local displacements can be written like this

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{bmatrix} \cos\propto & -\sin\propto & 0 & 0 \\ \sin\propto & \cos\propto & 0 & 0 \\ 0 & 0 & \cos\propto & -\sin\propto \\ 0 & 0 & \sin\propto & \cos\propto \end{bmatrix} \begin{Bmatrix} u_1' \\ v_1' \\ u_2' \\ v_2' \end{Bmatrix}$$

u is this coordinate Matrix multiplied by u Prime.

$$\{u\} = [\lambda^T]\{u'\} \text{ or } \{u'\}=[\lambda]\{u\}; [\lambda]=\text{direction cosine}$$

And u is Lambda transpose u Prime this is actually it is a matrix of direction cosines and that u is written as Lambda transpose u Prime are u is u Prime is Lambda times u where Lambda is the direction cosine Matrix. And the properties of this Matrix are like this Lambda times Lambda transpose is an is an identity Matrix and Lambda transpose is equal to the inverse of the inverse of this Matrix.

So, actually this relation is true only for the direction cosine Matrix but not for something else see previously in in the in one of the lectures we had discussed about the inverse of the matrix. But now we are saying that the transpose itself is equal to the inverse that is because of this trigonometric properties that cosine Alpha and sine Alpha and - sine Alpha and so on.

And how we get Alpha in a computer program is very simple tan Alpha is y 2 – y 1 divided by x 2 – x 1 w8 y 2 y 1 are the y coordinates of node 2 and Node 1 and x 2 x 1 are the x coordinates of the node 2 and node 1.

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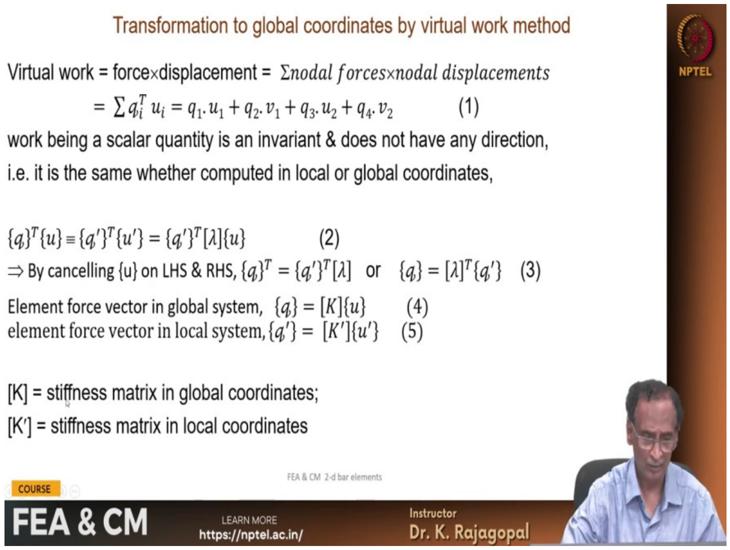

And we can transform these equations in the local directions to the global coordinates by considering the virtual work. Because the work being a scalar quantity it is independent of the coordinate system whether you calculate the work in the in terms of the element coordinates or in the global coordinates they should be the same. So, the virtual work is the force multiplied by displacement.

And let us say our displacement Vector is u i u 1 v 1 u 2 v 2 and the force Vector is a q i q i that is q 1 q 2 q 3 q 4 both are vectors and unless you take a transpose of one of them you cannot multiply.

Virtual work = force×displacement = $\Sigma nodal\ forces \times nodal\ displacements$

$$= \sum q_i^T u_i = q_1.u_1 + q_2.v_1 + q_3.u_2 + q_4.v_2 \qquad (1)$$

And say u is a is a vector of length 4 and q will be also vector of length 4 but q transpose is a vector of one row and 4 columns. So, the product is this and then this quantity is exactly the same even if you write as q 1 Prime u 1 prime + q 2 Prime v 1 Prime and so on.

So, whether you compute this virtual work in the local coordinates or the global coordinates it is the same. So, q q transpose u I can write as cube Prime transpose u Prime and u Prime is a Lambda times u that is what we have seen in the in the previous these things u Prime is Lambda times u and u is Lambda times Lambda transpose times u Prime and q Prime sorry q transpose times u is equal to q Prime transpose times u Prime and our u Prime is Lambda times U.

So, we can cancel out u on both the left hand side and the right hand side and we get that q Prime q transpose is a q Prime transpose times Lambda or by taking the transpose q is Lambda transpose times q Prime.

$$\{q\}^T\{u\} = \{q'\}^T\{u'\} = \{q'\}^T[\lambda]\{u\} \qquad (2)$$

$\Rightarrow$ By cancelling $\{u\}$ on LHS & RHS, $\{q\}^T = \{q'\}^T[\lambda]$ or $\{q\} = [\lambda]^T\{q'\}$ (3)

Element force vector in global system, $\{q\} = [K]\{u\}$ (4)

And so, the element force in the global system is q is equal to K times u where K is the stiffness Matrix and the global coordinates u is the vector of global displacements u and v in x and y directions the q are the nodal forces in the global coordinates.

And element Force Vector in the local system q Prime is K Prime Times u Prime right and the K is the stiffness Matrix in the global coordinates and K Prime is the stiffness Matrix and the local coordinates. Now we need to find a relation between the K and K Prime because the K is the one that we require.

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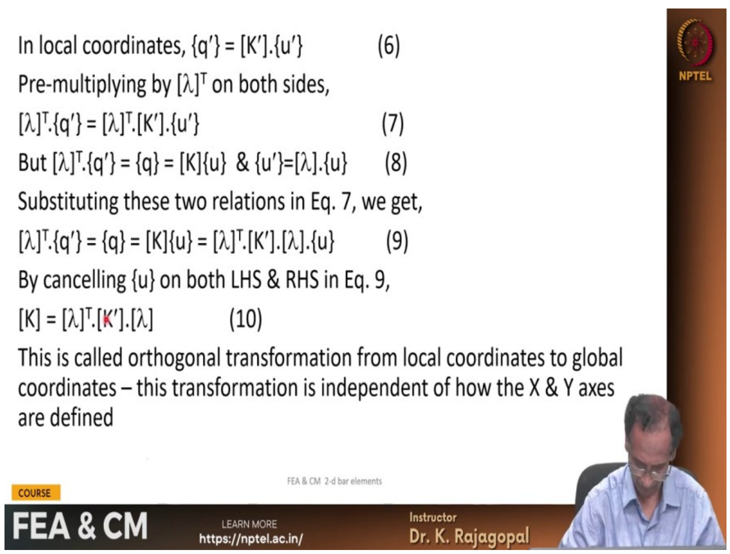

So, we can start with the force in the local coordinates q Prime is a K Prime times u Prime and let us multiply with the Lambda transpose on both the sides and we get Lambda transpose times q Prime is Lambda transpose K Prime times u Prime right but our Lambda transpose q Prime is nothing but q and the q is the force Vector at the global coordinates that is also equal to K times u and our u Prime is Lambda times u these are the Transformations that we derived earlier.

