Finite Element Analysis and Constitutive Modelling in Geomechanics
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Lecture - 36
Elastic - Plastic Constitutive Matrix

Hello all, I welcome you all for the lecture on finite element analysis and constitutive modelling in geomechanics. So, in this particular course I will be giving a guest lecture on constitutive modelling using ordinary cam clay and modified cam clay. So, hope you enjoyed this lectures in the previous enjoyed the previous lectures and this will be on mostly on constitutive modelling and how will you use some of the well-known constitutive models like ordinary cam clay and modified cam clay to predict the soil response.

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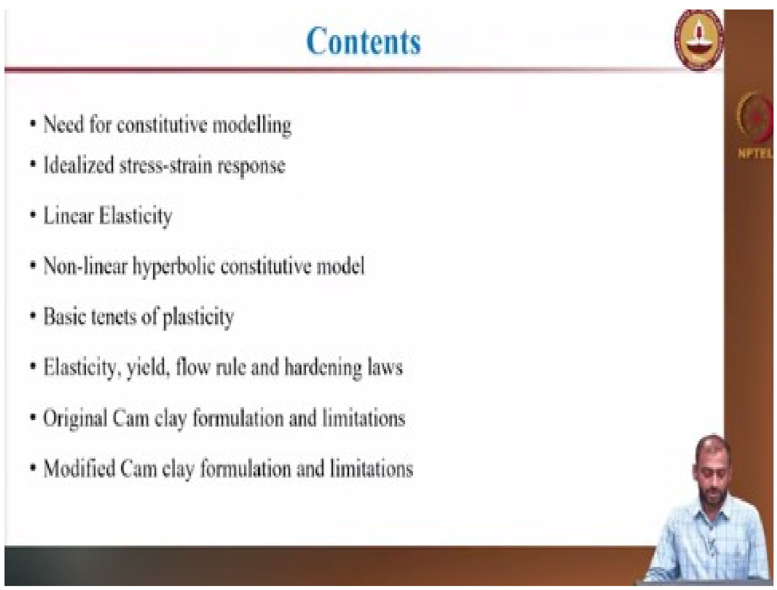

So, this is the layout of my contents I will start with why we need constitutive modelling and some of the idealized stress strain responses. I will just give a brief overview on the linear elasticity non-linear hyperbolic models and then I will just give basic tenets of plasticity theory and why it is important to have this sort of a nice framework which will be useful in modelling the soil response. And followed by the introduction on the cam clay models.

I will start with ordinary cam clay and then I will move on to modified cam clay. So, this will be the contents of this lecture followed by I will be giving one more lecture where you will be seeing the predictions of these cam clay models and what are its limitations and advantages will be discussed in detail.

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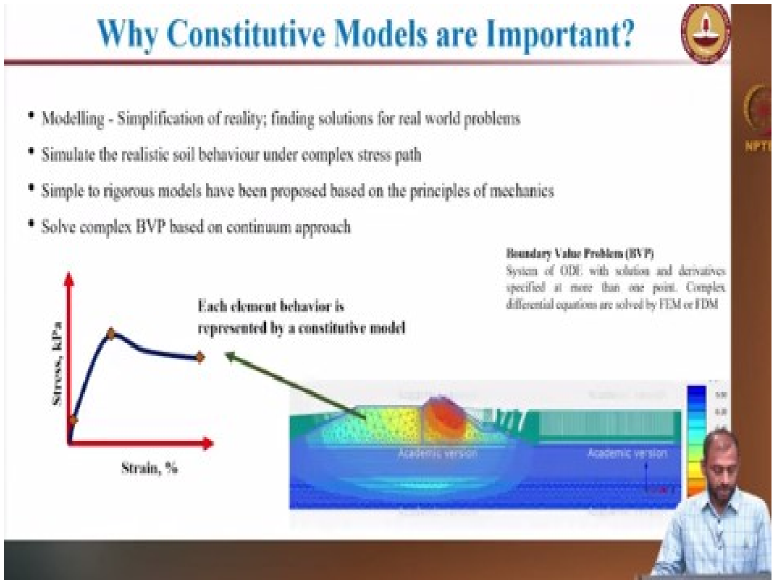

So, before going into why constitutive models are important, I will just give you what is before going into that like what is a constitutive model. A constitutive model is a mathematical framework which relates to physical quantities. So, for our case I think like we want have some sort of a relationship between stresses and strains. That is the models which we develop tries to relate stresses and strains.

And why modelling is important because you know it is hard to model any complex structures. And if you want to simplify the problem and instead of doing a lot of physical model studies if you want to numerically solve some complex problems we need to really go into modelling of any structure. So, we are trying to make it simplified and to solve some real-world problems. So, the models which we choose should be very so it can be very simple also it can be complex.

So, it can be in a wide variety of ranges. But so, we need to choose an appropriate model based on our requirements. So, the models can simulate a complex stress path and these rigorous models are actually proposed based on the principles of mechanics. So, once this continuum based constitutive models are developed this will be used in tandem with the numerical methods which has been discussed in this course on using finite element methods of finite difference methods.

You can use these constitutive models and solve some complex boundary value problems. So, this actually gives an idea why the material models or the behavioural models are really important.

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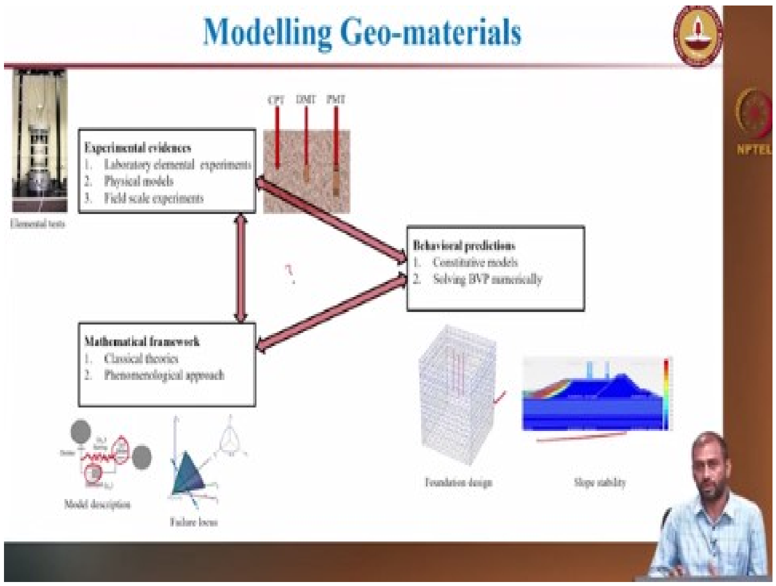

So, how we will model geo materials? So, that is the next important question which we need to ask how to model the behaviour of geo materials. So, for that you need to have as you are seeing on the screen you need to have three important framework one is the experimental evidences. So, you perform lot of experiments in the laboratories and understand the mechanical behaviour of geo materials.

