Finite Element Analysis and Constitutive Modelling in Geomechanics
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Lecture - 38
Introduction to Consolidation and Dynamic Analysis

Hello students, good morning. I hope you are doing well in the course and you remember what we did in the previous classes. We were looking at the different ways of correcting the stresses and then we had seen different methods of non-linear analysis using initial stress and then the tangent stiffness method and so on. And some examples of the bearing capacity calculations were seen in the previous class. And today's class let us look at the elastic plastic modelling of the soils.

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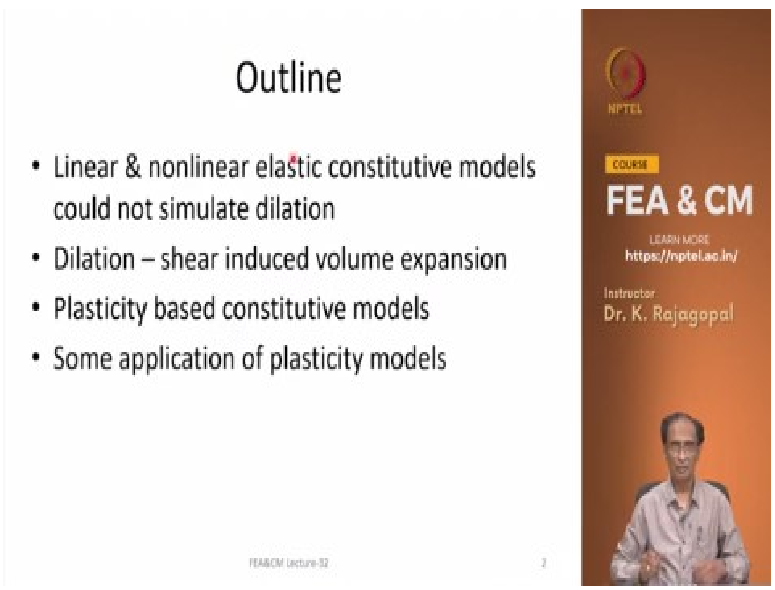

Because till date although we had looked at number of different constituted models all of them were good and some of them were stress dependent that they can simulate the influence of both the stiffness and the strength and the confining pressure. But then they were not able to simulate the shear induced dilation or the volume expansion and at that time itself we had seen that all the elastic volumetric changes will be only compressive.

If your axial strain is compressive the volumetric strain within the triaxial compression test will also be compressive. And now we are going to look at a new approach to the modelling by introducing the plastic strains and this will give us a more capability of simulating the volume expansion and then the strength increase under suppressed dilatancy and so on.

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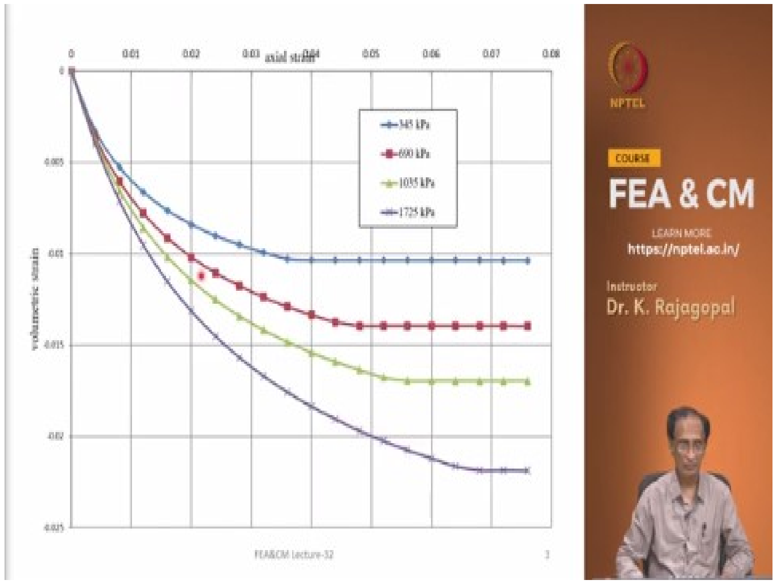

See this is what we had seen these are the volumetric strains during the triaxial compression test and the x axis we have the axial strain on the y axis we have the volumetric strains a different confining pressures. And all of them were only compressive at all the different confining pressures.

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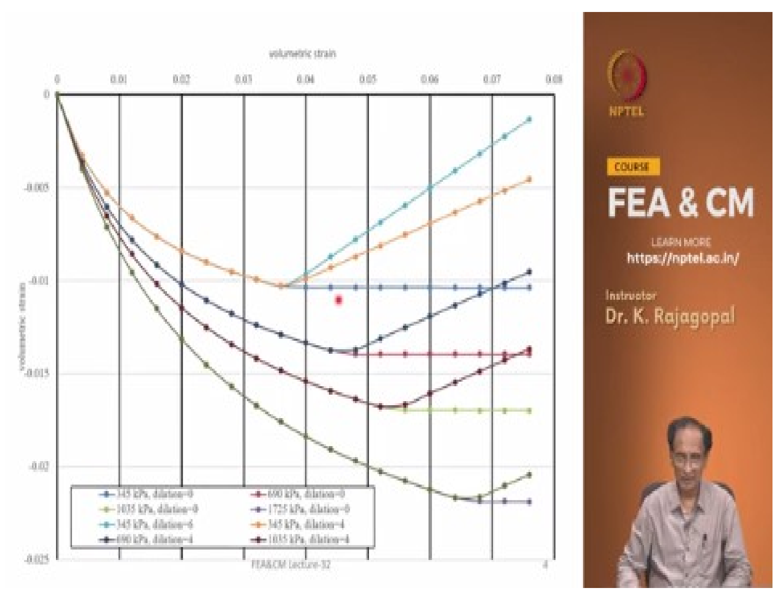

And then this is what we want to simulate there might be initial compression and then after the soil reaches the plastic limit state there should be volume increase this is what we want to predict and in fact these are showing a different confining pressures and different dilation angles. This particular one is for a dilation angle of 0 and then this is for 4 degrees. I think this must be I did not market here this is for six degrees a dilation angle at a confining pressure of 345.

And we see that at a higher confining pressure the soil undergoes the volumetric compression for a longer axial strain. So, for example at confining pressure of 345 after axial strain of about a 0.035 the soil started expanding our or it has reached a constant volume state but at this higher confining pressure of 1725 the plasticity has set in only at a axial strain of more than 0.06. And you notice that the slope of these lines is the same for the same dilation angle.

And although the confining pressure is different, we get the same slope that means that this particular model that we are going to discuss is still not realistic but then we can say that we are able to simulate volume expansion and so we should be happy.

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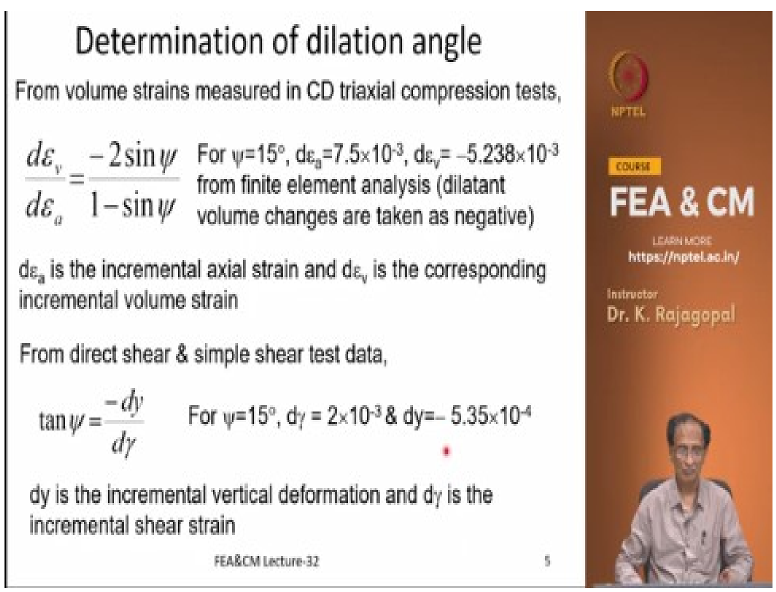

So, how do we determine the dilation angle? So, previously we had seen how to determine the and the friction angle by either drawing a common tangent to all the more circles or within the pq space we can do a best fit line and then determine the cohesion and then the friction angle. And the dilation angle we can determine from a consolidated drain test because we can measure the volume changes.

From volume strains measured in CD triaxial compression tests,

$$\frac{d\varepsilon_v}{d\varepsilon_a} = \frac{-2\sin\psi}{1-\sin\psi}$$

For ψ=15°, $d\varepsilon_a$=7.5×10^{-3}, $d\varepsilon_v$= –5.238×10^{-3} from finite element analysis (dilatant volume changes are taken as negative)

$d\varepsilon_a$ is the incremental axial strain and $d\varepsilon_v$ is the corresponding incremental volume strain

From direct shear & simple shear test data,

$$\tan\psi = \frac{-dy}{d\gamma}$$

For ψ=15°, $d\gamma$ = 2×10^{-3} & dy=– 5.35×10^{-4}

And these are some of the theoretical relations for triaxial compression test the ratio between the incremental volume change to the incremental axial strain is in terms of the psi its - 2 times sin psi by 1 - sin psi is actually this is from the we can easily derive it based on our fundamental definition for d epsilon v, that is the volumetric strain. And these are the typical results these are from and the finite element program for a dilation angle of 15 degrees and the d epsilon a is 7.5 times 10 to the power of – 3.

And the corresponding d epsilon v is - 5.238 times 10 to the power of - 3 and if you substitute this and solve this equation and this, I will come to approximately 15 degrees. So, that is how we can actually determine the dilation angle. So, within the from the triaxial compression test we get the increment of volumetric strain and increment of axial strain. And then by using this equation we can back predict the dilation angle.

And similarly, from the direct shear and the simple shear test as we are applying shear strain, we are going to measure the vertical deformations and tan psi is - dy by d gamma is actually in all these cases the minus sign is to say that the expansion is negative. In the geotechnical sign convention, the tensile stresses are taken as negative and for this particular case for a friction angle sorry for a dilation angle of 15 degrees d gamma is this and dy is this much.

And so, if you substitute this back and take psi of psi as a tan inverse of this quantity, we will get approximately 15 degrees.

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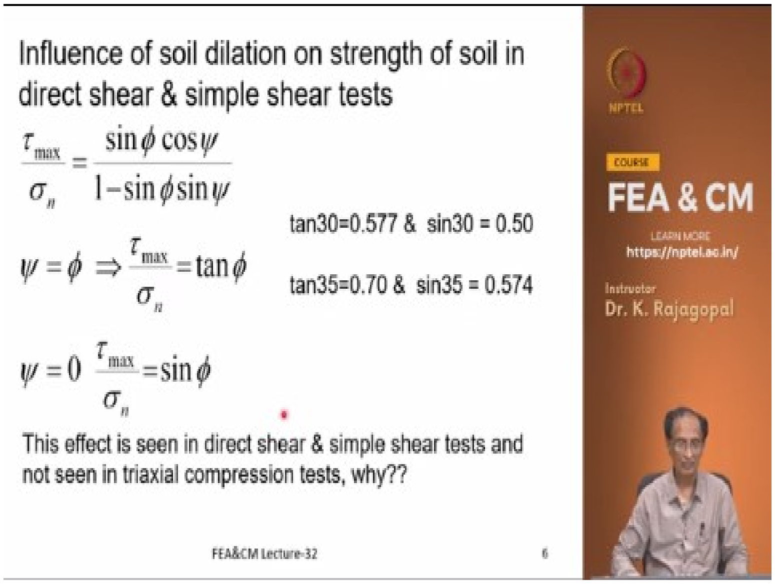
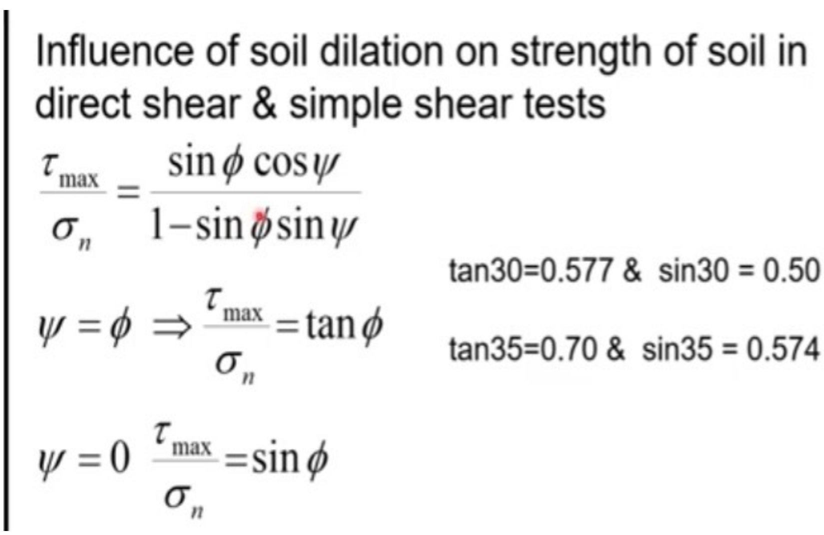


And what is the influence of a dilation angle and the strength of the soil? And this we can derive this is a theoretical equation and there is a long derivation based on the more circle analysis. So, this I will give the proof as an MS Word file because it is a very long one and its quite boring to explain but then it is quite easy if you do it yourself. The tau max by sigma n is sin phi times cosine psi by 1 - sin phi sin psi and so if you substitute there is psi of equal to phi this quantity becomes tan phi.



Because in the numerator we have sin phi times cosine phi by 1 - sin phi times sin phi that is 1 - sin square phi that is cosine square phi. So, end up with sin phi by cosine phi that is tan phi and if dilation angle is 0 this tau max by sigma n is a sin phi and for any friction angle phi the tan phi is more than sin phi. For example, for 30 degree tan 30 is 0.577 and sin 30 is 0.5 and for 35 degree tan 35 is 0.7 and sin 35 is 0.574.

And we see that as the dilation angle is increasing from 0 towards the friction angle, we will see some increase in the strength. And this effect we can see only in the direct shear and the simple shear test but not in triaxial compression test. Why? Because in the triaxial compression test the sample is free to expand in one direction so, we will not see the effect of the suppressed dilatancy in the triaxial compression test.

Whereas in the direct shear and simple shear we are doing the test in a rigid box and because of the rigid confinement the soil has to expand only in one direction and against and that expansion is happening against the applied pressure. So, because of that we will see some increase and it is easy to demonstrate through finite element analysis and that I will show you now.

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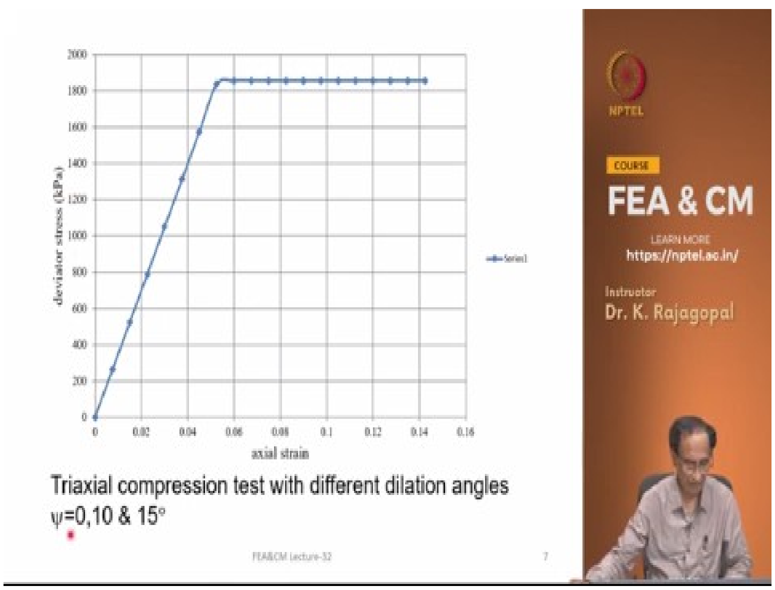

Say here this is the plot between the axial strain and the deviate stress for three different dilation angles dilation angle of 0, 10 and 15. For all of them you get the same stress strain response because the soil is free to expand in the other direction without doing any extra work.