And by substituting these relations we can get that Lambda transpose times q Prime is q that is K times u that is Lambda transpose K Prime Lambda u right and by cancelling the u on both the sides. Say by cancelling this u we end up with K is equal to Lambda transpose K Prime times Lambda right. And stiffness Matrix in the global coordinates K is Lambda transpose K Prime times Lambda and this is called as the orthogonal transmission from the local coordinates to global coordinates.

In local coordinates, $\{q'\} = [K'].\{u'\}$ (6)

Pre-multiplying by $[\lambda]^T$ on both sides,

$[\lambda]^T.\{q'\} = [\lambda]^T.[K'].\{u'\}$ (7)

But $[\lambda]^T.\{q'\} = \{q\} = [K]\{u\}$ & $\{u'\}=[\lambda].\{u\}$ (8)

Substituting these two relations in Eq. 7, we get,

$[\lambda]^T.\{q'\} = \{q\} = [K]\{u\} = [\lambda]^T.[K'].[\lambda].\{u\}$ (9)

By cancelling $\{u\}$ on both LHS & RHS in Eq. 9,

$[K] = [\lambda]^T.[K'].[\lambda]$ (10)

And actually this helps us in making our solution independent of the definition of the coordinates x and y.

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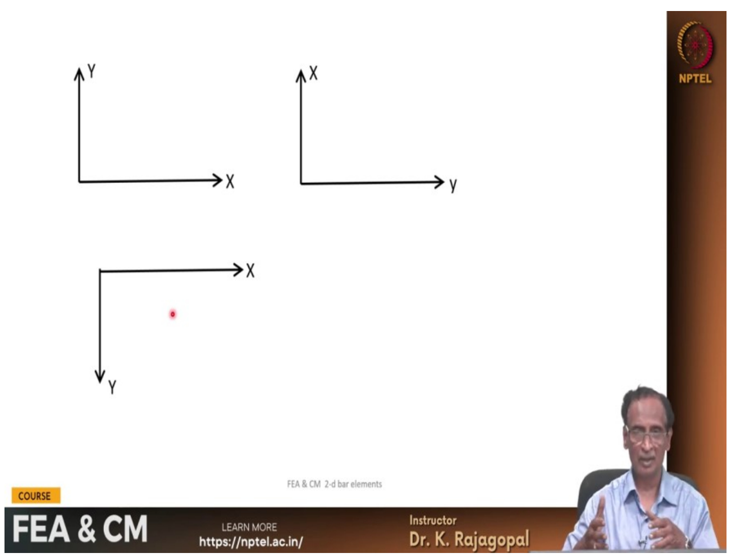

So, for example we could define the global coordinates as like this x and y with x in the along the horizontal direction y at the vertical direction are x in the vertical direction and y in the horizontal direction are the y going down instead of going up. Whatever you do the solution will be the same like if you get 10 millimeter displacement in this coordinate system you should get the same 10 millimeters here it will not be any different. That is what we mean by the coordinate independent solution.

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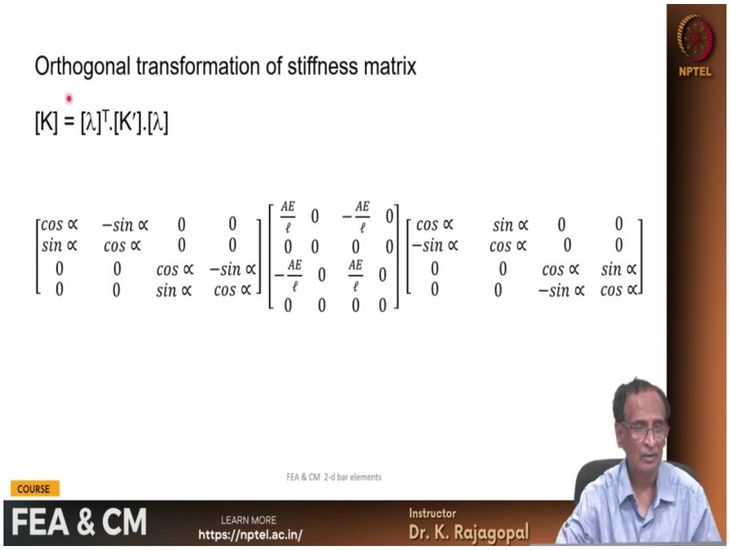

But the sign may change like if it is +10 maybe it will be -10 in some other coordinate system. So, this orthogonal transformation of the coordinate the stiffness Matrix K is Lambda transpose K Prime Lambda and our Lambda transpose is this K Prime is this and Lambda is this.

$$[K] = [\lambda]^T.[K'].[\lambda]$$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \frac{AE}{\ell} & 0 & -\frac{AE}{\ell} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{AE}{\ell} & 0 & \frac{AE}{\ell} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

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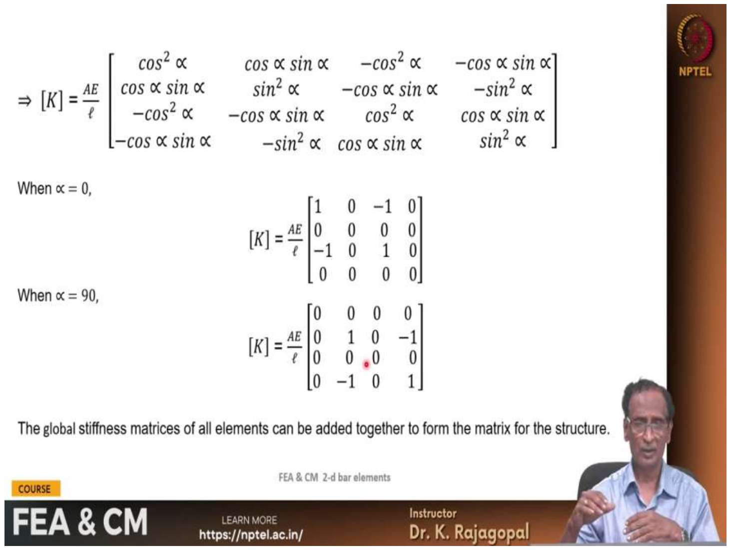

And if you go through the full multiplication your K is AE by L cosine Square Alpha and sine Square Alpha cosine Square Alpha sine Square Alpha and so on. And you see our stiffness Matrix is symmetric that is the upper diagonal terms are equal to the corresponding lower diagonal terms cosine Alpha sine Alpha and the same thing cosine Alpha sine Alpha - cosine Alpha sine Alpha here - cosine Alpha sine Alpha.

$$[K] = \frac{AE}{\ell}\begin{bmatrix} \cos^2\propto & \cos\propto\sin\propto & -\cos^2\propto & -\cos\propto\sin\propto \\ \cos\propto\sin\propto & \sin^2\propto & -\cos\propto\sin\propto & -\sin^2\propto \\ -\cos^2\propto & -\cos\propto\sin\propto & \cos^2\propto & \cos\propto\sin\propto \\ -\cos\propto\sin\propto & -\sin^2\propto & \cos\propto\sin\propto & \sin^2\propto \end{bmatrix}$$

So, when Alpha is 0 you get K like this

When $\propto = 0$,

$$[K] = \frac{AE}{\ell}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and when Alpha is 90 you get K like this.