Starting from as simple as a liquid limit test to a much complex directional simple shear test. So, you have a wide variety of laboratory tests which you can carry out and try to understand the behaviour and you also do a lot of physical model studies. So, again starting from 1G test and you can extend it to an NG or NG test using centrifuge models and there also you try to understand the behaviour.

So, in the laboratory you typically do an elemental test whereas by physical models you try to simulate the exact field condition. You try to replicate in the like using a model study and then solve or similarly try to solve a boundary value problem. Additionally, we also do a lot of field scale studies. So, we use cone penetrometer test, flat dilatometer test and pressure meter test in the field to understand the stress deformation response.

So, all these experimental evidences are key before developing any constitutive model because this gives you a clear picture of how the soil behaves under complex stress path complex environment. Already the soil or geo material is extremely complex. We will discuss about why it is complex in the latest slides but it is complex. So, if you want to realistically model it you need to understand and get all these experimental response from the laboratory.

So, that is one of the key aspect of before modelling any geo material. So, once you get all these details, we go back we will use a mathematical framework. So, we use some classical theories based on plasticity fundamentals of plasticity we also use thermodynamic framework. So, we use different mathematical framework to model this response. So, here actually what you are trying to see is a spring which shows the elastic which tries to predict the elastic behaviour of a material.

We also have a dashboard which will be useful for understanding the viscoelastic response of a material and here you are trying to see a slider because since the granular materials are made of lot of individual grains. It can slide so there is a frictional response that is taking place in in the system. So, you have a slider, you have a spring, you have a dashboard this is a simplified system.

So, you can actually use there are a lot of complex simple models like linear elastic spring-based models to a viscoelastoplastic. So, the complexity keeps on building up. So, but we just go for a basic framework mathematical framework from where we can develop these models. Also, on the side you see we try to use the fundaments of plasticity where you try to model any response using and you represent the material response in a three-dimensional principle stress space.

So, you have an e locus you try to find a failure criteria which actually models the soil response. So, you have a mathematical framework set you have the experimental from the experimental evidences you are having a mathematical framework. Now using these things these tools, you try to formulate a consolidated model trying to relate the stresses and strains. So, once you relate these stresses and strains you come up with a framework to model the response.

So, that will be inbuilt in a numerical schema which we I will be showing in the next slide. And you will try to solve any boundary value problem. So, for example what you are seeing here is a slope stability problem. So, here again the when there is a slip that is happening the soils tries to strain a lot. So, it undergoes a lot of plastic deformation along the slip line. So, and so you try to use the model which can predict the response much better.

So, you need to use an appropriate constitutive model to quantify the material response appropriately or material response and also the boundary value problem response. So, here again you are seeing a foundation problem. So, here again when you are inserting a pile into a surface or soil so what are the stresses that is getting developed. Those response you need to so these can be solved by choosing an appropriate quantitative mode.

So, that is the important key like how you are going to use all these evidences, framework and try to develop a model and solve a boundary value problem.

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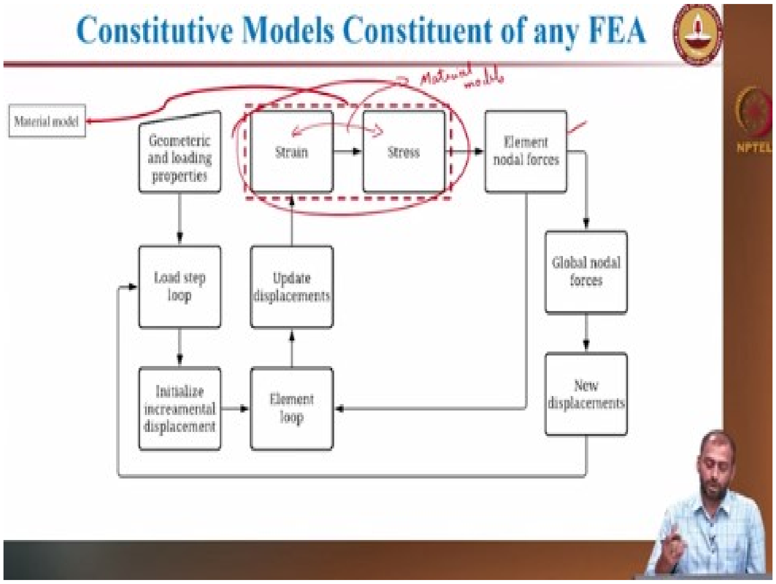

So, on the screen you are trying to see a finite element framework. So, you have a geometry so for example in the previous slide what you saw is a footing problem or a slope stability problem you try to make the geometrical scheme of that particular model and then you discretize and then generate the mesh. And once you do that you try to initialize with some incremental displacement.

So, once you have this incremental displacement you can actually update the displacement and you can find out the strains. And as you see here the strains are related to the stress. Here is what you use some material models real. So, here you use material models or constitutive models. From stress you are trying to transform it to strains and from there you can find out the element nodal forces then global flow forces then update the displacement and this keeps on continuing until equilibration is reached.

So, you try to solve so in a numerical scheme like in a finite element framework this relation between strain or stress or stresses and strain is key. So, this numerical tool is using a mathematical framework which models the geo material response. So, in any finite element framework when you are using different tools like plexus or abacus whatever tools you are using the key part of any numerical scheme is the model which you are choosing the constitutive model which you are choosing.

So, you have to be very careful in choosing an appropriate constitutive model for the geomaterials. And especially what we are looking for. If you are looking for large strain problems you need to choose a model which predicts the large strained response accurately. So, depending upon the problem depending upon the material you have to choose an appropriate constructively mode.