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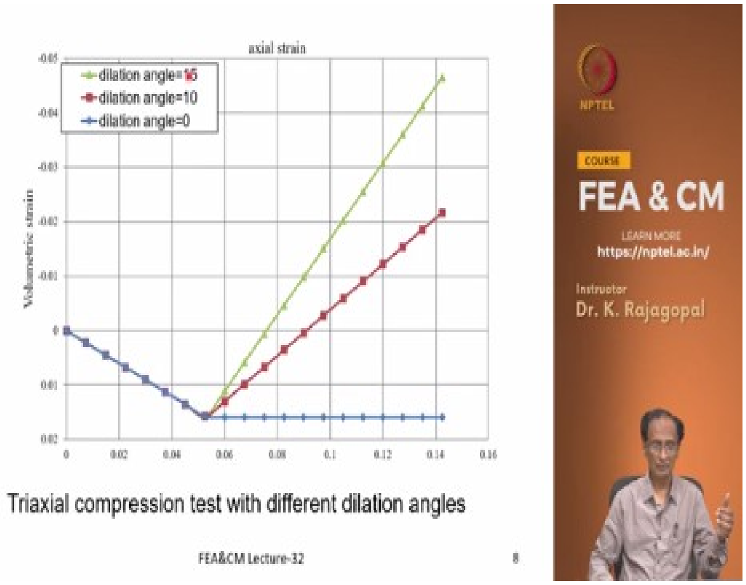

Whereas the stress strain response is the same but if you look at the volumetric strain response you get three different responses this is for a dilation angle of 0 this is for 10 degrees and this is for 15 degrees. And from the slope of these lines, we can back predict the dilation angle through the equation that I gave you earlier.

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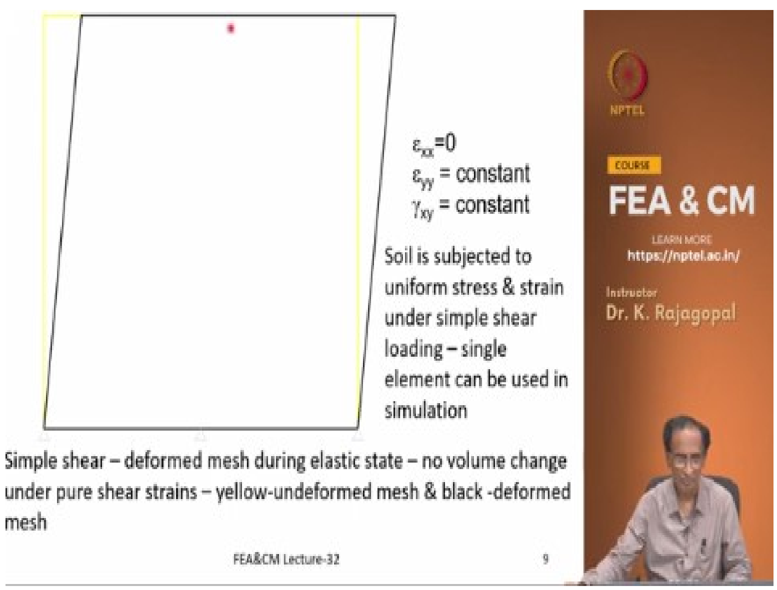

Whereas if we look at the simple shear test say the simple shear test is there are different varieties one is in a cubical box and that is one type of test and then there are other tests where we use cylindrical samples. But this is actually demonstrating only the and the cubicle sample test and there are some constraints within the simple shear test the epsilon xx is 0 that is the horizontal strain is 0.

So, that means that because basically we have a rigid box and that is both the vertical surfaces are moving by the same amount. That means that the relative displacement is 0 and so our epsilon xx is 0 and similarly in the y direction epsilon yy is constant and then the tau xy and the gamma xy is constant all through because the shear strain is constant. And we can since the stress state is constant, we can just use one single element for doing this simple shear test.

And this particular one that you are seeing this shows the deformed shape during the elastic state and you see that there is no volume change. Because it is basically it is a pure shear strain and under pure shear strain you will only get shear stresses and you will not get any volumetric strains either epsilon xx is anywhere 0 epsilon yy is 0 because we are not applying any strain or stress in the y direction. And so, you see that these two surfaces at the top they are exactly coinciding during the elastic state.

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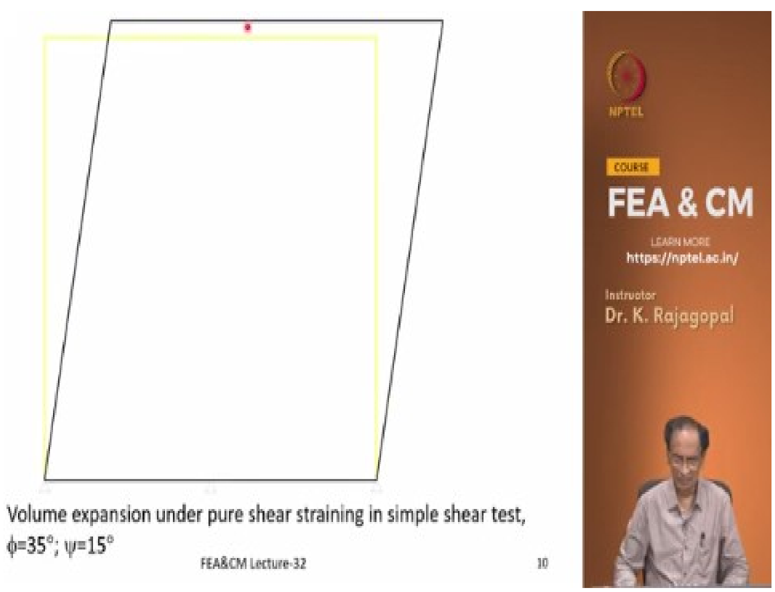

Whereas if you look at the during the plastic state there is some volume expansion, say this the top line has moved up. This is for a dilation angle of 15 degrees and for a friction angle of 35 degrees.

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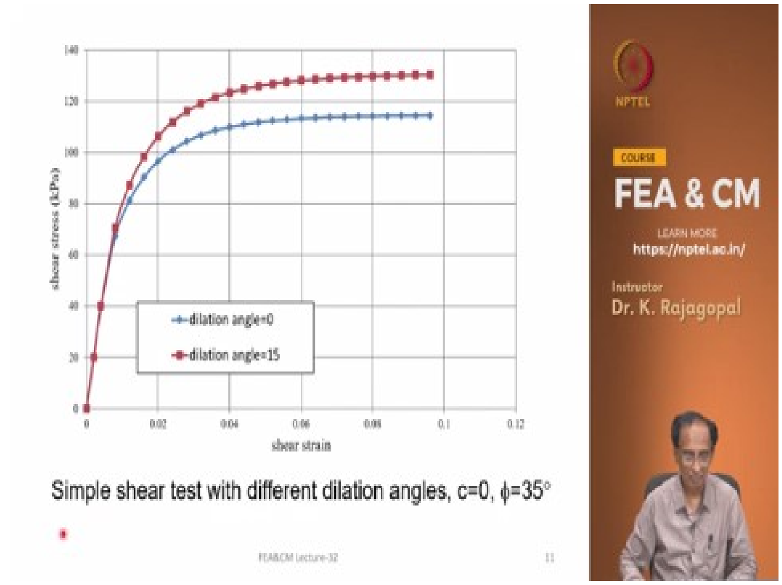

And you see two responses for a dilation angle of 0 this is the blue line and for the red for a dilation angle of 15 degrees you see an increased strength both for the same friction angle of 35 degrees and c of 0. So, you see the effect of dilation the soil wants to expand and this expansion is happening against the applied normal pressure, so because of that the soil has to do some additional work to induce this volume expansion or dilation. And that will result in a higher strength or higher stresses.

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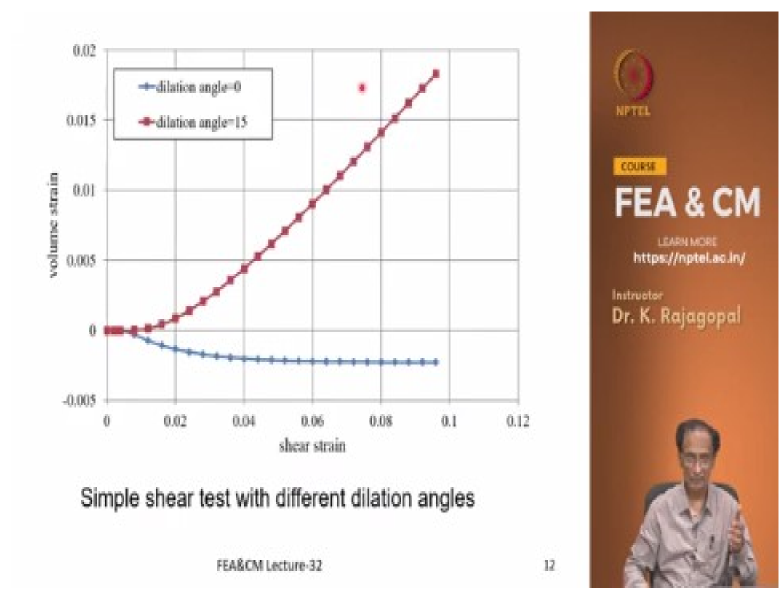

These are the volume strain responses, initially there is both of them have shown zero volume strain during the elastic state and then there is some compression with the dilation angle of 0 and then with the dilation angle of 15 degrees there is expansion.

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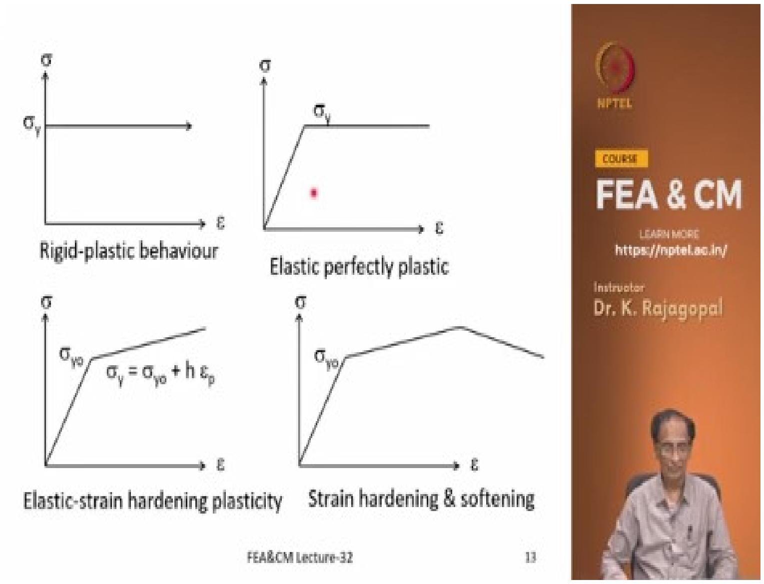

And what are the different types of plasticity that we can model? One is the rigid plastic behaviour this is typical of the brittle materials. There is no elastic strain, elastic strain is 0 during the stress application soil does not the material does not undergo any strains. And suddenly it will reach a plastic state when you might end up with the infinite strain at the same stress of sigma y. Sigma y is deal stress and then there is an elastic perfectly plastic.

Actually, in both of these models you see that the sigma y deal stress is remaining constant and that we call as perfect plasticity that is even with the plastic strains induced in the soil the strength is remaining constant. And this particular one is called as the elastic perfectly plastic, initially there are some elastic strains and then after the plastic limit is reached the stress remains constant and then we can have elastic strain hardening beyond an initial yield stress of sigma y 0.

The stress will go on increasing because there is a hardening or there is a stiffening of the soil because of the plastic strain induced hardening and we can also have a strain softening. Say this particular behaviour is a typical of our loose sands where the soil will undergo compression and then the deviate stress will go on increasing until reaching some asymptotic limit. And we can also have a strain softening initially its elastic and then strain hardening the yield stress will go on increasing.

And beyond a point the yield stress will go on reducing or the stress reduces and this particular behaviour is similar to our dense sands there is a nice peak and then beyond the peak the stress will go on reducing and that we call as the strain softening, this is the strain hardening part and then the strain softening part. And there are different ways of modelling this and one simple model is we assume that the increase in the deal stress is proportional to the plastic strains.

Because normally in a continuum there are different varieties of plastic strains in x direction y direction and z direction and then then in the xy, yz and so on like we take a measure. And that I think I am not going to explain in this course, that is too advanced but there are different varieties of hardening rules. But in this case, in this course we are going to look at only the perfect plasticity perfectly elastic plastic materials where the yield stress remains constant.

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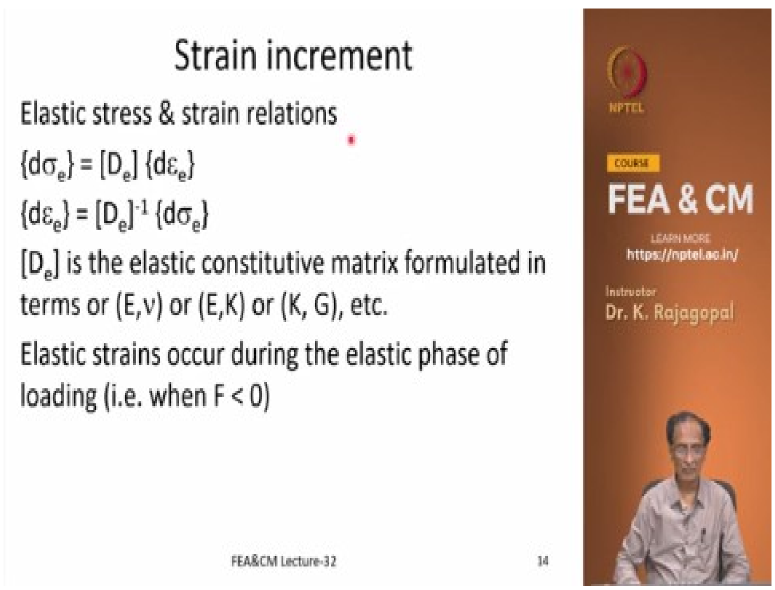

And let us look at how we can do this analysis. During the elastic state our stress and strain are related like this d sigma is D elastic times d epsilon elastic D is the elastic constitutive matrix and we can write d epsilon as d inverse of d sigma and D e is the elastic constituted matrix formulated either in terms of Young's modulus poisons ratio or bulk modulus and the shear modulus are the Young's modulus and bulk modulus and so on that we have seen earlier.

Strain increment

Elastic stress & strain relations

$\{d\sigma_e\} = [D_e]\{d\varepsilon_e\}$

$\{d\varepsilon_e\} = [D_e]^{-1}\{d\sigma_e\}$

$[D_e]$ is the elastic constitutive matrix formulated in terms or (E,ν) or (E,K) or (K, G), etc.