When $\propto = 90$,

$$[K] = \frac{AE}{\ell}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

And now once you transform all the equations to Global coordinate system are the common coordinate system then we can add up the contribution now for all these elements.

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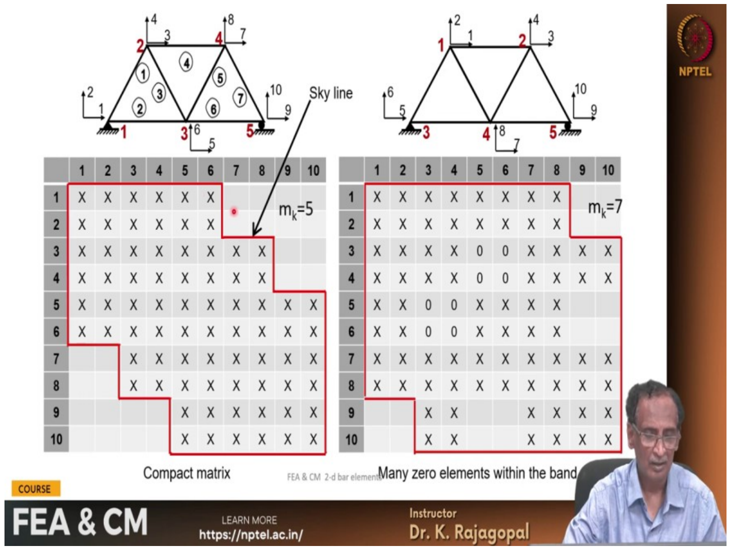

And if you contribute sorry if you add up you might get a system like this let us say for this structure with the 5 nodes and nodes one 2 3 1 2 3 4 5 and at each node you have 2 degrees of freedom one and 2. Your Global stiffness Matrix might look something like this actually these axes they refer to non-zero quantities and then this red line this red highlighted line is the it defines our band.

So, our degree of freedom one is connected to degrees of freedom 1 2 3 4 5 and 6 and it is not connected to 7 8 9 10. So, beyond column 6 in row 1 you will have zeros. And similarly in row 2 that is corresponding to degree of freedom 2 you have zeros. Then if you look at the degrees of freedom 5 and 6 they are connected to all the degrees of freedom through different elements. And so, you will have the fully populated row 5 and 6 all the ten columns are full.

And similarly 7 and 8 they are not connected to 1 and 2 and 9 and 10 they are not connected to and these degrees of freedom are not connected to 1 2 3 4. So, you get a very compact Matrix like this and we call this the half bandwidth M K that is I will Define that in the next slide and you can number the nodes something like this 1 2 3 4 5. And here our degrees of freedom one and 2 they are connected to up to eighth degree of freedom.

So, you are non-zero elements they will decrease previously the degree of freedom 1 is connected up to 6th degree of freedom but now it is connected up to 8th degree of freedom. And the bandwidth the half bandwidth is actually it is the maximum difference in the nodes or in this or in the degree of freedom for any element. Say for example if you take this element one it is sorry the degree of freedom 1 is connected to 6.

So, the half bandwidth is a 6 - 1 is 5 that is how it works up to 6th column you have non-zero elements and then when it comes to 5 and 6 they are connected to 9 and 10. So, the half bandwidth is 5 and so on. But if you look at here your degree of freedom 1 is connected to degree of freedom 8. So, the difference is 8 - 1 that is 7. So, the half bandwidth is is larger here and what difference does it make.

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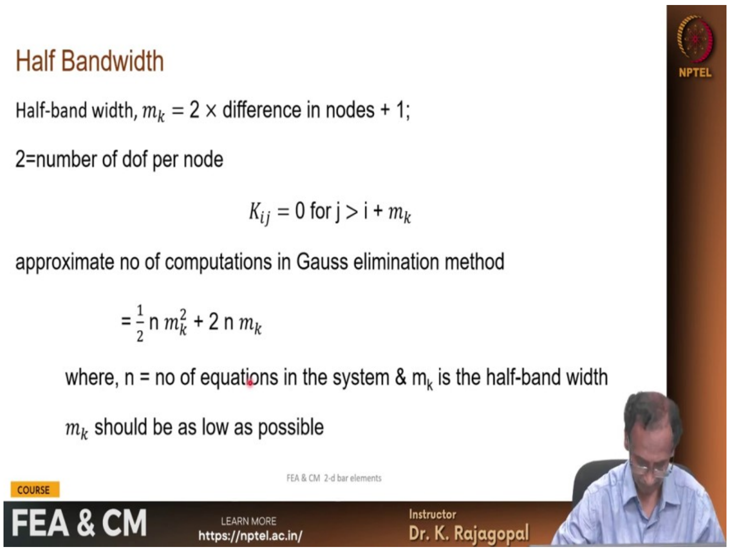

So, actually the half bandwidth m k is defined as the 2 times the difference in the nodes + one this 2 is the number of degrees of freedom per node

Half-band width, m_k = 2 × difference in nodes + 1

2=number of dof per node

and K ij is equal to 0 for j greater than i + m k.

$$K_{ij} = 0 \text{ for } j > i + m_k$$

And the approximate number of computations in the Gauss elimination method are one half n times m k square + 2 nm k and the first one one half nm k square is the number of operations were converting the matrix to an upper diagonal matrix.

And the second one is for back substitution here n is the number of equations the m k is the half bandwidth. And so, if you are half bandwidth is as small as possible then obviously your number of competitions will be lesser and then that much faster you can solve the system.

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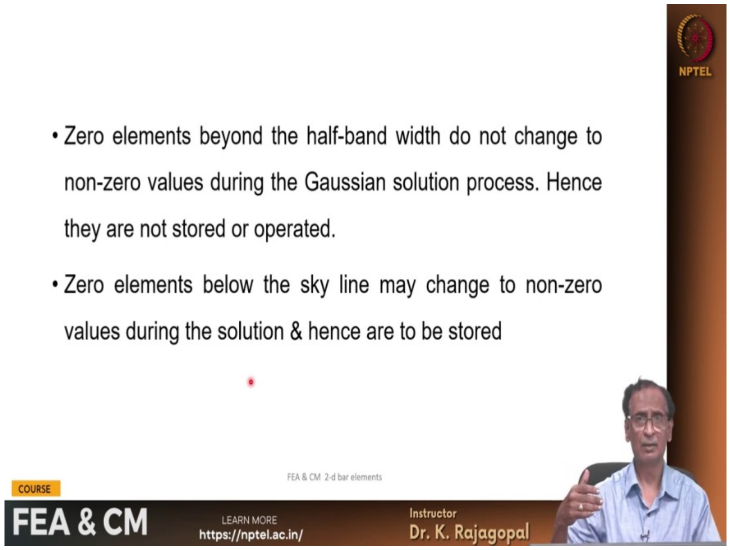

And these zero elements that we have beyond the bandwidth. So, we do not store them now we do not because they do not change to non-zero numbers during the Gauss elimination process.