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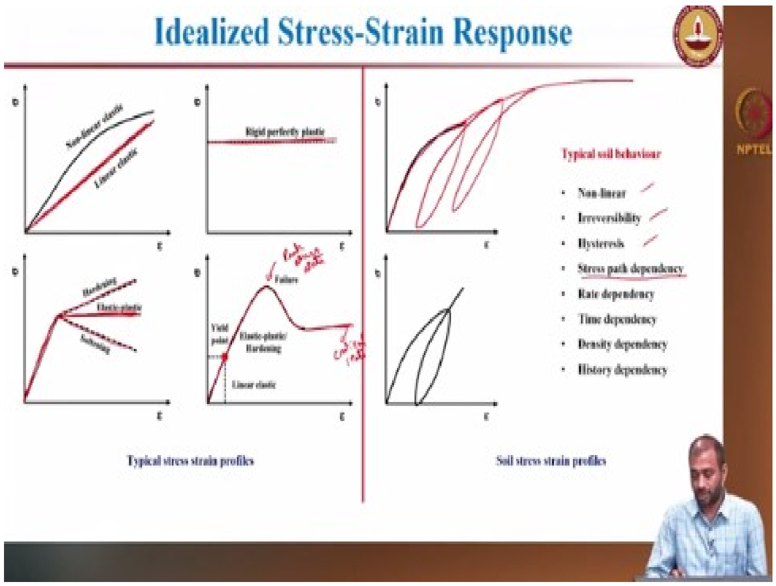

So, here now we will just discuss about some of the idealized stress strain response that you see in any material system. So, for example you try to see a linear elastic model. So, if you have a metal rod you try to apply some tension so you will have a linear stress strain relation. So, it does not have an yield point but as you apply some strains it will exhibit or it will exhibit some stress. So, this relation is sort of a linear.

But this is not applicable for many material system because many material systems are non-linear and it is also elastic. For example, if you take a rubber and rubber band or something and then if you try to stretch it so it the response is not linear maybe sometimes it is non-linearly elastic. So, you need a different constitutive model to model this particular response. You can use monoreblan and sort of a hyper elastic models to model this particular response.

So, again other idealized stress strain plot is you have a linear elastic and a perfectly plastic response. So, your conventional Mohr Coulomb failure criteria I will discuss about what is failure criteria and the latest slides. But so those sort of elastoplastic models it is actually modelling and linear elastic initial part after that the behaviour is perfectly plastic. So, but some materials exhibit hardening response, some material gain strength as you start sharing more and more.

And sometimes the material also shows a softening response. So, the material strength reduces as you start sharing at larger strings especially. So, you also see some sort of a rigid perfectly plastic response. So, if you have a bond between two particles and then if you try to pull those particles apart there is a sudden breakage. So, the there is a stress but there is not much of a deformation but once it breaks the bond breaks then it is like behaves like a rigid plastic.

So, there is no stress but the stress is there but this strain keeps on increasing. Again, these are all idealized model but many material system does not follow this idealized system especially when you are dealing with geo materials which is extremely complex. So, a typical stress strain profile also has a linear elastic zone you have an yield point and then after the yield point the material starts hardening reaches a peak and then it also starts softening and then at large strains it might reach a critical state.

So, critical state and this is a peak state. So, these are all idealized strain profiles what you are seeing on the left side of the screen. So, on the right side you see a typical soil behaviour. Why soil behaviour is extremely complex? Because as you all know that the soil behaviour is not linear the behaviour is non-linear and also it is irreversible. So, once you unload and reload, I think the behaviour is not reversible it will not go back to its initial position.

And you sort of see a hysteresis loop as you start unloading and reloading use sort of see a hysteresis loop that is getting formed that is because of some sort of a plastic dissipation that is happening in the material system. So, you have hysteresis, you have irreversibility, you also have stress path dependency. What is stress part dependency? so when you perform a compression test the material response is different.

So, you will have a different strength when you are performing a test on a soil specimen under compression and you will have a completely different response. So, the strength will be different and the volume change responses also will be different when you are performing a test under triaxial extension conditions. You can also do test at different stress path; you can perform at a simple shear test or different shear test.

So, you can transfer so we will see in the next slides how to do a different stress path test but under different stress path the material response is completely different. So, again that is why the soil behaviour is stress path dependent and also its rate dependent. So, if you do at a very small strain rate the material response will be different. Also, if you do at like very high strain rate the response will be different.

So, in addition to that you also have time dependency. So, again when you are doing an odometer or a consolidation test you will see a secondary consolidation that is happening over the period of time. So, with time I think the behaviour also changes. So, it is again a time dependent response again the soil behaviour is density dependent which you all know because a loose specimen behaves completely differently compared to a dense specimen.

In a dense specimen you will see a lot of dilation that is going on whereas in case of a loose specimen the material contracts. So, that is in addition to that the material also is history dependent. It also knows what is the previous history what is this the soil which has already has experienced some sort of a stress. So, the behaviour also depends on the stresses the pre-consolidation pressure which it has already experienced.

So, with all these things the behaviour the complexity of modelling a soil behaviour becomes even more challenging. So, we have all these non-linear irreversibility, hysteresis, stress path dependency, rate dependency, time dependency, density and history dependency. So, again that the challenge lies in modelling considering all these aspects and then try to model the material behaviour.

So, unlike any engineered material like steel or either a concrete or other materials this soil a natural material is extremely complex and its actually challenging to consider all these responses and model it.