Elastic strains occur during the elastic phase of loading (i.e. when $F < 0$)

And the elastic strains occur during the elastic phase of the loading that is when the yield function is 0 is less than 0. The deal function we have seen earlier that enables us to examine whether the given stress state is in the elastic state or the plastic state that is when F is less than 0 the soil is in the elastic state and when it is 0 it is in the plastic state or we can also draw a more circle. If your more circle is below the yield surface, we say that particular stress state is elastic.

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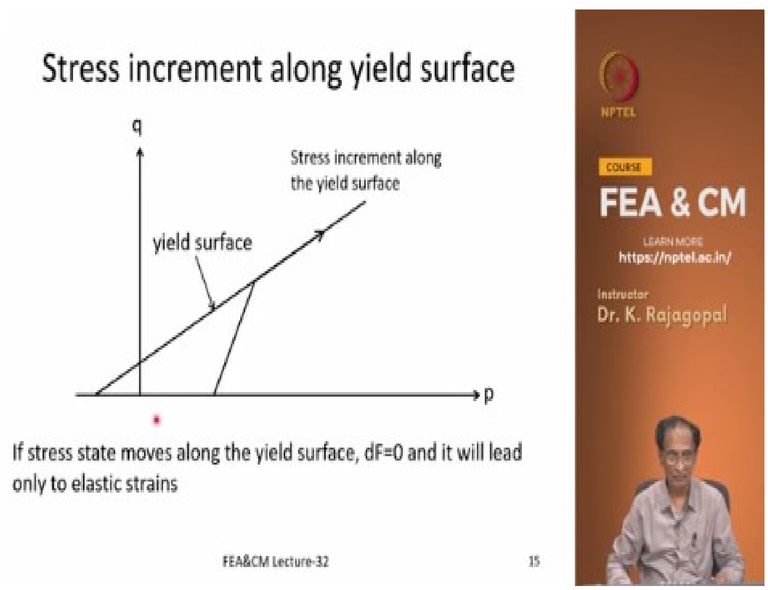
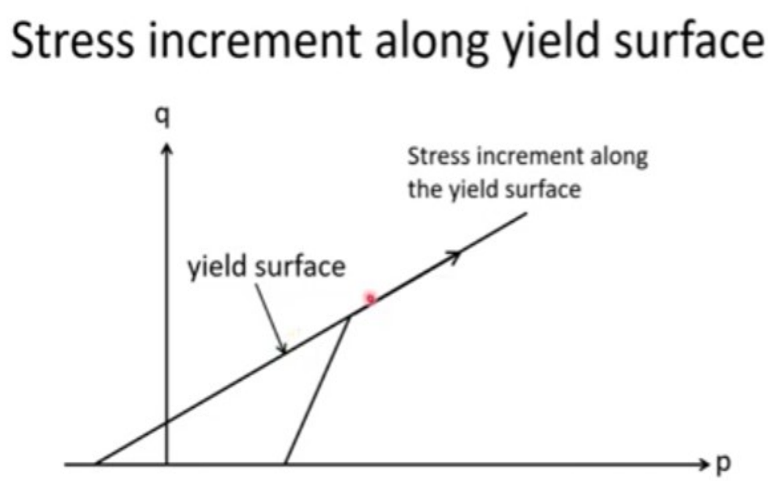


In the pq space our real surface a molecular meal surface is something like this and so if you plot any stress path initially our we have only normal stresses and the shear stress is 0, q is 0 and then the stress might increase then after reaching here the stress should move along the yield surface without exceeding the yield surface and this is the stress increment during the yield surface along the yield surface.



And because this surface is actually our yield surface F and we can say that F remains constant along the surface or dF is 0.

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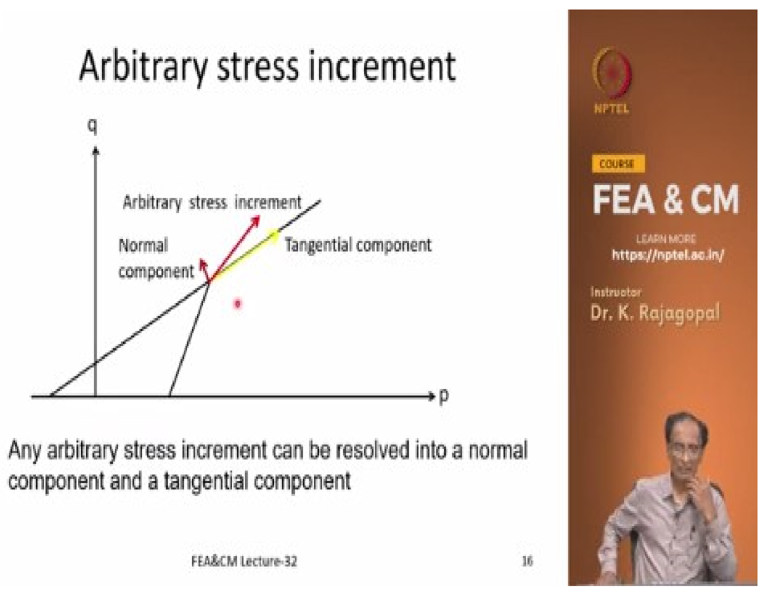
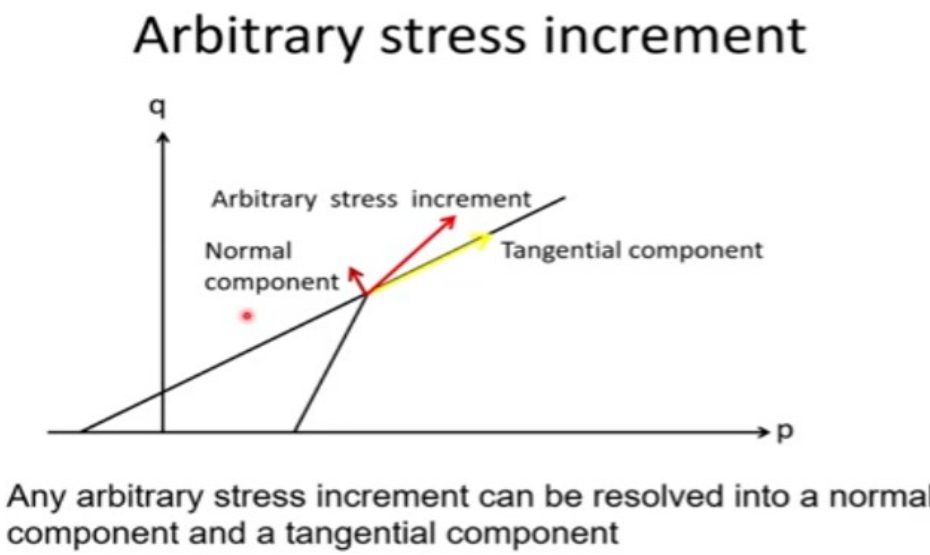


See we can apply any stress increment in any arbitrary stress increment like this say like this orange colour vector and we can resolve this into two components one is along the yield surface and the other is normal to the yield surface. And there is a tangential component then normal component and if our stresses are moving along the yield surface our yield function value remains constant and we will not be able to produce any plastic strains.



All these strains that happen when your F is less than or equal to 0 or elastic and only the normal component is going to produce our plastic strains because it is trying to cause some strain and so this normal component can be represented as the derivative of this yield function F with respect to different stress component sigma x sigma y tau xy and so on.

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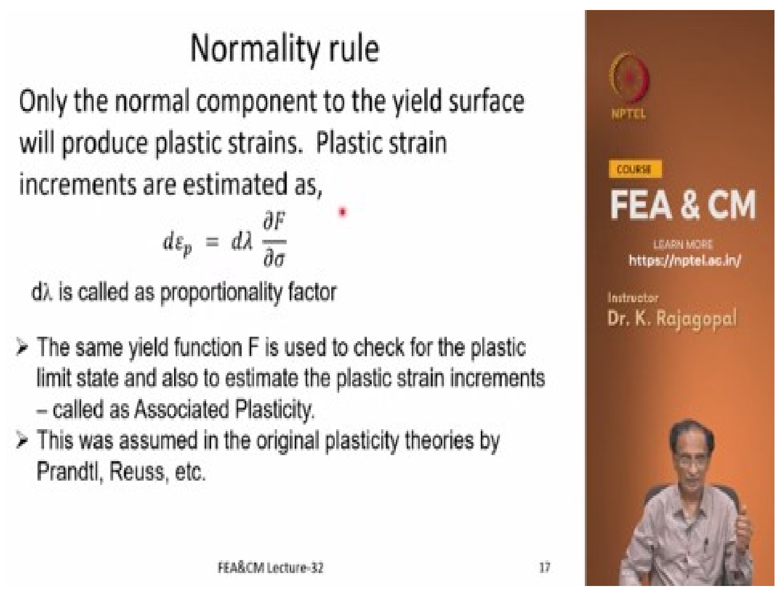

And the conventional plasticity or in the classical plasticity theories this rule is called as the normality rule the d epsilon p is a d lambda times dou F by dou sigma where dou F by dou sigma represents the normal yield surface and the pq diagram p is the normal stress and the q is the shear stress. In this diagram the yield surface is this very simple and dou F by dou sigma is your normal direction and the d lambda is actually it is a proportionality constant.

$$d\varepsilon_p = d\lambda \frac{\partial F}{\partial \sigma}$$

dλ is called as proportionality factor

And that we need to determine we do not know it but we can determine the d lambda later. So, if you know d lambda and if you are able to evaluate dou F by dou sigma the plastic strain increments can be evaluated. And this type of plastic theory is called as the associated plasticity because we are using the same yield function F for checking for the yield and also for calculating the plastic strains.

So, our ref is defining this surface and we are also calculating the plastic strains as d lambda times dou F by dou sigma and this was assumed in the original plasticity theory developed by Prandtl and Reuss and others.

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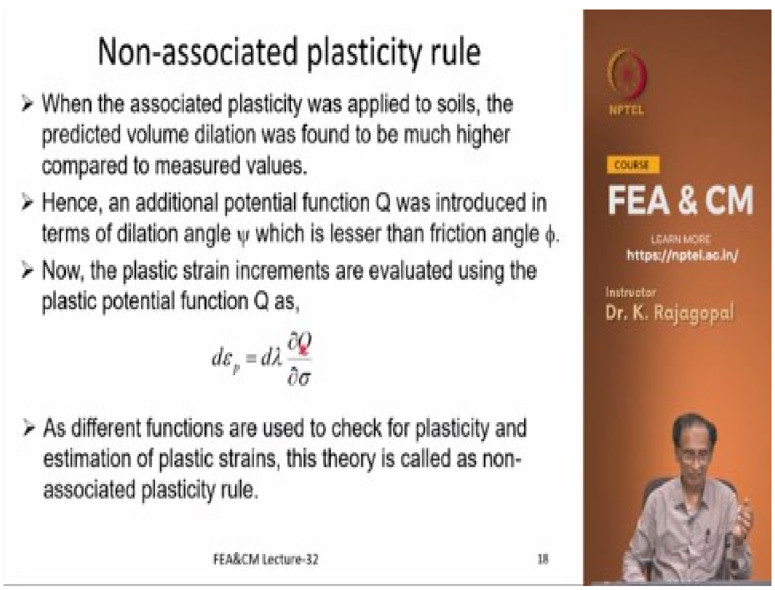

And when Terzaghi started applying those plasticity rules for the soils. See the metals do not have any friction angle because their strength is so much and their particles are so bounded together that any application of confining pressure will not change their shear strength, say for example you take the cubical samples sorry the cylindrical samples of metal like steel or some other material and then perform tetraxial compression tests at three different confining pressures.

We will see that the strength does not change its immaterial in it is independent of the confining pressure so we will get a horizontal line in the pq diagram. And so that means that your friction angle is 0. But then for soils we see that the friction angle could be significant depending on the sand content. And when the associated plasticity rule was applied to soils the predicted volume dilation was very high.

Now, the plastic strain increments are evaluated using the plastic potential function Q as,

$$d\varepsilon_p = d\lambda \frac{\partial Q}{\partial \sigma}$$

And Terzaghi and others worked on a slightly different concept and they introduced an additional potential function Q, that is defined in terms of dilation angle psi and usually it is less than the friction angle phi. And the now the plastic strains d epsilon p are written as d lambda times dou Q by dou sigma, where Q is our potential function that is defined in terms of dilation angle psi and since we are using two different functions.

One function for evaluating the yield, yield condition that is defining the yield surface and another surface Q for determining the plastic strains. This theory is called as the non-associated the plasticity rule and this is more applicable for soil mechanics problems where the dilation is a assumed to be lesser than what we get with the associated plasticity rules.

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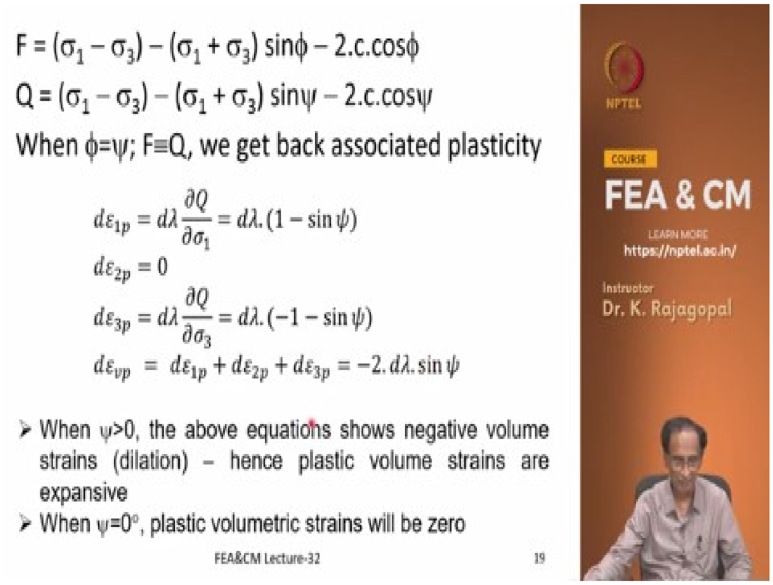

Say for example our yield function F was written in terms of sigma 1 and sigma 3 and c and phi like this and the same form of the function we can use for defining the Q but replace the friction angle phi with the dilation angle psi and when phi is psi their F and Q are the same and we get back our associated plasticity rule. And we can evaluate the plastic strains in different directions d epsilon 1p is a d lambda times dou Q by dou sigma 1.

$$F = (\sigma_1 - \sigma_3) - (\sigma_1 + \sigma_3)\sin\phi - 2.c.\cos\phi$$

$$Q = (\sigma_1 - \sigma_3) - (\sigma_1 + \sigma_3)\sin\psi - 2.c.\cos\psi$$

When $\phi=\psi$; $F\equiv Q$, we get back associated plasticity

$$d\varepsilon_{1p} = d\lambda\frac{\partial Q}{\partial \sigma_1} = d\lambda.(1 - \sin\psi)$$

$$d\varepsilon_{2p} = 0$$

$$d\varepsilon_{3p} = d\lambda\frac{\partial Q}{\partial \sigma_3} = d\lambda.(-1 - \sin\psi)$$

$$d\varepsilon_{vp} = d\varepsilon_{1p} + d\varepsilon_{2p} + d\varepsilon_{3p} = -2.\,d\lambda.\sin\psi$$

So, if you do this you get del lambda times 1 - sin psi and d epsilon 2p is 0 because we do not have sigma 2 in this potential function and d epsilon 3p is d lambda times dou Q by dou sigma 3 that is d lambda times – 1 - sin psi. So, our total plastic volumetric strain is a d epsilon 1p + d epsilon 2p + d epsilon 3p that comes to - 2 times d lambda sin psi and the negative sign is actually refers to volume expansion.