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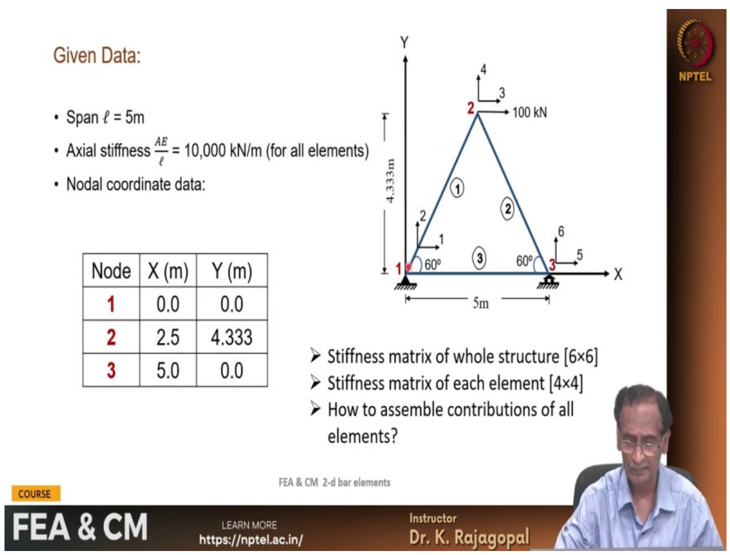
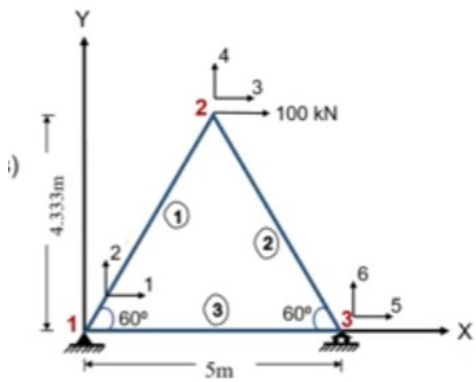


So, let us look at one numerical example so, that we can understand the concepts more clearly. Let us take a very simple structure having 3 nodes one 2 and 3 and 3 elements one 2 and 3 and the length of each of these elements is 5 units or 5 meters and AE by L the axial stiffness is ten thousand for each of these elements.



And the node one is defined at the origin 0 0 node 2 at 2.5 and 4.333 and node 3 is that x of 5 and y of 0 and the stiffness Matrix for the entire structure will be a 6 by 6 Matrix.

Because there are totally 6 degrees of freedom but at the element level we will have only a 4 by 4 Matrix because we have 4 degrees of freedom for each element. But then how do we assemble that we will see that we have to see later.

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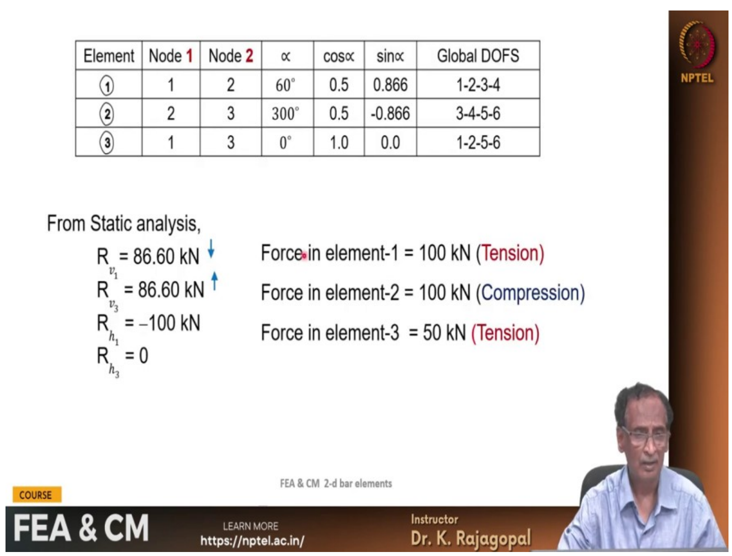

And we should have a systematic approach then only you will be able to do the correct computations and the first thing is when you have a structure we should number the nodes and also assign the coordinates. So, for example node 1 is x and y are 0 0 node 2 is 2.5 and 4.33 Note 3 is at x of 5 and y of 0 and we have 3 elements one 2 and 3 node one is each element has 2 nodes node one and node 2.

Element	Node 1	Node 2	∝	cos∝	sin∝	Global DOFS
1	1	2	60°	0.5	0.866	1-2-3-4
2	2	3	300°	0.5	-0.866	3-4-5-6
3	1	3	0°	1.0	0.0	1-2-5-6

So, for element one it is connected between nodes one and 2 and the element 2 is connected between nodes 2 and 3 at the element 3 is connected between 1 and 3. And this Alpha is defined in the anti-clockwise direction starting from the positive x coordinate towards the positive y coordinate.

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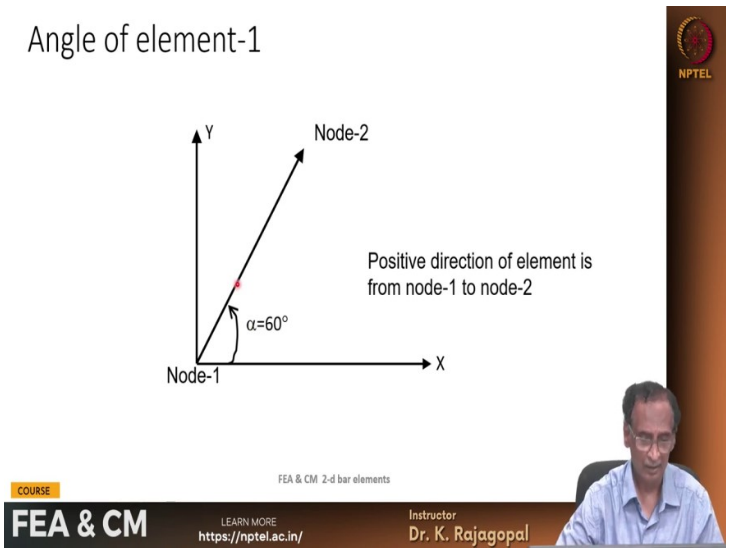
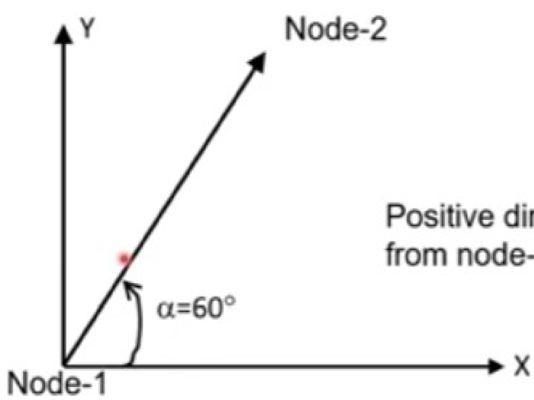


And that is defined like this say this element 1 is defined with node 1 here and node 2 there and this is the positive direction and the positive x coordinate is like this.