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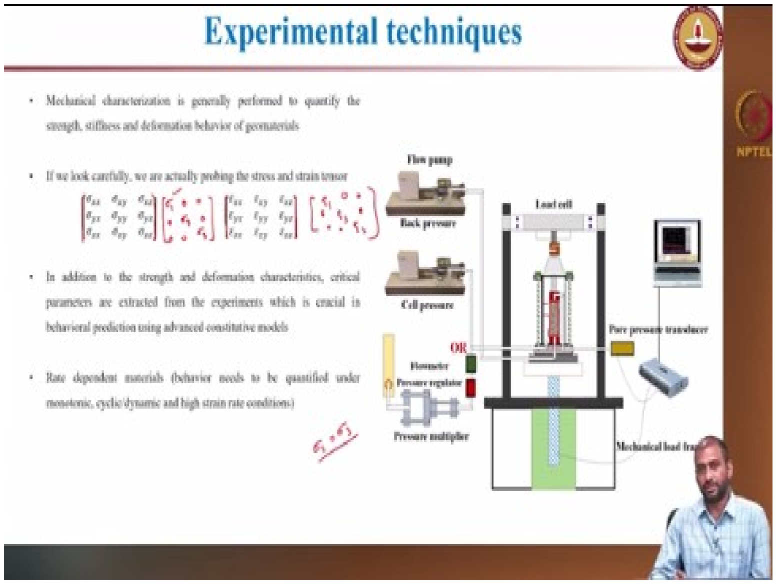
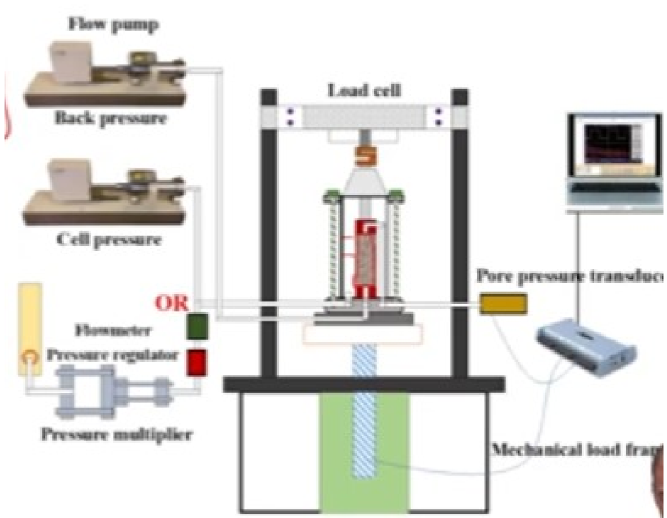


So, in the next few slides I will show you how what are the various experimental techniques which you use to quantify or relate the stresses and strains. So, you actually try to probe the stress tensor or plot the strain tensor and then try to get some information. So, conventionally we use triaxial conventional triaxial test you try to use in the laboratory where you have a specimen and then you try to apply a confining pressure.



So, when you are applying a confining pressure, you are equating sigma 2 = sigma 3 which is the intermediate and minor principle stress are equal and then you try to apply some deviatoric loading and then share the specimen. So, when you are actually doing it so as you see here this is your stress tensor so which is having an nine components. Again, you have a strain tensor which has nine components.

If we look carefully, we are actually probing the stress and strain tensor

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \qquad \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

So, when you are actually doing a triaxial compression test you are trying to probe sigma 1 sigma 2 = sigma 3. So, the other shear components are 0. So, you have a stress matrix where you have sigma 1, sigma 2 and sigma 3. So, you are actually trying to probe these stresses and then try to find the strain response. So, similarly you will calculate epsilon 1, epsilon 3 the other shear components are 0.

So, this is crucial in behavioural predictions using advanced constitutive volts. So, again here using a conventional triaxial compression test you can do a compression test also you can try to do some tracks extension test and you also can do a constant pre-primped. But you need to be extra cautious and it is bit complex if you have an appropriate system which can apply these sort of extension as well as a constant p prime test it will be easier. But conventionally you try to do a compression test.

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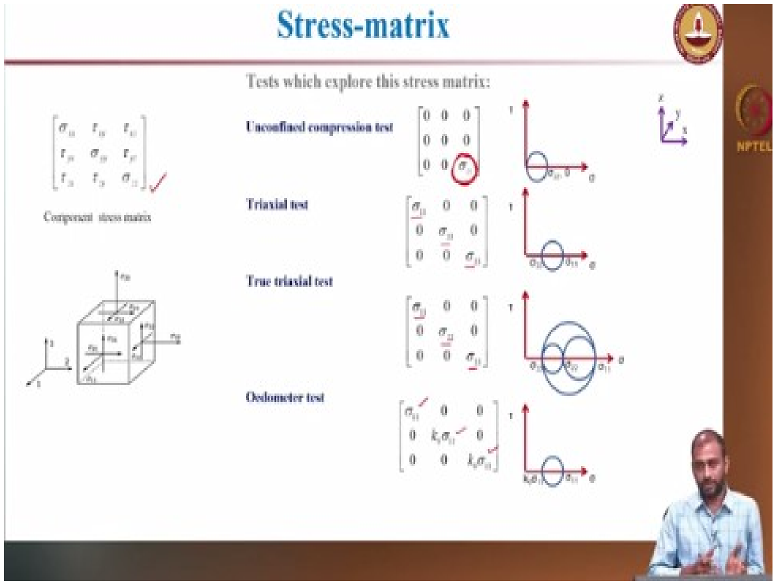
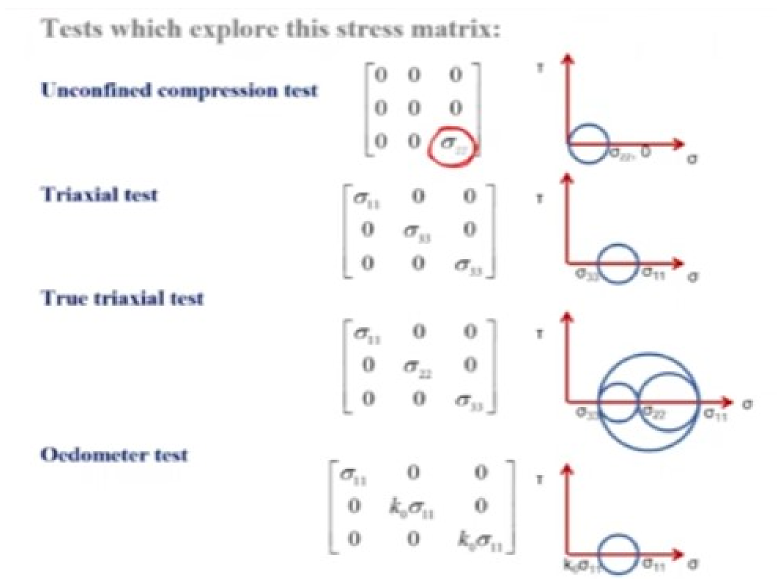
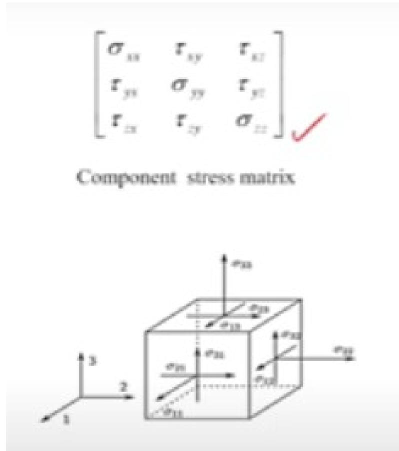


So, these are other various tests which you do to drop the stresses in the stress tensor. So, if you perform an unconfined compression test so the confining pressures are 0 you apply only the axial stress. So, you are probing one stress component of stress tensor or a matrix stress matrix. So, this is a component stress matrix what you are seeing here of that component stress matrix you are actually probing only one stress component that is sigma z.