And when psi is greater than 0, we see from the above equation that we will get some plastic volumetric strains. So, and this has a negative sign that means that we will get a expansion and when dilation angle is 0 we will not get any volume expansion the volume strain will be 0.

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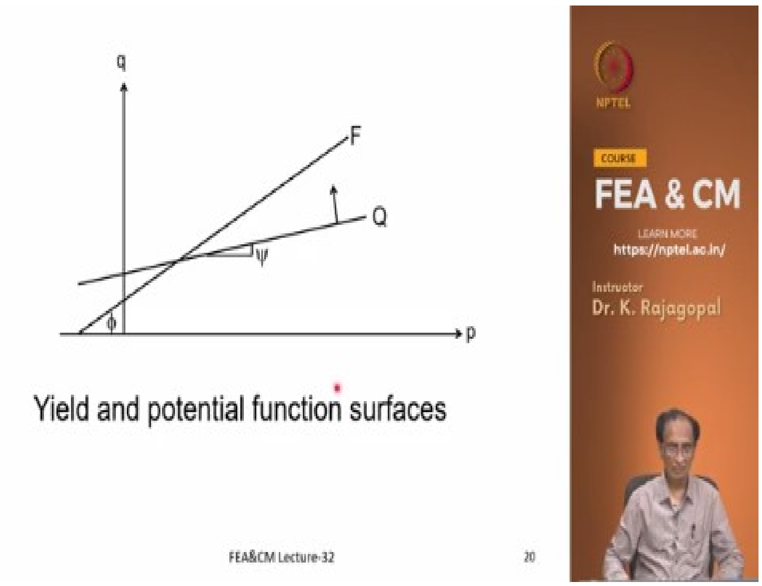
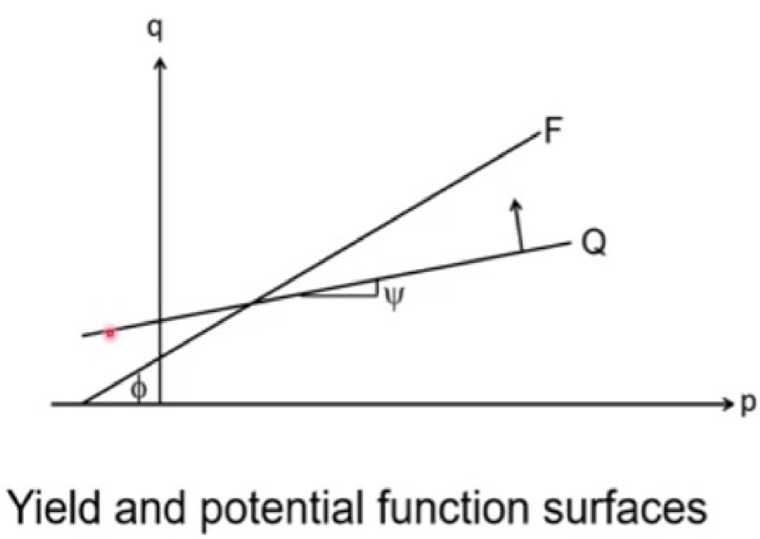


So, here we can we see in the pq plot we have one surface F that is the yield surface and the other is the plastic potential surface Q and that has a lesser slope. Actually, we can write Q as just simply sigma 1 - sigma 3 - sigma 1 + sigma 3 sin psi. We do not require this 2 c cosine psi because we do not require this term for evaluating our plastic strains. So, we show this Q with a lesser slope but then the intercept could be anything.



It is a non, I have shown with some arbitrary intercept it could be anything but we are not really interested in the intercept for the plastic potential function. We are only interested in the slope of that line.

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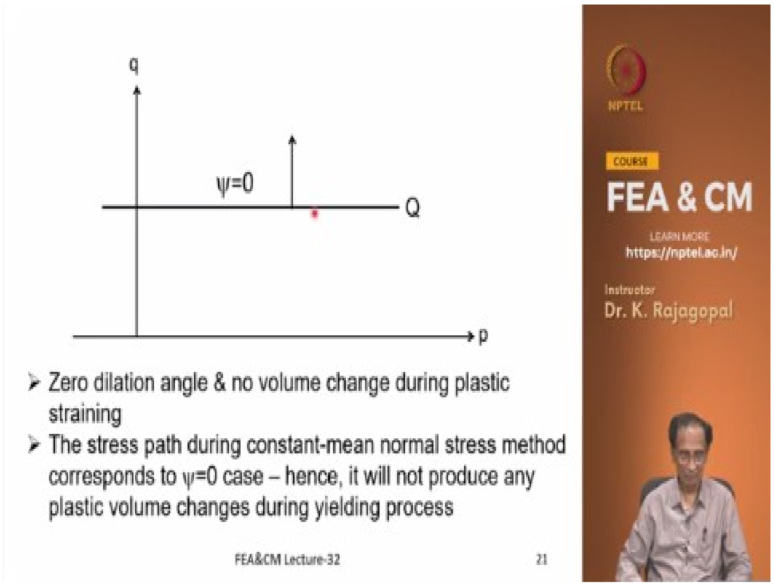

When dilation angle is 0, we get a horizontal plastic potential surface horizontal line and the normal to that is vertical and in fact this normal direction we have seen with the mean normal stress method of correction where we have seen that the stress path that is followed is vertical. It is a vertical line and that is perpendicular to this Q the Q of 0 represents the horizontal surface. So, we can say that our mean normal stress method of correction will not produce any volume strains a plastic volume strains.

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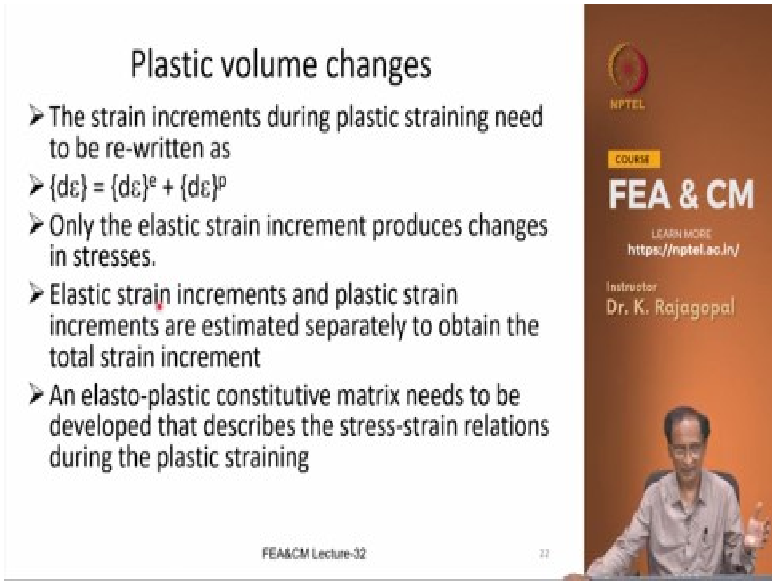

So, we can split the turtle strain tensor d epsilon as d epsilon e + d epsilon p and only the elastic volume elastic strains will produce some stresses and the plastic strains do not produce any stresses or distress increments. And we can write elastic strain increment and the plastic strain increment separately because we have two separate equations. And then we can get the total strain increment d epsilon as d epsilon e + d epsilon p.

$$\{d\varepsilon\} = \{d\varepsilon\}^e + \{d\varepsilon\}^p$$

And so based on this we can develop an elastoplastic constitutive matrix.

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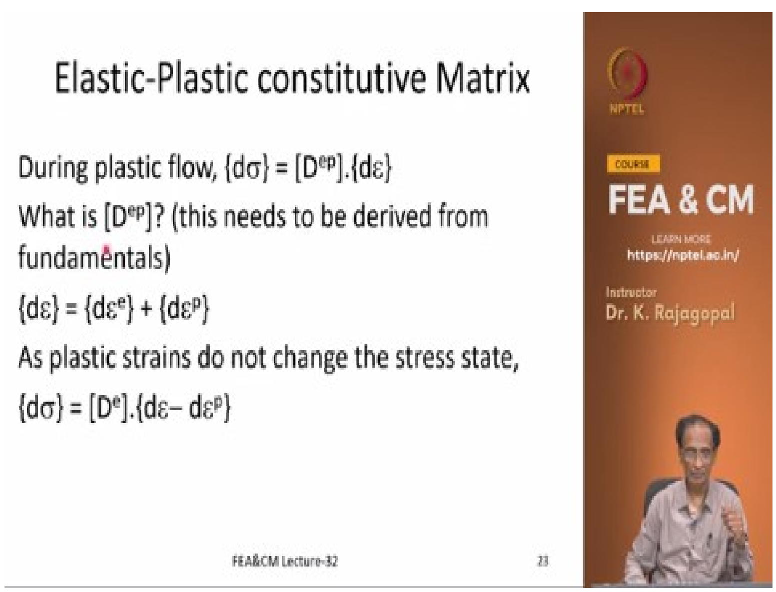

And during the plastic flow, d sigma is D ep times d epsilon and D ep is the elastoplastic constitute matrix and d epsilon is d epsilon e + d epsilon p and because our plastic strains do not involve any change in the stress state, we can write d sigma is a d elastic multiplied by d epsilon - d epsilon p because our this is your elastic strain increment multiplied by elastic constitute matrix is your incremental stresses.

During plastic flow, $\{d\sigma\} = [D^{ep}].\{d\varepsilon\}$

What is $[D^{ep}]$? (this needs to be derived from fundamentals)

$\{d\varepsilon\} = \{d\varepsilon^{e}\} + \{d\varepsilon^{p}\}$

As plastic strains do not change the stress state,

$\{d\sigma\} = [D^{e}].\{d\varepsilon - d\varepsilon^{p}\}$

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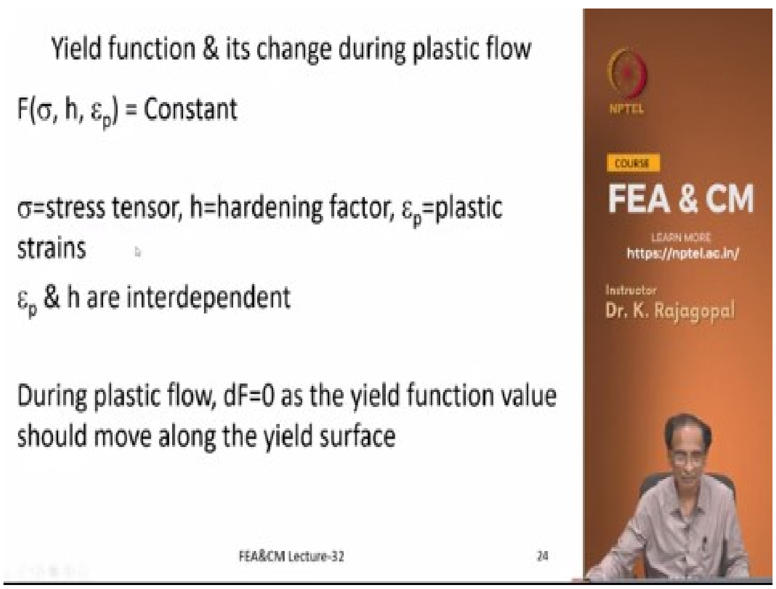

And in general, we can write the yield function as actually it should be constant. See during the plastic flow the yield function is constant the yield function could be a function of stresses and then the hardening factor hits and epsilon p and of course we also have the strength properties that have not listed. But the strength properties are we treat them as constants so we are not going to evaluate their derivatives.

Yield function & its change during plastic flow

$F(\sigma, h, \varepsilon_p) = 0$

σ=stress tensor, h=hardening factor, ε_p=plastic strains

ε_p & h are interdependent

During plastic flow, dF=0 as the yield function value should move along the yield surface

And sigma is the stress tensor h is the hardening factor the epsilon p is the plastic strains and epsilon p and h are interdependent because the hardening and then the plastic strains they are related to each other and during the plastic flow the dF is 0, the change in the if F is constant, we can say dF is 0. That is the change in the yield function 0 during the plastic flow because our stress state is moving along the yield surface.

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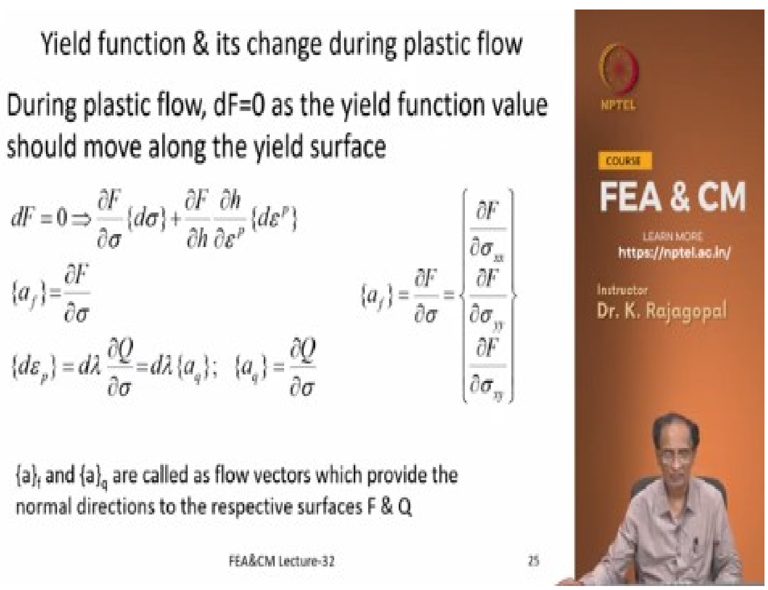

And dF of 0 means dou F by dou sigma times d sigma + dou F by dou h doe h by d dou epsilon p times d epsilon p by using the chain rule we can write like this. And our in this dou F by dou sigma is basically the normal yield surface and that let us designate with an a f that is a vector dou F by those dou sigma and d epsilon p is d lambda times dou Q by dou sigma. And that is d lambda times a q, a q is dou Q by dou sigma and a f is dou F by dou sigma for a simple plane strain case.

Yield function & its change during plastic flow

During plastic flow, dF=0 as the yield function value should move along the yield surface

$$dF = 0 \Rightarrow \frac{\partial F}{\partial \sigma}\{d\sigma\} + \frac{\partial F}{\partial h}\frac{\partial h}{\partial \varepsilon^{p}}\{d\varepsilon^{p}\}$$

$$\{a_f\} = \frac{\partial F}{\partial \sigma}$$

$$\{d\varepsilon_p\} = d\lambda \frac{\partial Q}{\partial \sigma} = d\lambda\{a_q\}; \quad \{a_q\} = \frac{\partial Q}{\partial \sigma}$$

$$\{a_f\} = \frac{\partial F}{\partial \sigma} = \begin{Bmatrix} \frac{\partial F}{\partial \sigma_{xx}} \\ \frac{\partial F}{\partial \sigma_{yy}} \\ \frac{\partial F}{\partial \sigma_{xy}} \end{Bmatrix}$$

$\{a\}_f$ and $\{a\}_q$ are called as flow vectors which provide the normal directions to the respective surfaces F & Q

It is dou F by dou sigma xx dou F by dou sigma yy and dou F by dou sigma xy and our a f and a q are called as the flow vectors which provide the normal direction to the respective yield surfaces F and Q.