And so, we define this Alpha by measuring the angle from the positive x coordinate towards the positive direction of the element in the anti-clockwise direction. So, Alpha is 60 degrees.

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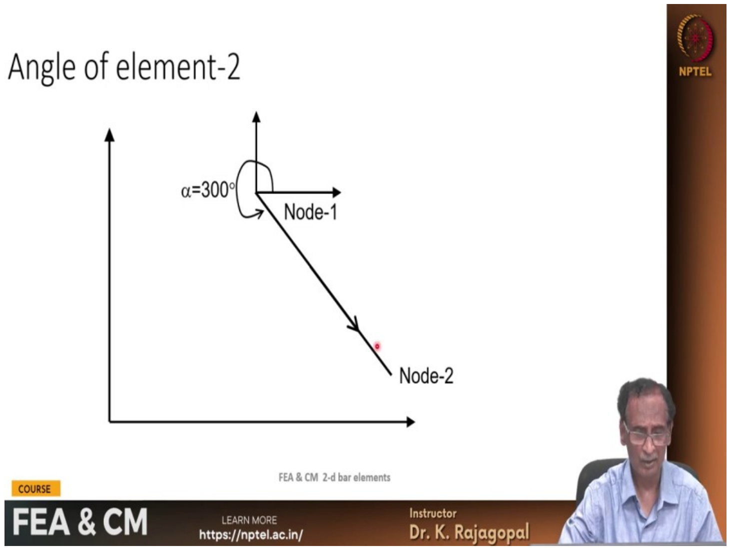

And but then say for and for this element to 2 node 1 is defined here and node 2 is defined here and this is how we measure the alpha starting from positive to x Direction in the negative direction answering the anti-clockwise direction towards the positive direction of the element. Positive direction of the element is in this direction node 1 to node 2. So, this is your Alpha right or we could define this as node 1 and node 2 there.

If you define it like this your Alpha is only 120 degrees. So, we should have a systematic system. And so, the element one is defined with the nodes 1 and to node 1 is 1 and node 2 is 2 and Alpha is 60 degrees. So, cosine Alpha and sine Alpha and then the global degrees of freedom connected to element one or 1 2 and 3 and 4 that is what we have seen here element one is connected between degrees of freedom 1 2 3 4.

And element 2 is connected to degrees of freedom 3 4 5 6 and element 3 connected between nodes 1 and 3 1 2 and 5 6 right and this is the basic data that we need so, that we can compute the stiffness matrix and the global coordinates because our stiffness matrix is AE by L times cosine Square Alpha cosine Alpha sine Alpha and so on. So, for that we require Alpha we require cosine Alpha sine Alpha.

And it is actually because this is a statically determinate structure we can do the analysis by force equilibrium methods and we can determine the force in element 1 as 100 kilo newtons that is tension force in element 2 is 100 kilo newtons compression the force in elementary is

50 kilo newtons tension. And these are the reaction forces this is actually these arrows they show the direction.

From Static analysis,

$R_{v_1} = 86.60\text{ kN} \downarrow$

$R_{v_3} = 86.60\text{ kN} \uparrow$

$R_{h_1} = -100\text{ kN}$

$R_{h_3} = 0$

Force in element-1 = 100 kN (Tension)

Force in element-2 = 100 kN (Compression)

Force in element-3 = 50 kN (Tension)

And so, if you look at this because we have played a force of 100 kilonewtons here and this particular node 3 is supported on rollers so that there will not be any horizontal reaction here and the node one is supported on a hinge. And so, that will develop force in both vertical direction and horizontal direction and the horizontal Force should be exactly equal to 100. So, we get a reaction force at one at node one as -100.

And then the reason why we do not keep both ends on hinges is these structures could be made of metal and during temperature changes these elements may elongate our contract and if we support one of the ends on hinge they can freely deform. So, that you do not develop undo stresses due to the temperature changes. That is the reason why we always prefer keeping one side on a hinge on the other side and a roller.

And so, actually these are some known values that we can compare with after we perform the analysis actually basically we are doing structural analysis but then these steps are common with finite element analysis. So, we might as well call it as finite element analysis and these are the different definitions for the directions this we have already seen.

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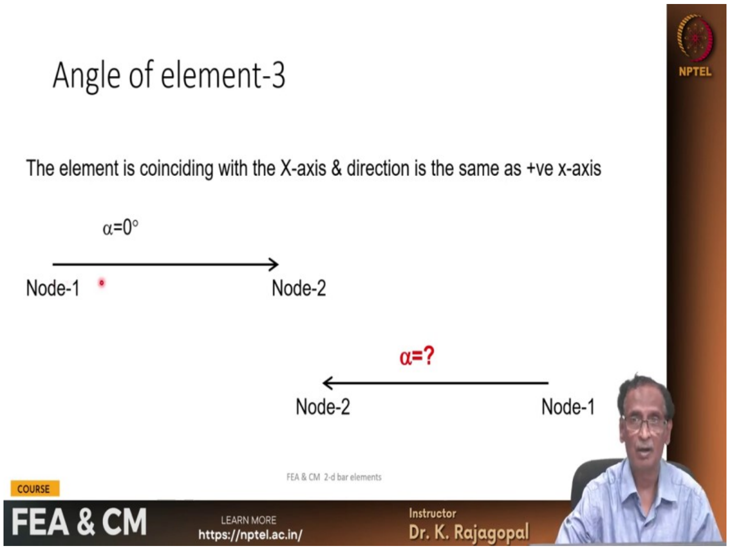

And element 3 is oriented along the horizontal direction in the along the x axis with the node 1 and node 2. So, Alpha is zero but if we number the node one here and node 2 here what will be the alpha that you can think about and tell me later.