When you are trying to do a triaxial stress as we discussed earlier you are trying to probe sigma 1 and sigma 3. Also, if you are performing a true triaxial test you can independently control sigma 1, sigma 2 and sigma 3 and try to understand its material behaviour. So, you are trying to apply a true three-dimensional stress and then you can vary different stress path and obtain the response. In this true triaxial test again you are making the shear stress components zero.

You are applying only the normal stresses. In again in odometer test you are applying sigma 1 and then sigma 3 is nothing but K 0 into sigma 1. So, this is your stress matrix and if you see all these stress matrices you are not trying to probe the shear components. So, if you want to probe a sheer component you either use hollow cylinder torsion testing apparatus where you try to probe the two shear components or you try to use a directional simple shear where you can probe all in fact all the shear stress components.

So, the more stress components you are trying to probe you will get better understanding on the material response under complex response. So, again the complexity of operating these devices also increases as you try to control more and more stress components. So, you use all these techniques basically to understand the material response by applying complex stress path.

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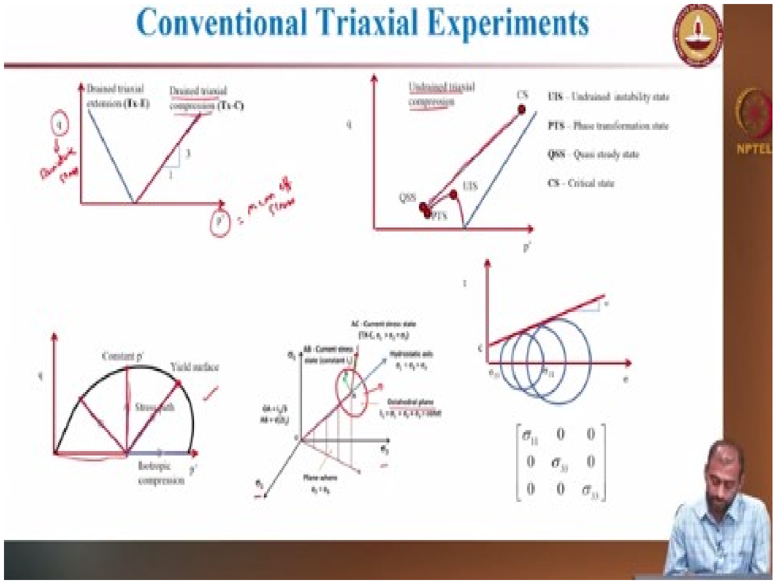
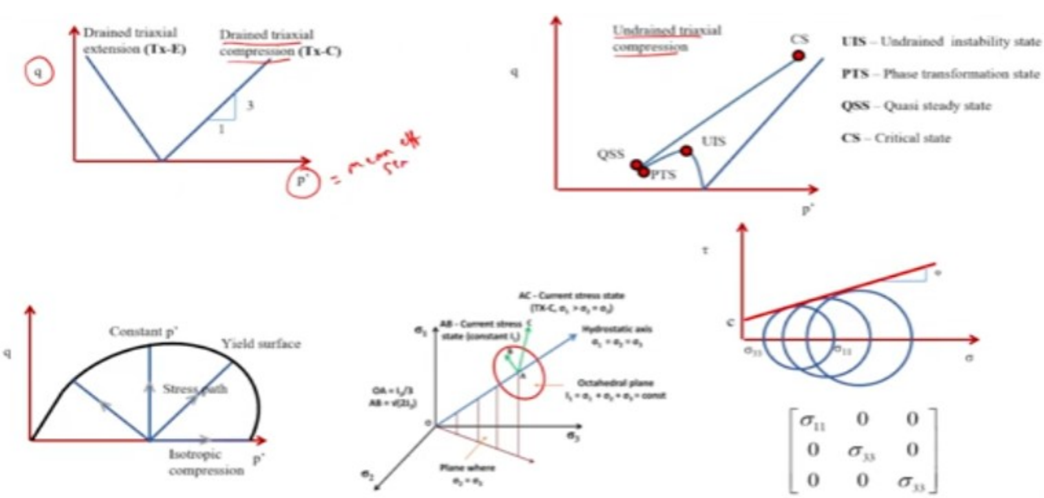


So, when you are using a conventional triaxial experiments you either do a drained triaxial compression experiment or an undrained triaxial compression experiment. So, when you are performing a drain compression triaxial compression experiment you try to move on this. So, in this p prime q space this is p prime and q both are invariance. So, this is the mean effective stress and this is the deviatoric stress.

So, when you are moving on this particular line on the total stress path it will be always making an slope of about three. But if you are performing an undrained typically the material behaviour starts here and then goes to an undrained instability state and then again it contracts to reach a phase transformation state and then reaches this quasi-steady state before reaching the critical state. So, it contracts after that it starts dilating and reaches the critical state.



So, this is a typical response under drained and undrained condition and also it varies depending upon the density depending upon the stress state it is in the mean effective stress that you are choosing. So, typically this is how the response will be. And if you are doing this in the p prime q space you are trying to isotopically consolidate the specimen after that you are trying to share the specimen and find the failure point.

So, if you want to construct the entire failure locus on this p prime q space you need to perform test a different stress path that is what I was emphasizing on applying different stress part to construct the failure locus. So, when you are so you can also perform triaxial extension test to get a point here and a constant p prime test. So, as I said performing triaxial extension and constant p prime test you need to have a sophisticated equipment to control it.

And also, extreme care should be taken by applying the stress increments at every stages. But once you perform it you will get a failure point at different stress path. This is under compression this is certain constant p prime and this is at extension conditions. So, if you want to represent that so what you are seeing is a two dimensional representation of the failure or a yield surface. I will define what is yield surface clearly in the next slides but assume that this is a failure point.