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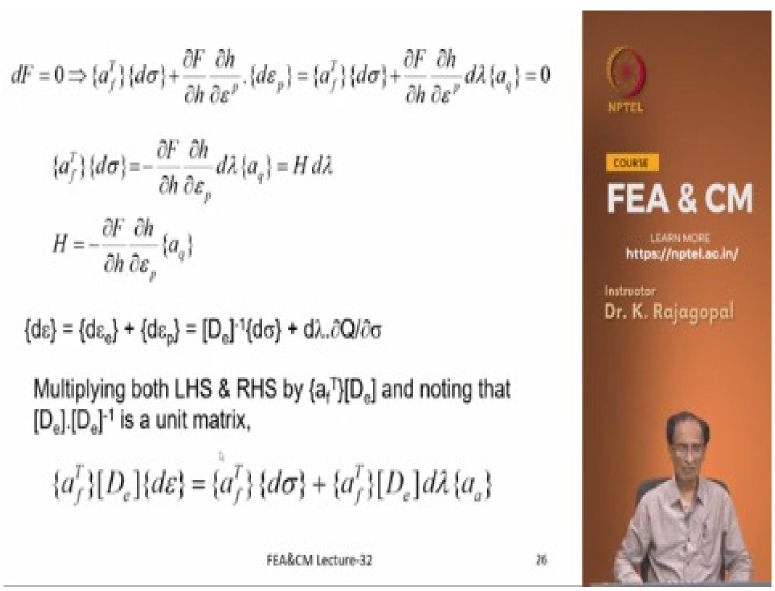

And during the plastic flow because our F is constant and dF is 0 and we have already written that that is a f transpose d sigma + dou F by dou h dou h by dou epsilon p d epsilon p and d epsilon p can be written as d lambda times a q this is 0 and let us set a f transpose d sigma. See by equating this to 0 we get a f transpose d sigma is - dou F by dou h dou h by dou epsilon p d lambda times a q and let us set all this quantity to some constant h.

So, that our writing becomes more simpler so a f transpose d sigma can be written as some h multiplied by d lambda and if you see this a f transpose d sigma is a scalar quantity because it is actually it is a vector product transpose of this vector a f and multiplied by d sigma. So, actually our H is a constant value and d lambda is also a constant value. So, that product is equal to a f transpose d sigma and our this is actual.

$$dF = 0 \Rightarrow \{a_f^T\}\{d\sigma\} + \frac{\partial F}{\partial h}\frac{\partial h}{\partial \varepsilon^p}.\{d\varepsilon_p\} = \{a_f^T\}\{d\sigma\} + \frac{\partial F}{\partial h}\frac{\partial h}{\partial \varepsilon^p} d\lambda\{a_q\} = 0$$

$$\{a_f^T\}\{d\sigma\} = -\frac{\partial F}{\partial h}\frac{\partial h}{\partial \varepsilon_p} d\lambda\{a_q\} = H\, d\lambda$$

$$H = -\frac{\partial F}{\partial h}\frac{\partial h}{\partial \varepsilon_p}\{a_q\}$$

$$\{d\varepsilon\} = \{d\varepsilon_e\} + \{d\varepsilon_p\} = [D_e]^{-1}\{d\sigma\} + d\lambda.\partial Q/\partial\sigma$$

Multiplying both LHS & RHS by $\{a_f^T\}[D_e]$ and noting that $[D_e].[D_e]^{-1}$ is a unit matrix,

$$\{a_f^T\}[D_e]\{d\varepsilon\} = \{a_f^T\}\{d\sigma\} + \{a_f^T\}[D_e]d\lambda\{a_a\}$$

This is our consistency equation dF of 0 and d epsilon the total strain increment is d epsilon e + d epsilon p and that we can write as d inverse times d sigma that is D elastic inverse times d sigma + d lambda times dou Q by dou sigma. And we can multiply both left hand side and right-hand side by a f transpose D e and D e times D e inverse is unit vector a unit matrix one. So, a f transpose D e times d epsilon is a f transpose d sigma that is D e times d epsilon e is d sigma + a f transpose D e d lambda times a q, a q is our dou Q by dou sigma.

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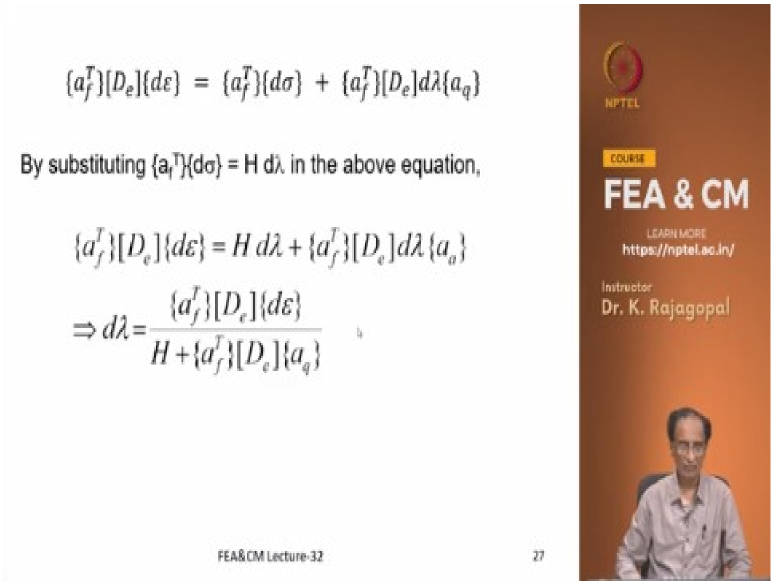

And so, we can write this equation as a f transpose D e times d epsilon is a f transpose d sigma + a f transpose D e d lambda times a q and our a f transpose d sigma was earlier written as some constant hits multiplied by d lambda. So, if you substitute this that in this equation and then now, we can determine the d lambda is as a f transpose D e times d epsilon divided by H + a f transpose D e times a q.

$$\{a_f^T\}[D_e]\{d\varepsilon\} = \{a_f^T\}\{d\sigma\} + \{a_f^T\}[D_e]d\lambda\{a_q\}$$

By substituting $\{a_f^T\}\{d\sigma\}$ = H dλ in the above equation,

$$\{a_f^T\}[D_e]\{d\varepsilon\} = H\,d\lambda + \{a_f^T\}[D_e]d\lambda\{a_a\}$$

$$\Rightarrow d\lambda = \frac{\{a_f^T\}[D_e]\{d\varepsilon\}}{H + \{a_f^T\}[D_e]\{a_q\}}$$

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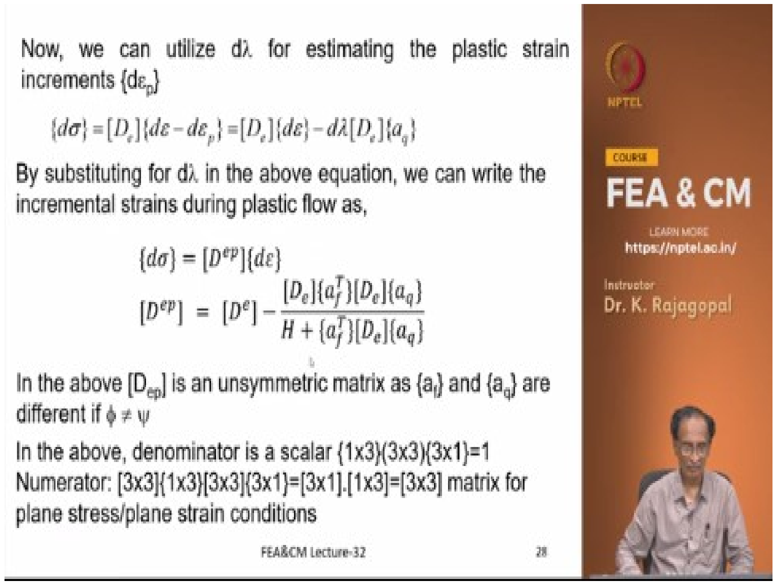

And so now we are writing during the plastic straining d sigma is the elastic times d epsilon - d epsilon p that is D e times d epsilon - d epsilon p is d lambda times a q and so if you substitute that your d sigma is a D e times d epsilon - d lambda D e times a q and our d sigma is ultimately we want to know what is elastic plastic constitutive matrix d epsilon p d epsilon. So, from this see previously we had determined the d lambda and server d sigma is a sum matrix multiplied by d epsilon.

And that matrix is a D elastoplastic and by substituting all these parameters we get a d elastoplastic is D e minus all this quantity D e times a f transpose D e a q divided by H + a f transpose D e a q is actually this a f transpose and a q these are the derivatives of the yield function f with respect to different stress components. And in general, our elastoplastic constitutive matrix is unsymmetric because our a f and a q are different.

$$\{d\sigma\} = [D_e]\{d\varepsilon - d\varepsilon_p\} = [D_e]\{d\varepsilon\} - d\lambda[D_e]\{a_q\}$$

By substituting for dλ in the above equation, we can write the incremental strains during plastic flow as,

$$\{d\sigma\} = [D^{ep}]\{d\varepsilon\}$$

$$[D^{ep}] = [D^e] - \frac{[D_e]\{a_f^T\}[D_e]\{a_q\}}{H + \{a_f^T\}[D_e]\{a_q\}}$$

In the above $[D_{ep}]$ is an unsymmetric matrix as $\{a_f\}$ and $\{a_q\}$ are different if $\phi \neq \psi$

In the above, denominator is a scalar {1x3}(3x3){3x1}=1

Numerator: [3x3]{1x3}[3x3]{3x1}=[3x1].[1x3]=[3x3] matrix for plane stress/plane strain conditions

Because a f is a dou F by dou sigma whereas a q is dou Q by dou sigma and in general the friction angle and dilation angle are not the same. So, we can actually look at what these the dimensions of these terms represent a f transpose is 1 by 3 vector. Now this is for a plane strain or a plane stress case and D elastic is 3 by 3 a q is a 3 by 1 that entire product is 1. And if you look at the numerator D e is a 3 by 3 matrix a f transpose is 1 by 3 vector D e is 3 by 3 a q is 3 by 1.

And we get a 3 by 3 matrix that is corresponding to your constitutive matrix for plane stress and plane strain.

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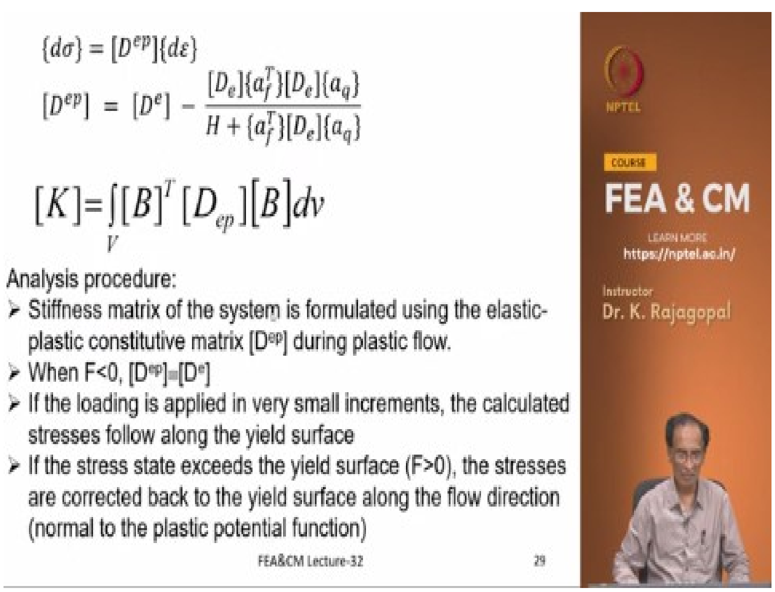

And our in general during the plastics training we can write d sigma as D elastoplastic times d epsilon where D elastoplastic is D elastic - this whole quantity. And we can evaluate the stiffness matrix K as B transpose D elastoplastic. So, instead of D e we write D ep and so B transpose D ep B integrated of the volume and this will give us our stiffness matrix for the continuum during the elastic plastic state.

$$\{d\sigma\} = [D^{ep}]\{d\varepsilon\}$$

$$[D^{ep}] = [D^{e}] - \frac{[D_e]\{a_f^T\}[D_e]\{a_q\}}{H + \{a_f^T\}[D_e]\{a_q\}}$$

$$[K] = \int_V [B]^T [D_{ep}][B] dv$$

And the stiffness matrix can be formulated using D ep and when F is less than 0 that is when the soil is in the elastic state the elastoplastic is just simply D e and then we need to apply the load in very small increments so that we can follow the stress we can follow the yield surface. And so, but then we cannot always do that like we apply some arbitrary stress increments and then it is more preferable to correct the stress state back to the yield surface rather than using very very small stress and strain increments.

Save the stress stating the exceeds the yield surface with F greater than 0. The stresses are corrected back to the surface along the flow direction normal to the plastic potential function.

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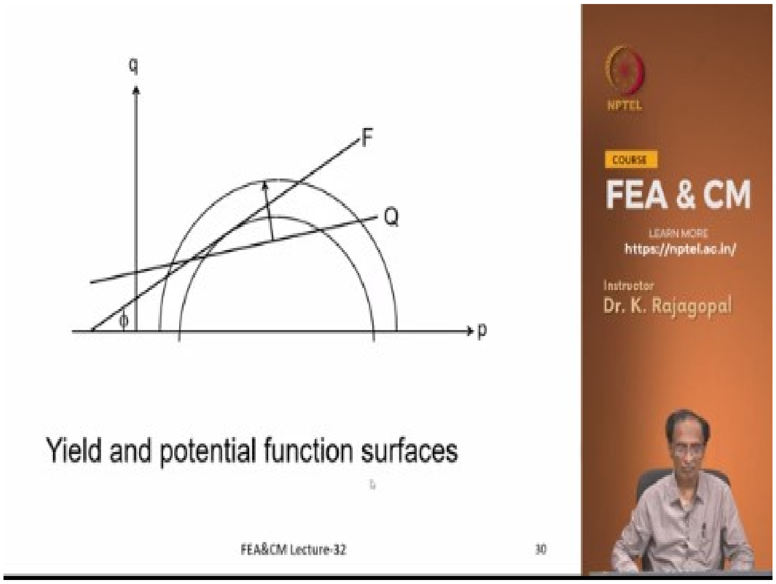

And it is something like this actually this is our Mohr circle that has crossed the yield limit and we have another Mohr circle that is within the yield surface. It is just simply tangent and if you look at the pq diagram that is normal to this to this Q, Q is our potential function.

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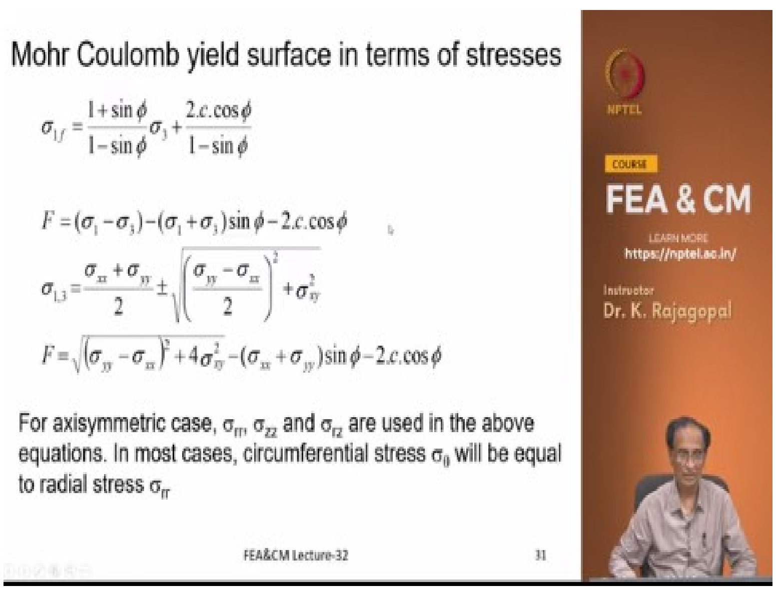

And our Mohr Coulomb yield surface can be written like this sigma 1 f is 1 + sin phi by 1 - sin phi sigma 3 + 2 c cosine phi by 1 - sin phi and in terms of stresses we can write like this and for the axisymmetric case instead of sigma xx we have sigma rr and instead of sigma yy we have sigma zz and instead of having sigma xy we have sigma r theta r z and then we have other sigma theta that is the circumferential stress.