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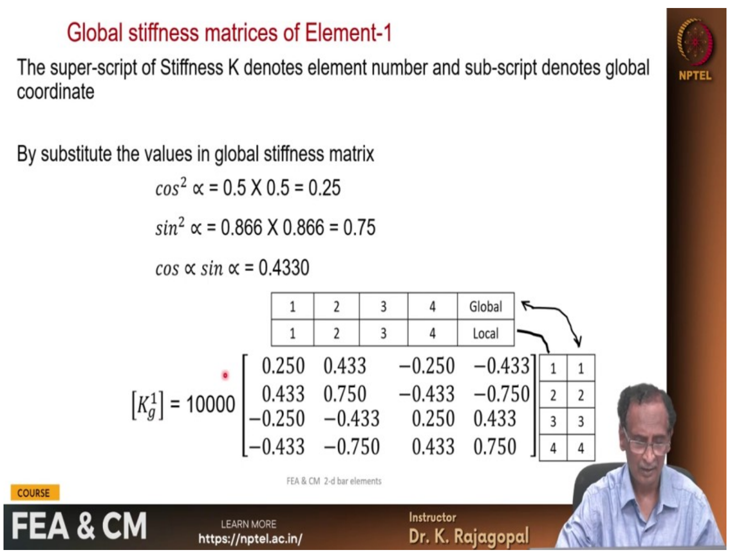

And so, for element one your cosine Square Alpha is 0.25 and sine Square Alpha is 0.75 and this there are 2 index values given.

By substitute the values in global stiffness matrix

$$\cos^2 \propto = 0.5 \text{ X } 0.5 = 0.25$$

$$\sin^2 \propto = 0.866 \text{ X } 0.866 = 0.75$$

$$\cos \propto \sin \propto = 0.4330$$

The first row is corresponding to local degrees of freedom 1 2 3 4 and the second row is corresponding to global degrees of freedom 1 2 3 4 1 2 3 4 and it just so, happens that the first element the degrees of freedom one 2 3 4 at the local level they correspond to 1 2 3 4 at the global level. So, we can directly place it in the global Matrix in the first 4 rows and 4 columns.

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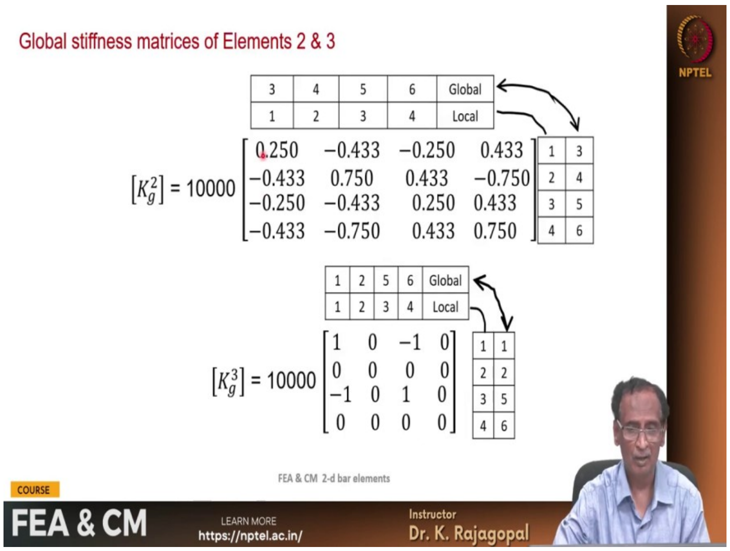
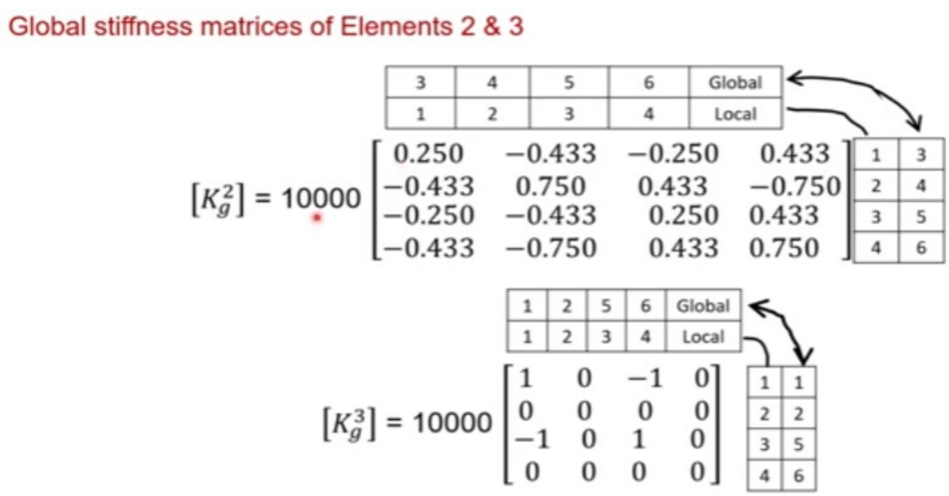


Then for element 2 we have the degree the global degrees of freedom 3 4 5 6 3 4 5 6 and one 2 3 4 at the local degrees of freedom. So, we can directly place this the contribution of element 2 in the last 4 rows and last 4 columns whereas element 3 the we have the local numbers 1 2 3 4 and the corresponding global coordinate numbers are 1 2 5 1 6. So, we place the contribution of this third element in rows 1 2 and then rows 5 6 and columns 1 2 and columns 5 6.



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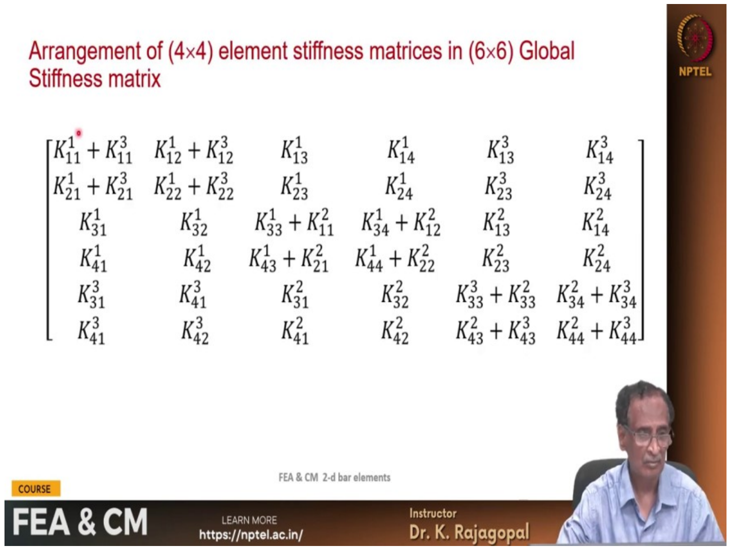

So, if you do this exercise and this is our combined Matrix in the global coordinates and in the subscript 11 refers to the element numbers whereas the super script 1 means that it is the corresponding to element 1.