So, once you join all the failure points you construct a nice failure locus that is in p prime q space. If you want to draw the same construct the same failure locus in the principal stress space sigma 1, sigma 2 and sigma 3 you try to move on the hydrostatic axis that is your actually isotopically consolidating the specimen and you are actually knowing on this particular point. So, if you are performing a triaxial compression test obviously the stress path will be not on the deviated plane or octahedral plane.

So, this is a octahedral plane perpendicular to the hydrostatic axis. So, you have to project it back on this particular plane to construct the failure locus. So, again here we can probe only using conventional triaxial tests you can probe you can get three stress path easily. But if you want to construct the entire failure locus you need to look out for other testing devices.

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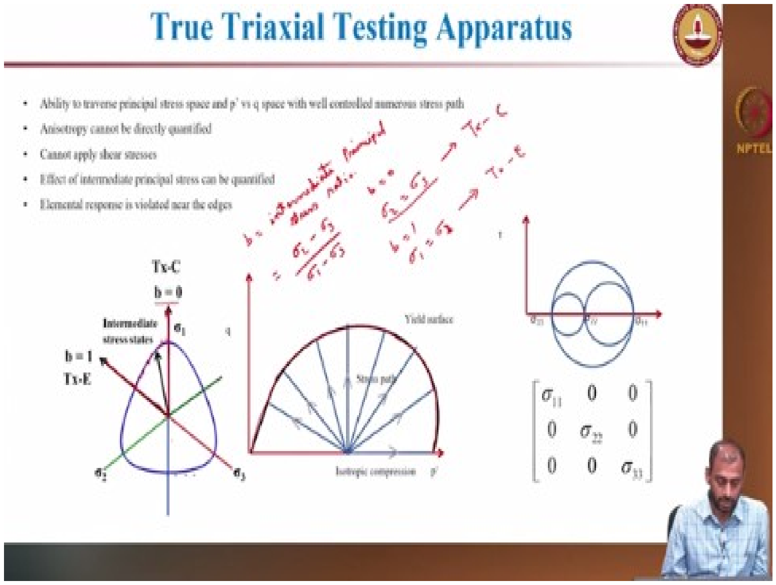
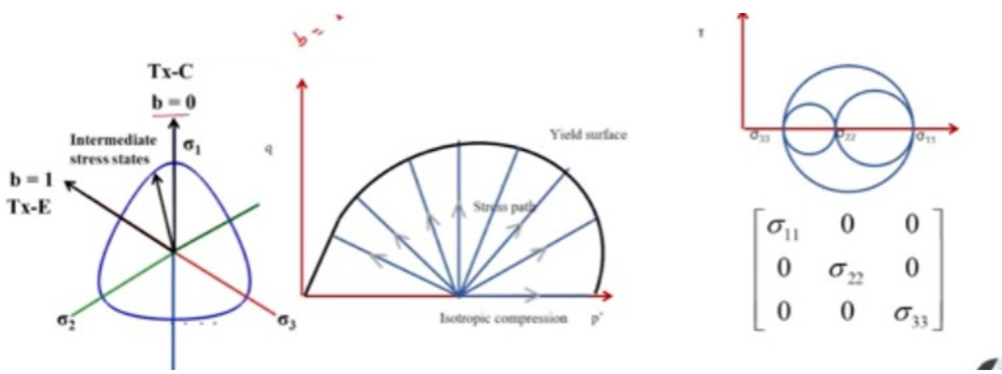


So, here I am going to show about the true triaxial testing apparatus where you can apply multiple stress path by varying b. What is b? This b is nothing but the intermediate principle stress ratio this is nothing but the ratio of sigma 2 - sigma 3 pi sigma 1 - 6. So, b = 0 means sigma 2 = sigma 3 so which is triaxial compression test b = 1 means sigma 1 = sigma 3 which is triaxial extension C = sigma 2.



So, you have triaxial compression so this is the octahedral plane or the deviatoric plane which I have discussed in the previous slide it is perpendicular to the hydrostatic axis. So, after reaching a hydrostatic axis you can actually apply a triaxial compression test and move on this particular line triaxial compression sigma 2 = sigma 3. Similarly, if you are performing a triaxial extension test you try to move on this particular plane and then just locate your failure point.

So, what is the different stress state in between? So, this is one sector of the failure locus which is about 60 degrees and you perform triaxial or this is triaxial compression three this is triaxial extension if you perform different stress path you will get the failure point on the octahedral.

Similarly in p prime q space this is how you get the failure points. So, you finally construct the failure locus.

So, again this anisotropic cannot be directly quantified you need to do other different types of test to quantify the anisotropy. But at least like you will try to get the failure locus by performing test and difference stress path.