Mohr Coulomb yield surface in terms of stresses

$$\sigma_{1f}=\frac{1+\sin\phi}{1-\sin\phi}\sigma_3+\frac{2.c.\cos\phi}{1-\sin\phi}$$

$$F=(\sigma_1-\sigma_3)-(\sigma_1+\sigma_3)\sin\phi-2.c.\cos\phi$$

$$\sigma_{1,3}=\frac{\sigma_{xx}+\sigma_{yy}}{2}\pm\sqrt{\left(\frac{\sigma_{yy}-\sigma_{xx}}{2}\right)^2+\sigma_{xy}^2}$$

$$F=\sqrt{(\sigma_{yy}-\sigma_{xx})^2+4\sigma_{xy}^2}-(\sigma_{xx}+\sigma_{yy})\sin\phi-2.c.\cos\phi$$

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J_1 & J_{2d} in terms of stresses

$$J_1=\sigma_1+\sigma_2+\sigma_3\equiv\sigma_{xx}+\sigma_{yy}+\sigma_{zz}$$

$$J_{2d}=\frac{1}{2}s_{ij}:s_{ij}=\frac{1}{2}(s_{xx}^2+s_{yy}^2+s_{zz}^2+2s_{xy}^2+2s_{yz}^2+2s_{zx}^2)$$

$$=\frac{1}{6}\left\{(\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2\right\}+s_{xy}^2+s_{yz}^2+s_{zx}^2$$

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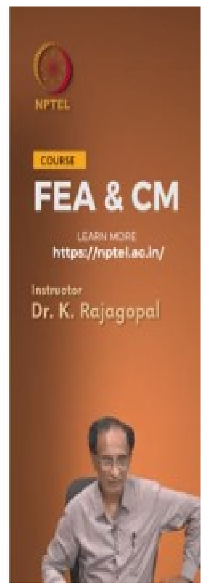

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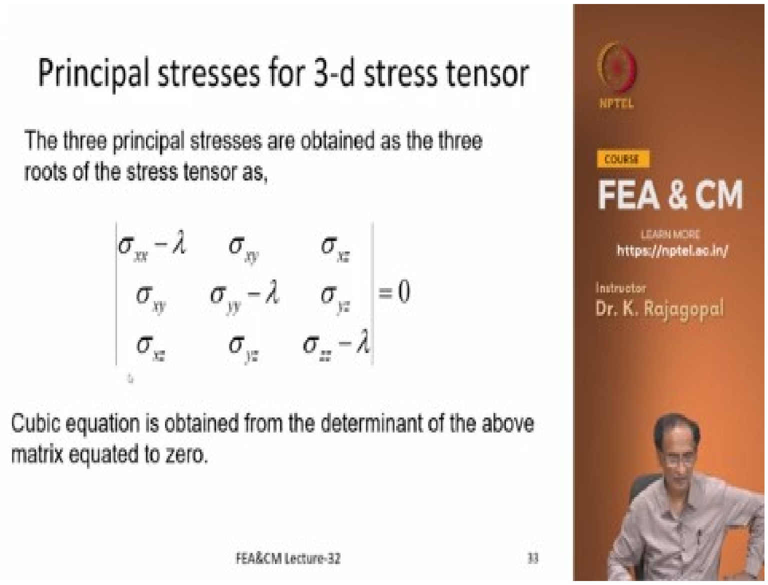

I think this we have already seen all these J 1 J 2d definitions and this also we have seen earlier.

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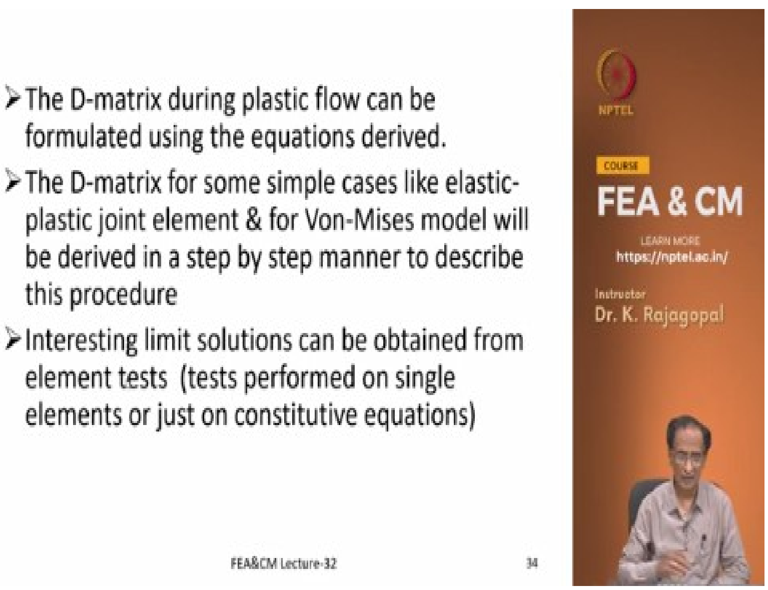

So, the D matrix during the plastic flow can be formulated using the previously derived equations and we will see how to determine the D matrix for very simple cases like the next class I will demonstrate step by step how to determine the elastoplastic constitutive matrix. Then we can also do these calculations for the Von-Mises model. Von-Mises model we will see is actually it is a very simple cylindrical surface and finding normal to that is very simple.

But then when it comes to our Mercola model, we will see that it is very difficult the reason we will see later. And we can get very interesting limit solutions from after implementing this elastoplastic analysis.

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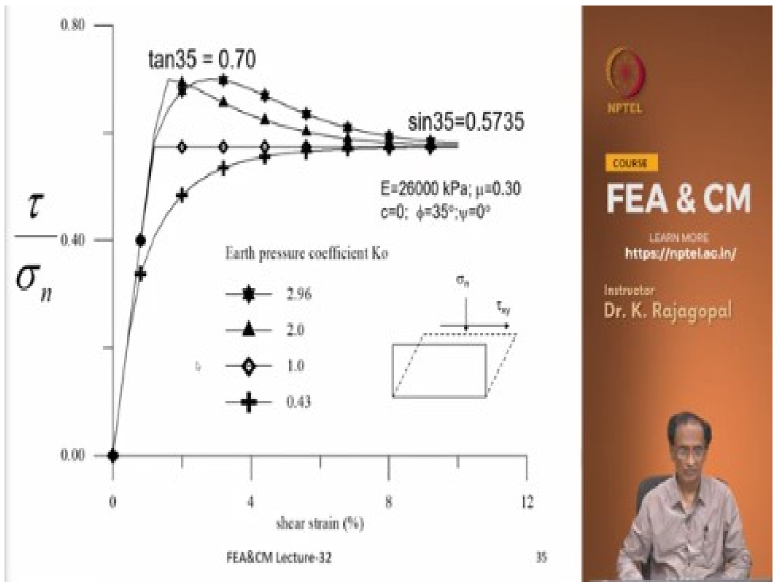

And here we see the stress strain response of the direction of the simple shear test the properties are the young's modulus is at 26000, poisons ratio is 0.3 c is 0, phi is 35 degrees and dilation angle is 0. And the dilation angle 0 means the soil is non-dilatant and I have taken a cubicle sample and then applied the simple shear strain conditions at four different confirm K 0 values K 0 of 0.43 that is 1 - sin phi because our friction angle is 35 degrees 1 - sin phi is 0.43 sin 35 is 0.57 and K 0 of 1 K 0 of 2 and K 0 of 2.96.

So, all these very high K 0 values what we do is we can initiate them by applying with the dummy poisons ratio as Mu is K 0 by 1 + K 0 and then when we are doing the shearing, we reset the poisons ratio back to 0.3. And say if you perform the simple shear test, we get very interesting results with the K 0 of 0.43 the tau is increasing and gradually it is increasing to sin 35 because our dilation angle is 0 and the tau by sigma n is sin phi times cosine psi by 1 - sin phi times sin psi.

And for a psi of 0 the tau by sigma and max = sin phi that is 0.57 and then when your K 0 is 1 exactly equal to 1 we get a bilinear elastic response. So, with the K 0 of less than 1.43 we get a nice curved surface like this reaching the asymptotic limit. But then when K 0 is 1 the stress is increased and then it has reached the yield state and then after that the stresses have remained constant.

But if you look at K 0 of 2 and 2.96 the shear stress has gone beyond this limit of sin phi it has gone up to tan phi tan 35 that is 5.7 then after eating after the shear stress is increased beyond your critical state the shear stress started falling down and with the K 0 of 2.96, we get a very nice peak a curvilinear surface and then after reaching tan 35 the shear stress will go on decreasing with further increase in the shear strain.

And with the K 0 of 2 is actually it is another interesting result the stress has suddenly increased to tan 5 level tan 35 level and then after that the stresses have fallen down. Then at a very large shear strain all the four responses are corresponding to sin phi tau by sigma n of sin phi. That is what we discussed as the critical state or the critical state both the dense sand and the loose sand they have the same strength because all the interlocks have broken and both behave in a similar manner. Because your void ratio tends to the same value at a large critical state.

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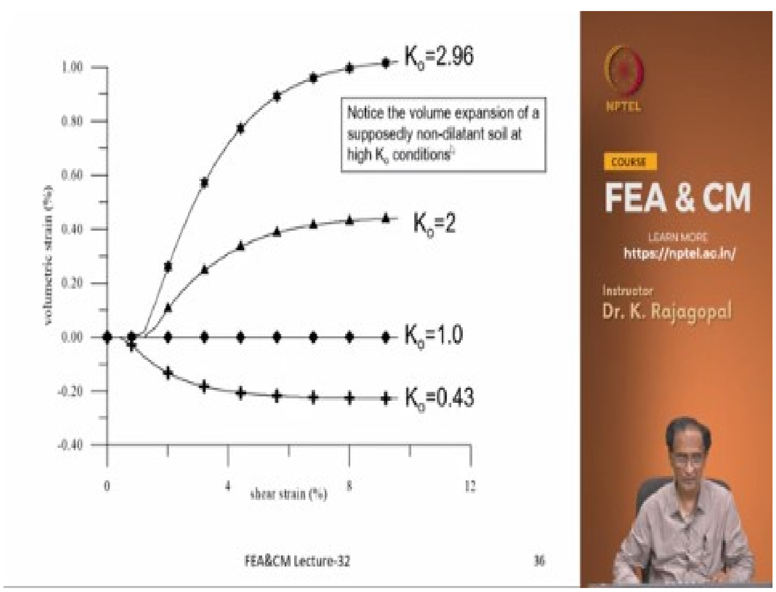

And if you look at the volume strain graphs you get very interesting results with the K 0 of 0.43 there is volume compression and with the K 0 of 1 is absolutely no volume change. The plastic straining happen without any volume change and with the K 0 of 2 and 2.96 initially during the elastic state there is no volume change and then after the plasticity started there is a volume expansion and then at a critical state the volume has remained constant.

And this volume expansion is actually it is happening for a so-called non-dilatant soil with a dilation angle of 0. So, that has happened because our initial K 0 is very high and you get when the initial stresses are dispersed you get volume expansion. That is what we see here and this is similar to your highly interlocked soil and as the interlocks are breaking you get volume expansion.

That is what we see here as the K 0 are the K gradually reduces from K from say 2.96 to 0.43 because our strength of the soil is phi. And the K 0 of 1 - sin phi is consistent with the plastic equilibrium. So, it will come back then once that happens this is what we have. We have constant volume condition.

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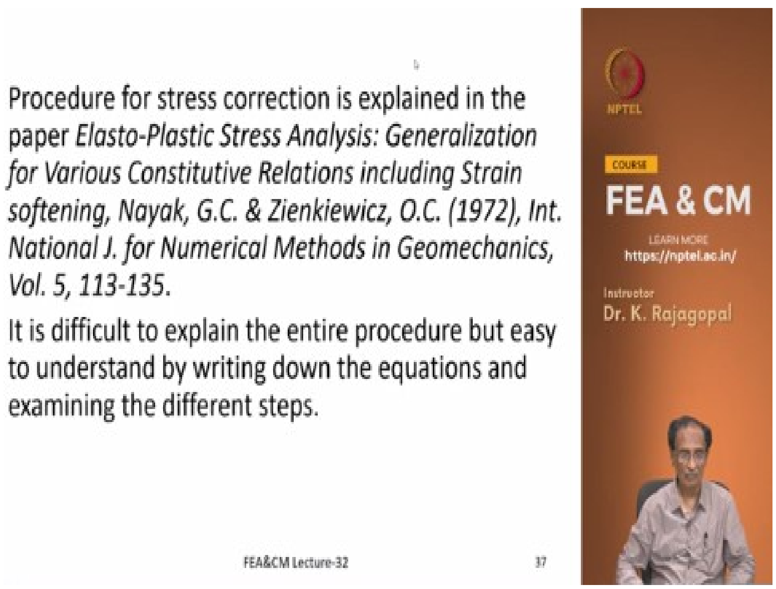

And more details of this derivation and how to do these computations are explained in the paper by Nayak and Zienkiewics 1972 and this paper is titled as elastoplastic stress analysis and generalization for various constitute relations including strain softening. I have tried to explain some part of this paper through this lecture have not have skipped some mathematical details because they become too complicated.

But then for some simple cases I will do step by step calculations so that we are not carried away by those complicated mathematical equations, that we will see in the next class. And is actually these equations could be very long and so it is a difficult to explain but if you sit down with that paper and work out all these calculations they appear to be not so difficult.