$$\begin{bmatrix} K_{11}^1 + K_{11}^3 & K_{12}^1 + K_{12}^3 & K_{13}^1 & K_{14}^1 & K_{13}^3 & K_{14}^3 \\ K_{21}^1 + K_{21}^3 & K_{22}^1 + K_{22}^3 & K_{23}^1 & K_{24}^1 & K_{23}^3 & K_{24}^3 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 + K_{11}^2 & K_{34}^1 + K_{12}^2 & K_{13}^2 & K_{14}^2 \\ K_{41}^1 & K_{42}^1 & K_{43}^1 + K_{21}^2 & K_{44}^1 + K_{22}^2 & K_{23}^2 & K_{24}^2 \\ K_{31}^3 & K_{41}^3 & K_{31}^2 & K_{32}^2 & K_{33}^3 + K_{33}^2 & K_{34}^2 + K_{34}^3 \\ K_{41}^3 & K_{42}^3 & K_{41}^2 & K_{42}^2 & K_{43}^2 + K_{43}^3 & K_{44}^2 + K_{44}^3 \end{bmatrix}$$

And so, actually see the correspondence between the local numbers and the global numbers. And accordingly the contributions are placed the contribution of element 1 is in the first 4 rows and 4 columns because the local numbers one 2 3 4 they correspond to Global numbers 1 2 3 4 whereas for element 2 they correspond to 3 4 5 6.

So, we are placing the contribution of element 2 in the last 4 rows and 4 columns whereas the contribution of element 3 is in the first 2 columns and the last 2 columns and the first 2 rows and last 2 rows. So, 3 is here and then once again here not in the second sorry the third and 4th rows are third and 4th columns.

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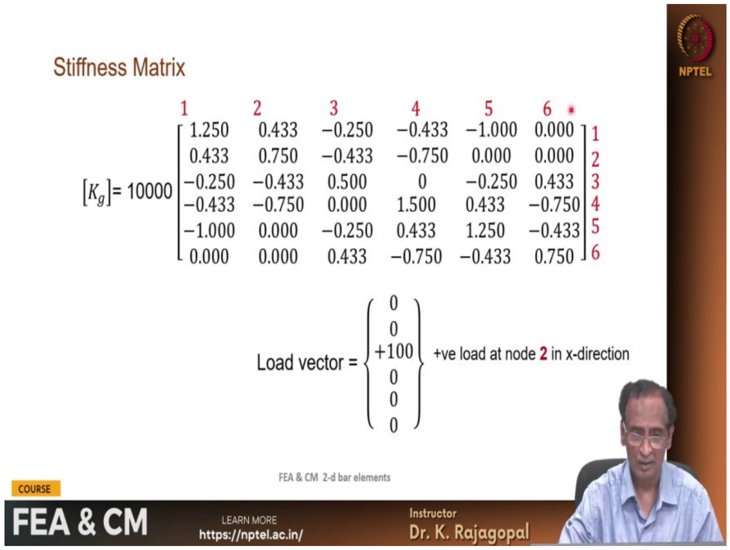

So, if you add up this is your total stiffness matrix and then the load Vector there is only one load 100 kilonewton supplied at node 2 in the positive x direction.

$$
[K_g] = 10000 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1.250 & 0.433 & -0.250 & -0.433 & -1.000 & 0.000 \\ 0.433 & 0.750 & -0.433 & -0.750 & 0.000 & 0.000 \\ -0.250 & -0.433 & 0.500 & 0 & -0.250 & 0.433 \\ -0.433 & -0.750 & 0.000 & 1.500 & 0.433 & -0.750 \\ -1.000 & 0.000 & -0.250 & 0.433 & 1.250 & -0.433 \\ 0.000 & 0.000 & 0.433 & -0.750 & -0.433 & 0.750 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

So, this is our load vector and this is our stiffness matrix and actually I forgot to mention one more thing. See for us to avoid rigid body deformations we need some boundary constraints. So, here on the left hand side we have provided a hinge support. So, that both x and y displacements are zero at node 3 it is unroller.

So, our x direction displacement it can take place but y direction it is constrained and this is a stable system because whatever force that you apply the supports are there to prevent the rigid body deformations. But suppose we place this node 1 also in a hinge then it can have infinite lateral deformations without any without any restraint. So, here we have sufficient number of constraints. So, we will not have any rigid body displacements and the degrees of freedom 1 2 and 6 are constrained.

So, we can eliminate rho and column of 1 2 and 6. So, the degrees of freedom one 2 6 are fixed. So, we can delete the corresponding rows and columns and we get a 3 by 3 Matrix like this and by solving you get a u 3 of this u 2 and u 2 and v 2 and u 2 and then the forces in element 3 and this is our stiffness Matrix the same in both global and local coordinates because it is coinciding with this global coordinate.

Solving by Gauss elimination procedure

$u_3 = +0.0050$

$v_2 = -0.001443$

$u_2 = +0.0225$

So, if you calculate the forces are -50 0 + 50 and 0. -50 at node 1 and +50 at node 2 and this indicates a tensile force because we are pulling on both the sides. So, + 50 is the tensile force.

Forces in Element-3:

$$\begin{bmatrix} 10000 & 0 & -10000 & 0 \\ 0 & 0 & 0 & 0 \\ -10000 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.005 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} -50 \\ 0.0 \\ +50 \\ 0.0 \end{Bmatrix}$$

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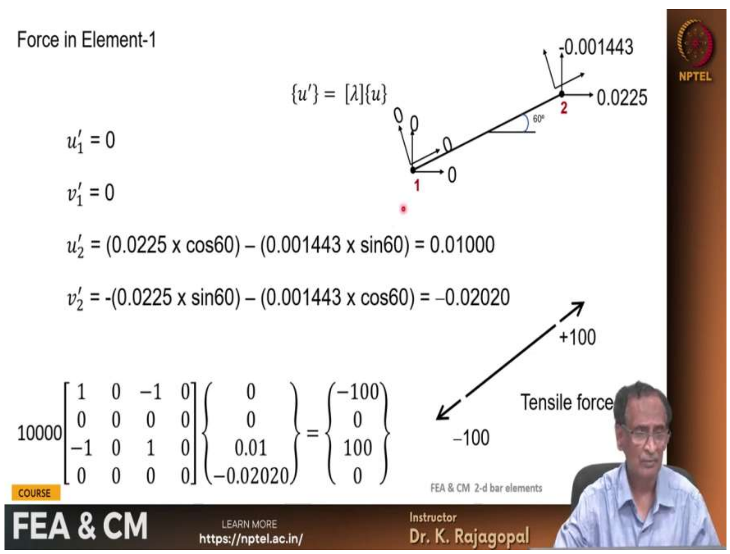

And then similarly that in element one we need the node one is fixed. So, whether it is global or local displacements they are the same and then at element 2 we have these displacements. And so, we need to resolve them to find the local displacements u 2 Prime and v 2 Prime and these are the displacements in the in the local directions and the stiffness matrix in the local direction multiplied by u Prime will give you the forces the axial forces. So, we see that it is -100 here and +100 there that is once again indicating the pull.