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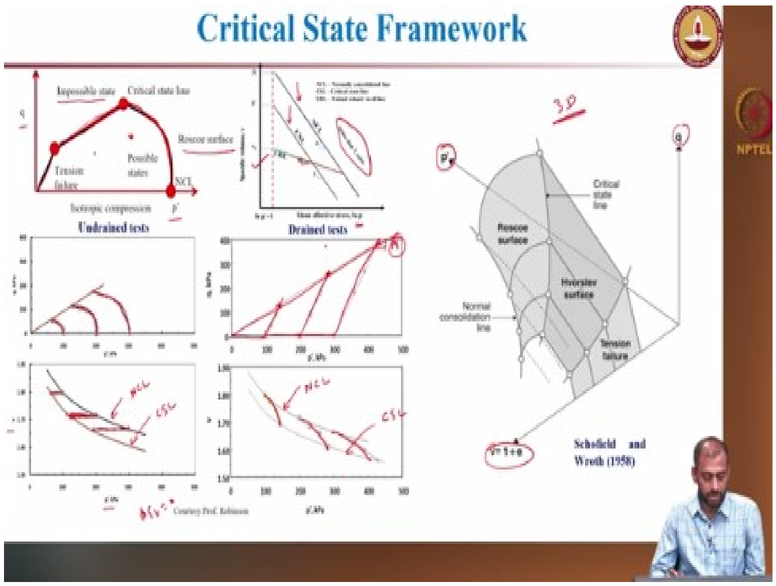
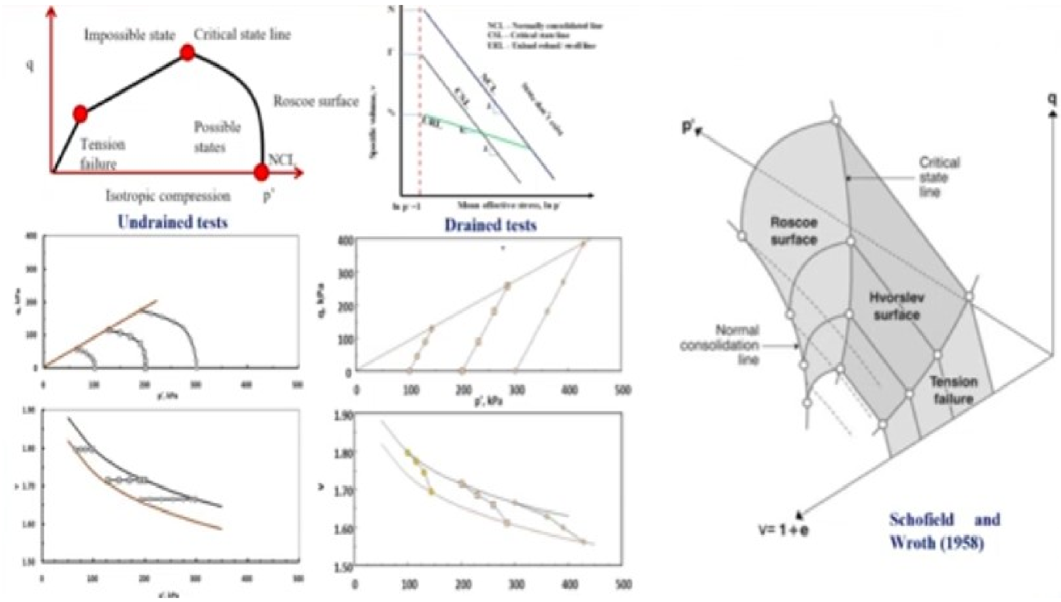


So, now you have the test data now you have to fit all these experimental result in a nice critical state framework. So, this slide will just explain on what is the framework or what is this framework all about and the entire soil response can be modelled in this particular framework. So, in 1950s Schofield Wroth they have came up with a state boundary surface which is in the three dimensional space which is nothing but like you have the deviator express axis the mean effective stress axis.

And the specific volume which is nothing but the void ratio. You have these three axes and the entire soil response they try to like fit in all the experimental results in this particular state boundary locus. So, this is a two dimensional representation of this 3D surface. So, you see a tension cut off you also have a Hvorslev surface you have a Roscoe surface. So, this surface need not be linear this can be curved depending upon the material.



But all the stress or all the behaviour will be inside the entire response will be inside this locus. There is no stress state that can exist beyond this particular locus. There is an impossible stress state beyond this locus but you can have any stress state that is inside the locus. So, this is again experimentally they determine they performed a lot of experiments and then they tried to construct this failure.

So, this is p prime versus q space mean effective stress and D matrix best space also you can represent all these things in like specific volume and mean effective stress space. So, you have a normally consolidated line a critical straight line and an unload reload line. So, you have this factor lambda and kappa. Beyond this normally consolidated line you do not have any stress state, stress states are not possible.

So, you try to relate this mean stress, deviatoric stress and the specific volume and try to come up with this three dimensional locus this is called as state boundary locus. So, how did you arrive at this particular locus? So, you perform lots of tracks and drain tests so you like isotopically consolidate the specimen after that you start sharing and reach a critical state. Similarly, for different confining pressure you try to reach the critical state.

This is by performing triaxial drain test. So, you just reach this failure points connected and then you will get the slope of this line which is nothing but M. This is called as critical state stress ratio M. Similarly, you can perform a undrained test and then it material contracts and reaches critical state. So, again if you represent the same thing in specific volume p prime space if you perform an undrained test again and undrain test the volume changes zero as you all know.

So, the void ratio is constant throughout the test. So, it moves on this particular line with increase in or increase or decrease in mean effective stress and reaches the critical state locus. Whereas if you are performing a drain test, I think the void ratio the volume change happens and then it tries to reach the critical strings. So, this is your normally consolidated line and this is your critical state line. So, you try to construct the entire response in this critical state framework.

This will be useful when you are discussing your critical state framework based constitutive models which are nothing but the cam clay models. So, before going into the cam clay based models, I will just give a brief overview on the elastic models elasticity and the non-linear elastic models.

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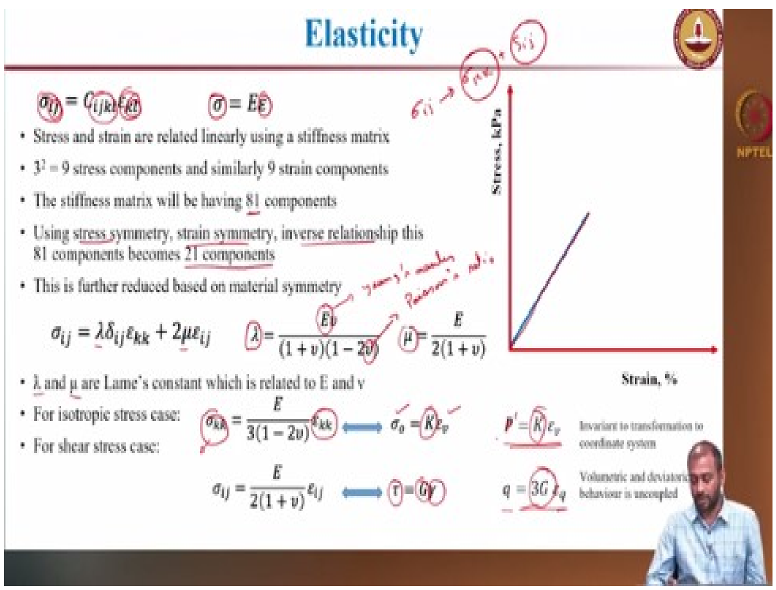

So, as you all know I think a material response can be modelled linearly elastically. So, you have a stress tensor and a strain tensor. So, again this is a second order tensor that is why you are having two indices and this is again a second order. And so, if you want to relate two second order tensor you have a fourth order tensor which is C ijkl. So, that is your stiffness matrix and this fourth order tensor will have about 81 components.