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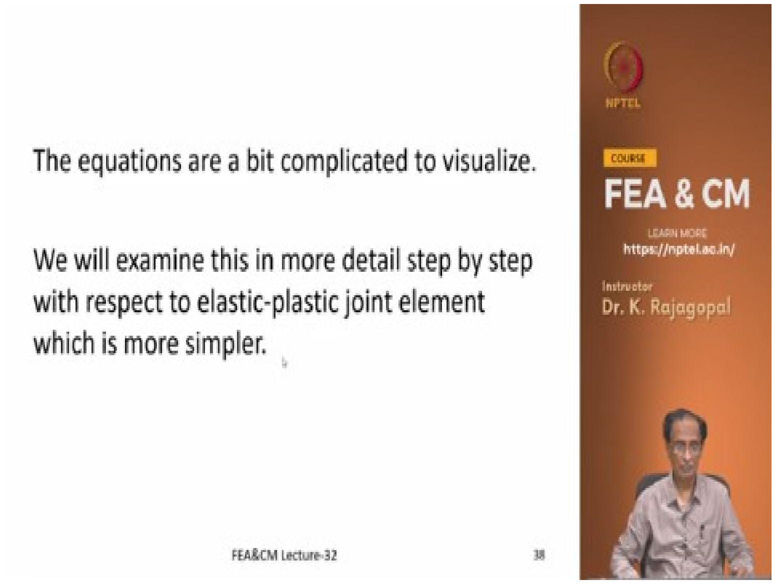

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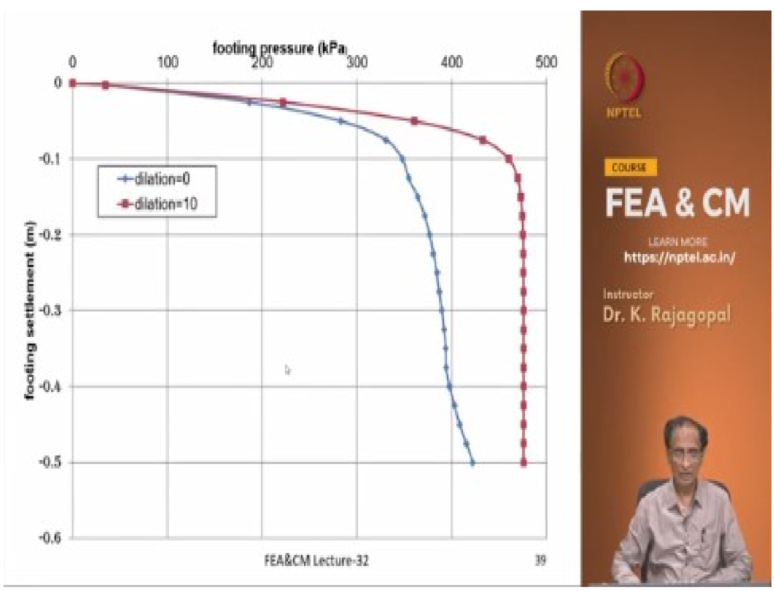

So, the application of this theory for the bearing capacity problem with the dilation angle of 0 and the dilation angle of 10 you get two different responses with a higher dilation angle we get a higher bearing capacity. And that is what we see here on the x axis we have the bearing pressure and then the y axis we have the settlement.

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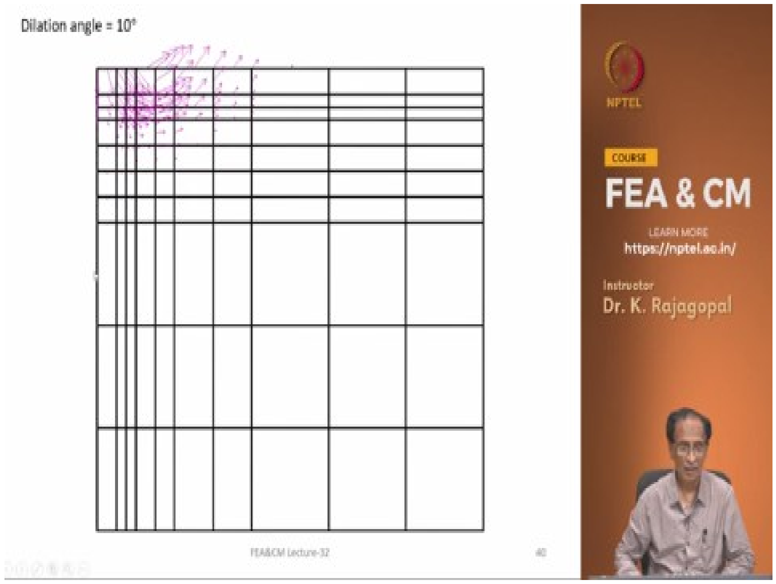

And with the dilation angle of 10 degrees there is a nice volume expansion or the ground heaving.

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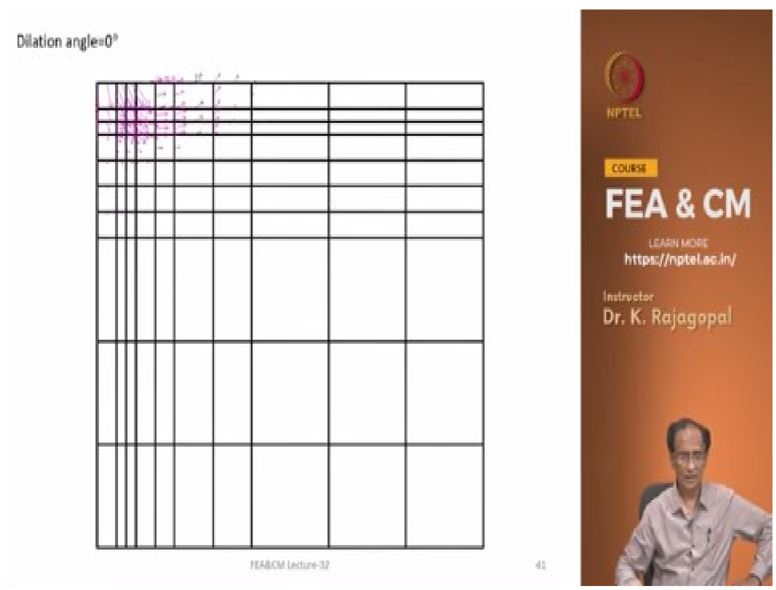

And with the dilation angle of 0 you see these arrows are very small compared to the size of the arrows with the dilation angle of 10 degrees.

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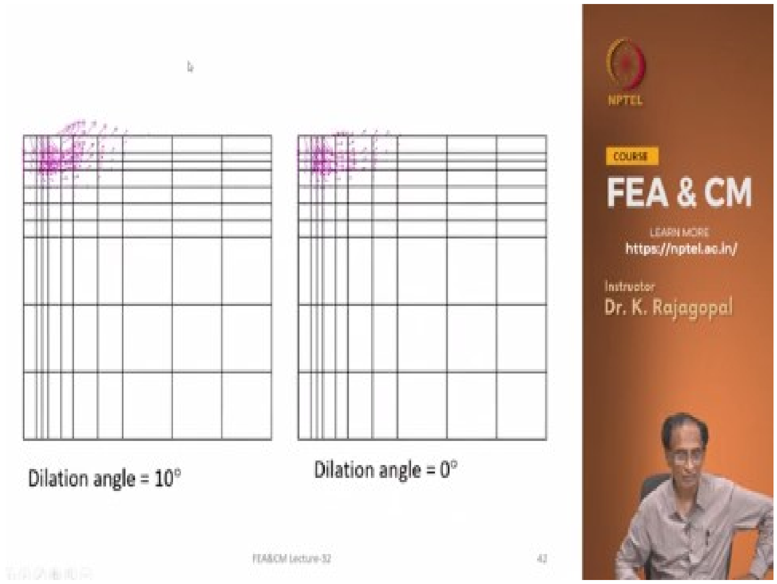

This is the comparison on the left hand side we have dilation angle of 10 degrees on the right hand side we have a dilation angle of 0.

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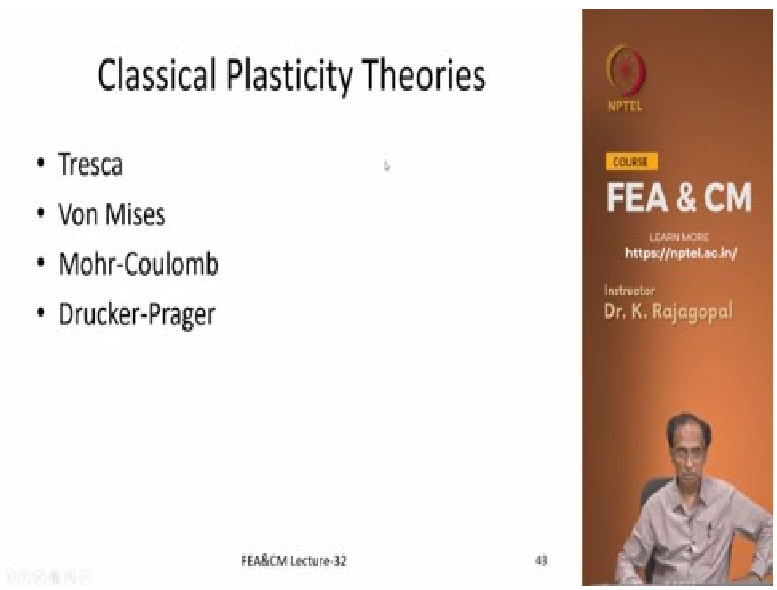

And this generalized procedure we can extend to any type of plasticity theories some of the classical plasticity theories are Tresca, Von Mises, Mohr-Coulomb and Drucker-Prager.

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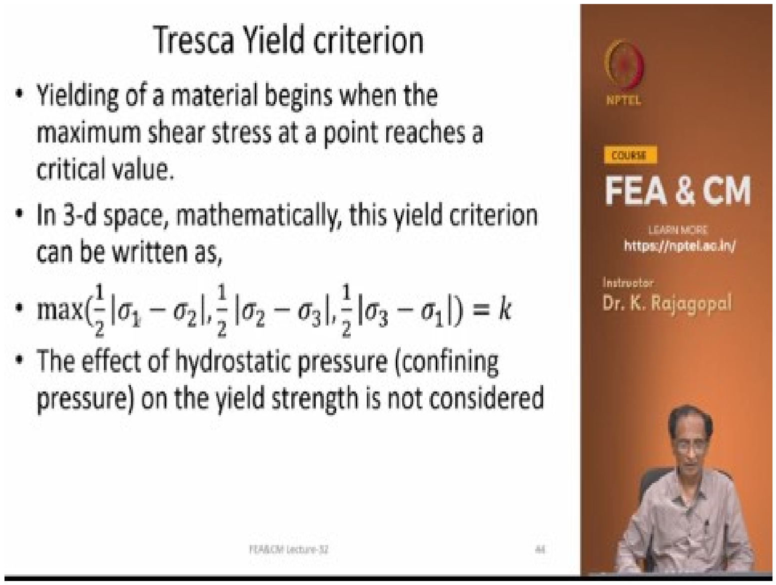

Tresca is one of the earliest models the in which we say that material starts yielding when the maximum shear stress reaches a particular limit or a critical value. And in the three-dimensional space we can write this a Tresca yield criterion as maximum of sigma 1 - sigma 2 by 2 or sigma 2 - sigma 3 by 2 or sigma 3 - sigma 1 by 2. This k the k is some yield limit and actually when we do the triaxle compression test, we plot the Mohr circle and then the radius is defined as the cohesive strength.

In 3-d space, mathematically, this yield criterion can be written as,

$$\max(\frac{1}{2}|\sigma_1 - \sigma_2|, \frac{1}{2}|\sigma_2 - \sigma_3|, \frac{1}{2}|\sigma_3 - \sigma_1|) = k$$

And this k is similar to our cohesive strength and in this Tresca model the effect of hydrostatic pressure is not considered. The yield strength is not a function of J 1.

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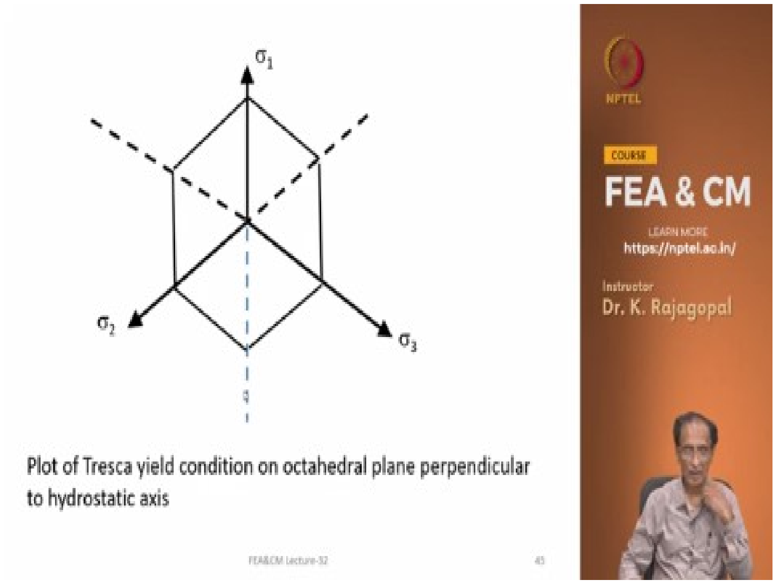

And say if you plot this in the principal stress space say the if you take a space of sigma 1 and sigma 2 sigma 3 and the droid and hydrostatic axis line and then perpendicular to that we call that as the octahedral plane. And if you plot your Tresca yield surface in the principal stress direction sigma 1, sigma 2, sigma 3 we get a surface something like this. This is an octahedral plane perpendicular to the hydrostatic axis.

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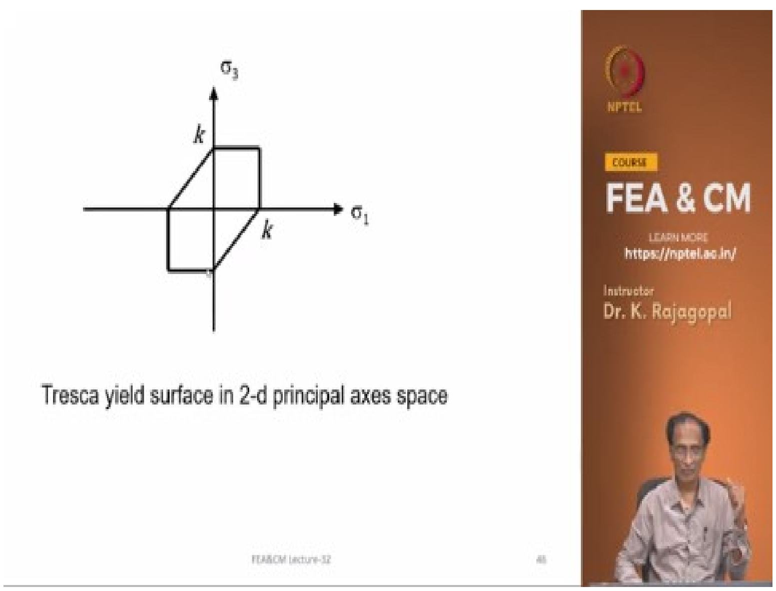

And in the 2d space where you have only sigma 1 and sigma 3 and sigma 2 is not considered we get a more simple shape something like this and if you have this type of surface and if I ask you to determine the normal to this yield surface at the corners, how do we do? We have no idea because we know how to calculate the normal along this vertical surface or horizontal surface or inclined surface but then at this point it could be infinite.

It is there are too many directions we can draw that means that we will have some we should expect some numerical difficulties.

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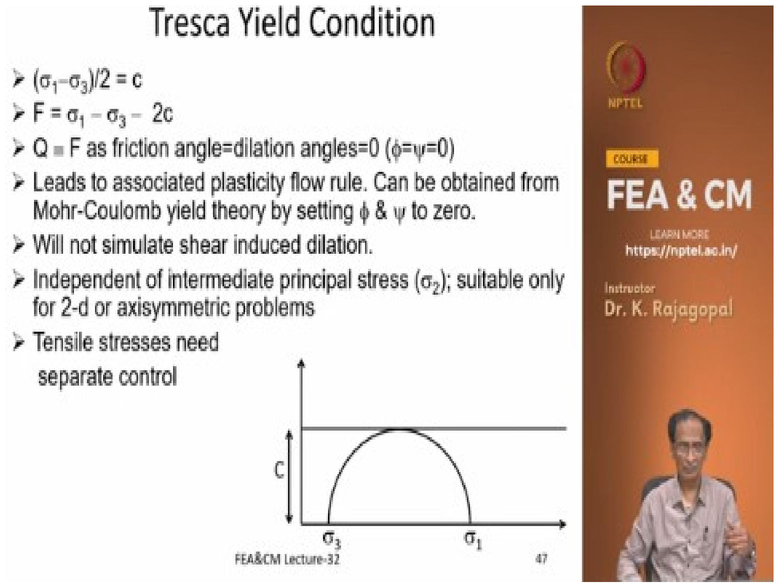

And in the pq space our Tresca yield criterion is very simple it says it is a line it is a horizontal line and we can write sigma 1 - sigma 3 by 2c the c is the cohesive strength and the yield function F can be written as sigma 1 - sigma 3 – 2c and there is no friction angle phi and the c is not a function of hydrostatic pressure. Whatever may be the hydrostatic pressure or the strength of the soil is not going to change.

$$(\sigma_1 - \sigma_3)/2 = c$$
$$F = \sigma_1 - \sigma_3 - 2c$$
$$Q = F \text{ as friction angle=dilation angles=0 } (\phi = \psi = 0)$$

And in this particular case both dilation angle and the friction angle are 0 we can say it is an associated plasticity rule. But then it is a misleading one because you will not get any plastic volumetric strains because our dilation angle is 0. And we see that the effect of sigma 2 is absent we do not have the effect of the intermediate principle stresses sigma 2. So, this particular Tresca yield criterion is good for 2d or axisymmetric problems.