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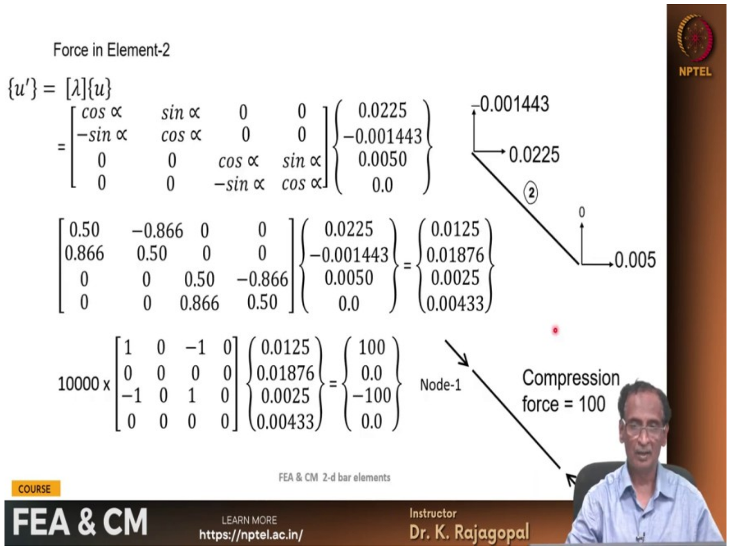

And then for element 2 we can calculate u Prime then we when we calculate the element forces at node 1 the force is positive and at node 2 the force is negative and this is the nature like we are compressing the element. So, we can think of that as a compression force of 100.

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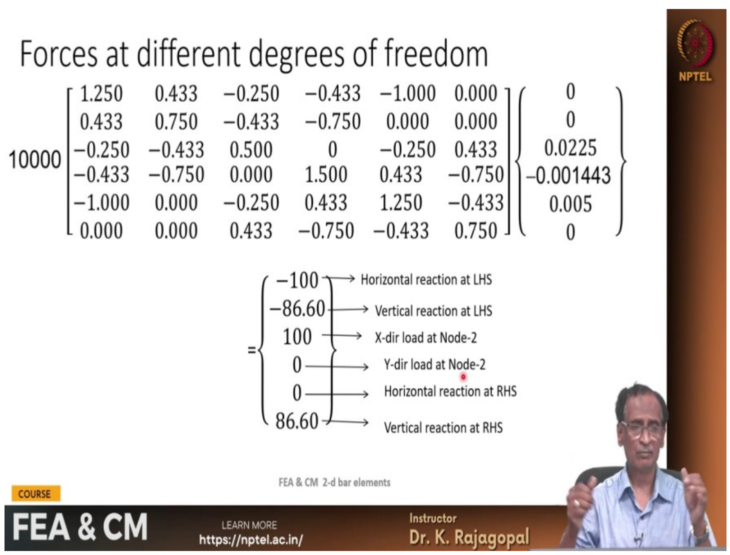
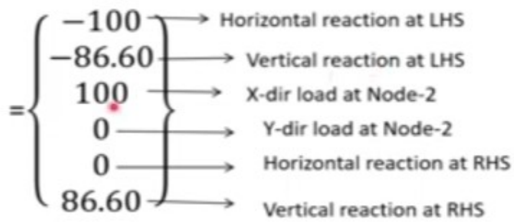


And we can also get the check for the global equilibrium and by multiplying this stiffness Matrix with the global displacements we get this. And at degree of freedom one we get -100 that is corresponding to horizontal reaction at the left hand side support yeah and -86.6 that is the vertical reaction at left hand side support +100 that is corresponding to the applied force at node 2 and 0 0 because there are no forces in these 2 degrees of freedom.



And then this corresponds to vertical reaction at the right hand side support actually we required these reaction forces at the supports. So, that we can decide or we can design the foundation. So, our foundation that we provide should be able to comfortably resist a force of 100 in the horizontal direction and force of 86.6 in the vertical direction and this is downward acting this is an upward acting.

So, obviously if you have an upward acting force we will go in for pile foundation like a tension pile or something like this or a ground anchor to support this. So, this analysis has given us the displacements and then the element forces and then also the the reaction forces. So, that completes our analysis. So, if you are not happy with these displacements if you think these displacements are too large then we can reduce them by by changing the element properties.

Like we can go in for a stiffer element like we might go in for a bigger cross section to increase our stiffness AE by L or change the material. And then we can get our displacements within the control and we need to design our supports for these reaction forces and then we should make sure that our elements they are they are able to resist these forces without undergoing any tensile yielding or buckling.

Because whatever element is in compression we should check for buckling and with if it is in tension we need to check for tensile yield stress. So, that completes our analysis and we got all the required data displacements and then the forces and then the reaction forces.

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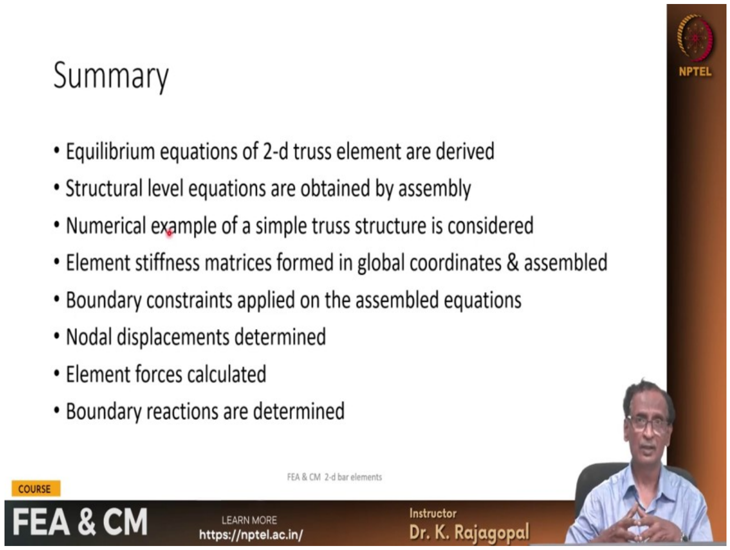

Just to summarize we have gone through the entire system of calculations for a 2-dimensional trust structure and we have assembled all the element equations at the structural level to form the global equations then by solving we can get our displacements and then we can get our element forces and then everything else. And we have also seen the calculations through a simple numerical example.

And so, this is how we do these calculations in this particular example we have a simple structure with 3 nodes and 3 elements and we have totally 6 degrees of freedom. And after eliminating the fixed degrees of freedom we ended up with a 3 by 3 Matrix. But let us say you have a bigger structure like 100 elements or 100 nodes and you will have a 200 degrees of freedom and that type of system you cannot solve by hand because it is too big then we have to go in for this type of analysis.

The later I will show you one computer program that you can utilize for doing any of these analyses. So, thank you very much and if you have any questions you please write an email to this to this address and I will reply back. So, thank you very much.