And as we discussed earlier, we can plot this Mohr circle and without exceeding the yield limit either in the compression space or in the tension space. So, we require some control on the tensile stresses so whenever you develop two very large tensile stresses, we need to correct the stresses so that we end up in the compression space.

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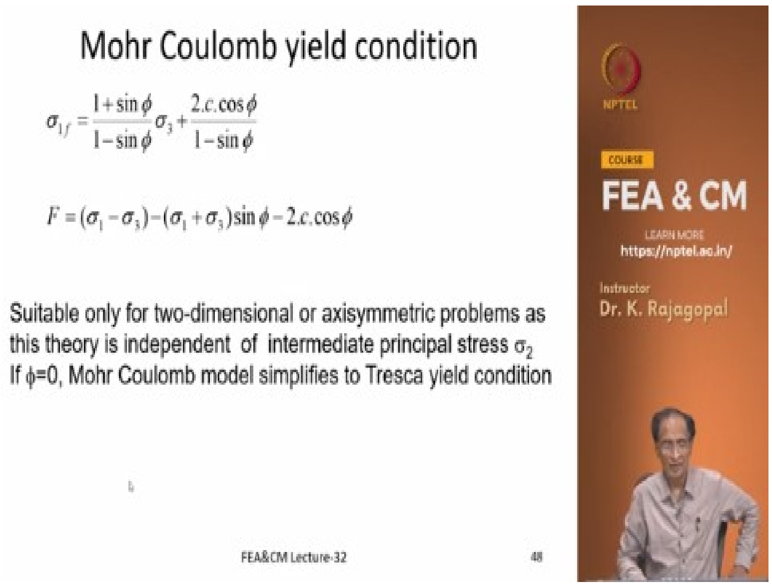

The Mohr Coulomb yield surface is something like this. This is in terms of c and phi and the yield function F can be written like this and if you substitute friction angle phi of 0, we get back our Tresca yield criterion. So, the Mohr Coulomb yield surface with the phi of 0 is nothing but the Tresca yield criterion.

Mohr Coulomb yield condition

$$\sigma_{1f} = \frac{1+\sin\phi}{1-\sin\phi}\sigma_3 + \frac{2.c.\cos\phi}{1-\sin\phi}$$

$$F = (\sigma_1 - \sigma_3) - (\sigma_1 + \sigma_3)\sin\phi - 2.c.\cos\phi$$

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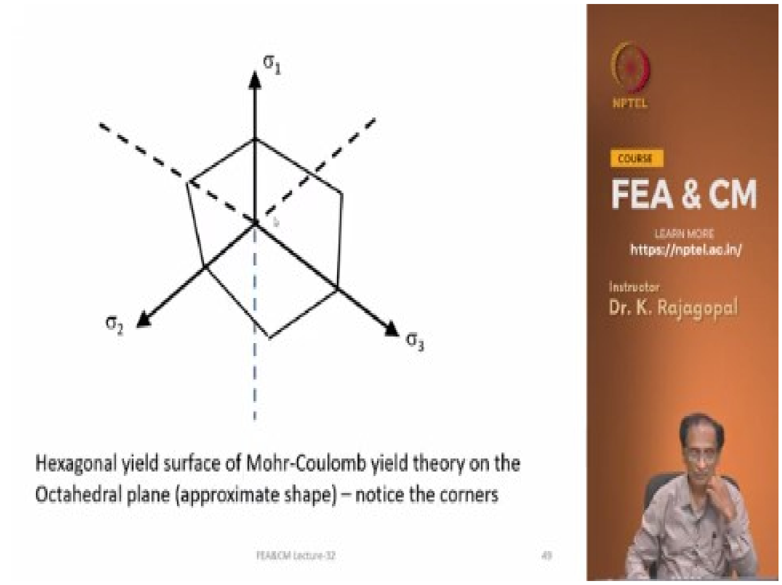

And the octahedral plane the if you plot the Mohr-Coulomb yield surface it will be something like this. It is actually the shape is correct but then the its only approximation and if you want more exact one you refer to any textbook any good textbook. But it is basically it is a hexagonal shape because you have 1, 2, 3, 4, 5, 6 surfaces given for Tresca we had six surfaces. But then that is a regular hexagon and this is a hexagonal it is not exactly hexagon.

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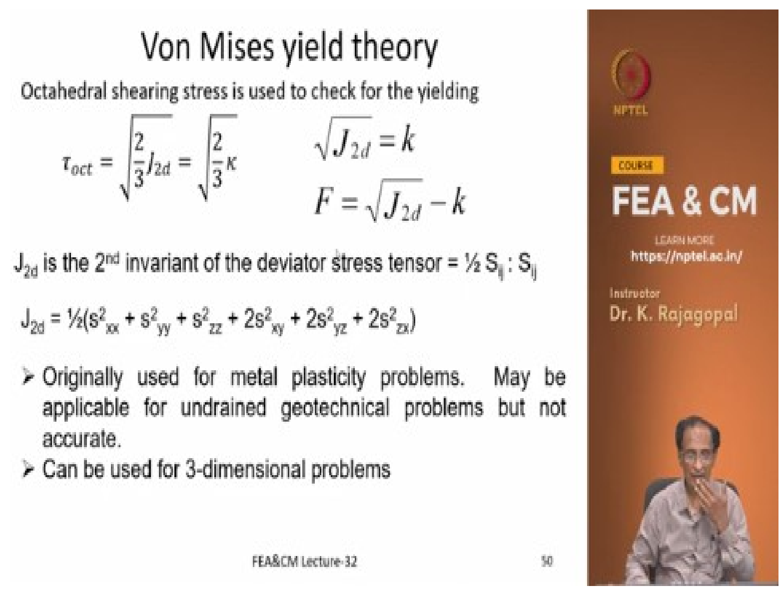

And Von Mises theory is actually the octahedral shearing stress is used and basically, we can say the yield function F is the square root of J 2d - k where k is some constant and J 2d is our second invariant of the deviator stress tensor. And this Von Mises floral was used in the metal plasticity theories and it is not really applicable for geotechnical problems but it is by experience they have seen that this model is good for unruined problems.

Von Mises yield theory

Octahedral shearing stress is used to check for the yielding

$$\tau_{oct} = \sqrt{\frac{2}{3} J_{2d}} = \sqrt{\frac{2}{3}\kappa} \qquad \sqrt{J_{2d}} = k$$

$$F = \sqrt{J_{2d}} - k$$

J_{2d} is the 2^{nd} invariant of the deviator stress tensor = ½ $S_{ij} : S_{ij}$

J_{2d} = ½($s^2_{xx} + s^2_{yy} + s^2_{zz} + 2s^2_{xy} + 2s^2_{yz} + 2s^2_{zx}$)

- Originally used for metal plasticity problems. May be applicable for undrained geotechnical problems but not accurate.
- Can be used for 3-dimensional problems

And the c and k they are not directly related to it they are somehow related to each other that we need to find the relation that we will see later. And because this J 2d is involving all the stress components sigma x sigma y sigma z tau xy tau yz tau zx. So, this Von Mises theory can be applied for 3d problems. Whereas the Mohr Coulomb model and Tresca they are only good for two dimensional problems either plane strain or axisymmetric but not for 3d because we do not have the sigma 2.

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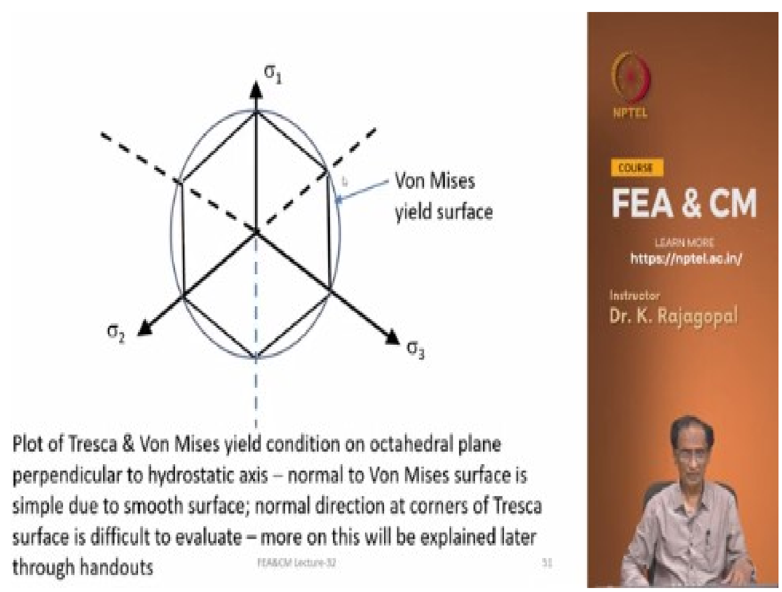

Whereas the Von Mises theory is good for the 3d problems and if you plot the Von Mises is actually get a nice circle it is a circular surface on the octahedral plane and circle means it is a smooth surface so we can easily draw the normal to this von Mises surface.

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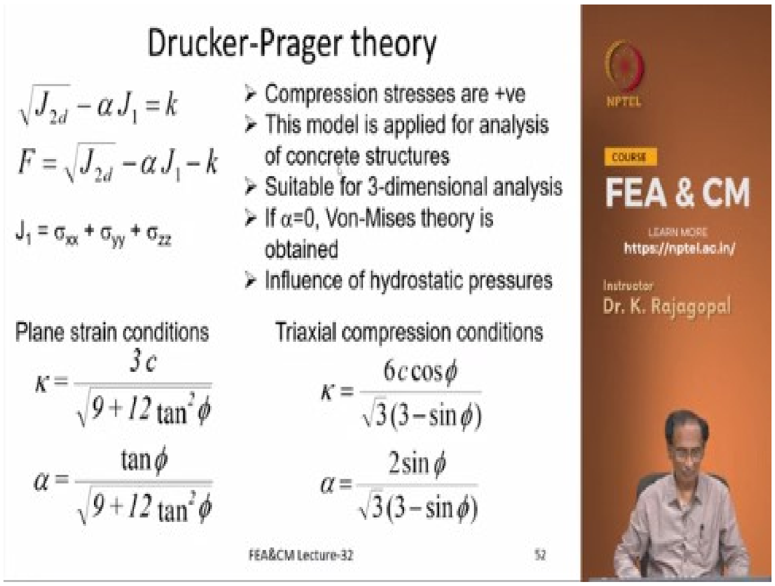

And the Drucker Prager theory is a slight extension of the Von Mises and this is the square root of J 2d - alpha J 1 = k where alpha and the k are the yield parameters or the strength parameters. And this the Drucker Prager model is slightly different where in our yielding is controlled by the hydrostatic pressures the J 1 whereas in the Von Mises theory the J 1 was not there. And the alpha and k can be related to c and phi by matching.

Drucker-Prager theory

$$\sqrt{J_{2d}} - \alpha J_1 = k$$

$$F = \sqrt{J_{2d}} - \alpha J_1 - k$$

$J_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$

- Compression stresses are +ve
- This model is applied for analysis of concrete structures
- Suitable for 3-dimensional analysis
- If α=0, Von-Mises theory is obtained
- Influence of hydrostatic pressures

Plane strain conditions

$$\kappa = \frac{3c}{\sqrt{9 + 12\tan^2\phi}}$$

$$\alpha = \frac{\tan\phi}{\sqrt{9 + 12\tan^2\phi}}$$

Triaxial compression conditions

$$\kappa = \frac{6c\cos\phi}{\sqrt{3}(3 - \sin\phi)}$$

$$\alpha = \frac{2\sin\phi}{\sqrt{3}(3 - \sin\phi)}$$

So, for the plane strain conditions this is the matching and for the triaxial compression test this is the matching. The actual this matching you can find in most of the plasticity textbooks but I am not explaining here because we are not really interested in deriving these equations but we will see the applications. And depending on the problem that you have whether you have a triaxial or axisymmetric or a plane strain case we can relate the c and k say for a friction angle of 0 the k is exactly equal to c for the plane strain case.

Whereas for triaxial compression it is 6 by 3 root 3 see that is a 2 by root 3 so 2 by root 3 means slightly more than 1. So, the k is 2 divided by root 3 times c so, this the Drucker Prager by using the c and the by relating k and alpha to c and phi we can get a good match between the predictions of Drucker Prager and then the Mohr Coulomb models. And the Drucker Prager model is also applicable for three dimensional problems.

And the Drucker Prager model is good for concrete whereas it is not so popular for soil mechanics but although some people use the Drucker Prager model for 3d problems by matching the c and phi to k and alpha like this, this may be my last slide.

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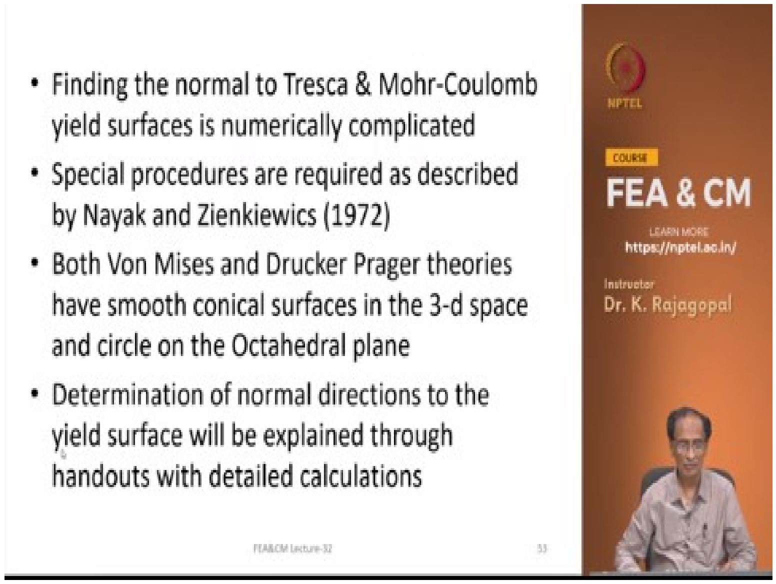

And see when we do the numerical computations, we see that we cannot easily determine the normal to the Tresca and Mohr coulomb yield surfaces especially at corners. And how do you know that you are at a corner? So, that requires one more parameter theta that I have not explained but you can find in this paper by Nayak and Zienkiewics and both Von Mises and Drucker Prager they have a smooth conical surfaces in the 3d space.

And in the octahedral plane is a circle so drawing a normal to the circle is very simple. And so, this determination of the normal I will explain through handouts and then I will try to reproduce the lines from the computer code. So, that you see how these normals are evaluated. So, this is my last slide. So, thank you very much, if you have any questions, please send an email to this profkrg gmail.com so, thank you very much until we meet next time bye for now.