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad \sigma = E\varepsilon$$

So, you have a second order tensor which has nine components and you have another second order tensor which is having another nine components. If you want to related the stiffness matrix will have about 81 components. Then you start using your stress symmetry, strain symmetry and inverse relationship and then you simplify it to have about 21 components in your stiffness matrix.

Once you have this stress symmetry, strain symmetry and inverse relationship you reduce the number of components. You can further reduce the number of components by taking the material symmetry. So, if you are using a material which is isotropic then you will come down to only two components which is nothing but the Lame's constant which is a lambda and mu. So, this Lame's is constant lambda and mu are actually related.

And you can actually determine this lambda on mu are related to your Young's modulus and Poisson ratio. So, this is your young’s modulus and Poisson ratio. So, again so the total stress tensor any total stress tensor is divided into the volumetric part plus the deviatoric part. This is your volumetric part of the stress tensor this is your deviated part of stress tensor. So, if you want to relate volumetric part of the stress to volumetric strains.

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \qquad \lambda = \frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)} \qquad \mu = \frac{E}{2(1+\upsilon)}$$

$$\sigma_{kk} = \frac{E}{3(1-2\upsilon)}\varepsilon_{kk} \longleftrightarrow \sigma_o = K\varepsilon_v$$

$$\sigma_{ij} = \frac{E}{2(1+\upsilon)}\varepsilon_{ij} \longleftrightarrow \tau = G\gamma$$

$P = K\,\varepsilon_v$ Invariant to transformation to coordinate system

$q = 3G\,\varepsilon_q$ Volumetric and deviatoric behaviour is uncoupled

This is your sigma kk is your volumetric stress and your epsilon kk is your volumetric strain. If you want to relate the volumetric stress and volumetric strain you have a constant which is called as bulk modulus. So, again if you want to relate the shear stress versus shear strain you have shear modulus. Typically, since in critical state framework you use p and q so p this is p prime = k into epsilon V.

Similarly, q = 3G into epsilon q, so this is the relation between mean effective stress and volumetric strain, deviatoric stress and deviatoric strain. So, again we can write all these things nicely but again you can see clearly that K is a constant G is a constant. This is not there is no pressure dependency nothing. I think it is constant and the variation is linear but as you all know that the soil response is not linear. So, you need to bring in some pressure dependency to this stiffness values.

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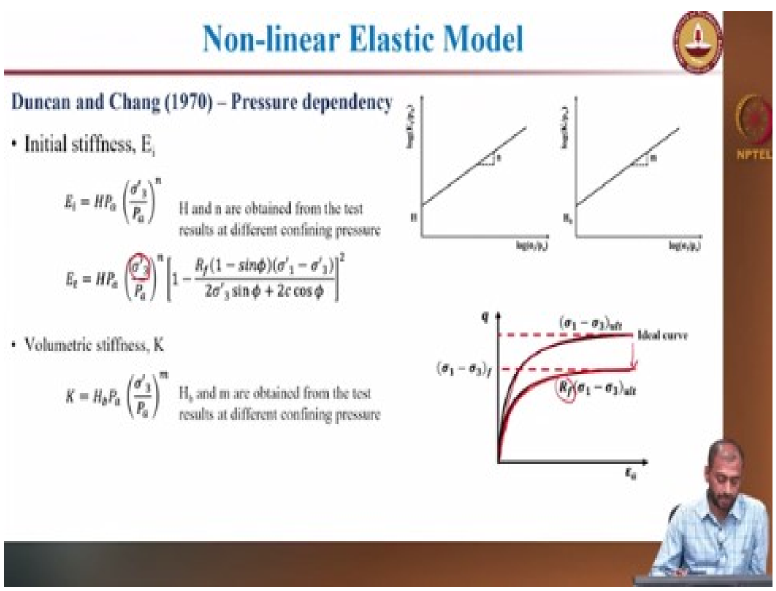
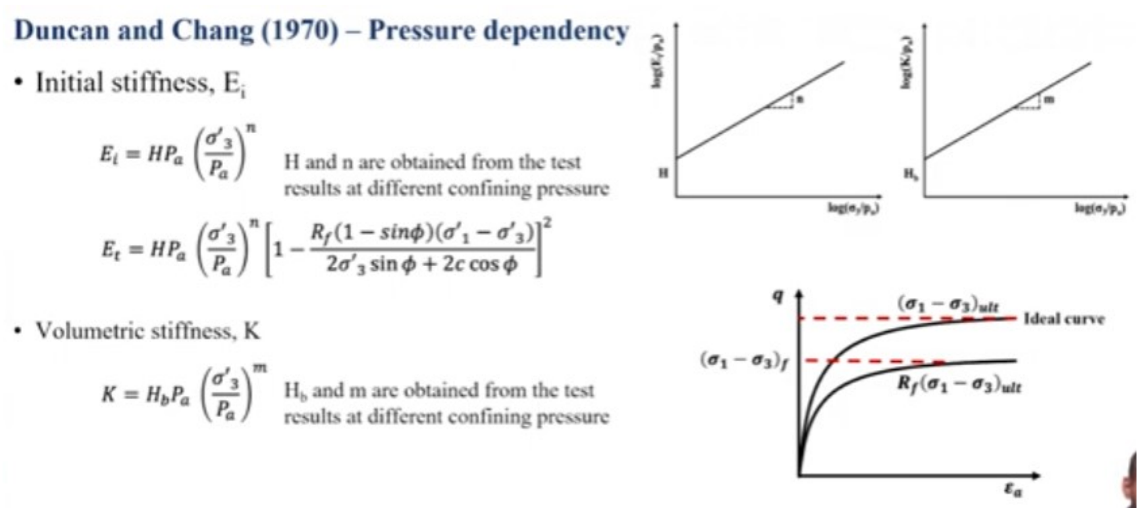


So, there comes some of the non-linear elastic models. So, in 1970 Duncan and Chang proposed some pressure dependent models. So, they try to modify elastic stiffness by bringing in so they model based on the hyperbolic equation and they brought in this pressure dependency sigma 3.

So, you have this equation and you can clearly see that the behaviour is non-linear. So, this is R f is again stress ratio.



And this is actually the stress the R f value can be tuned such that the hyperbola is trying to model the ideal curve is brought down to the actual material. Similarly, you can do it for the volume change response also.

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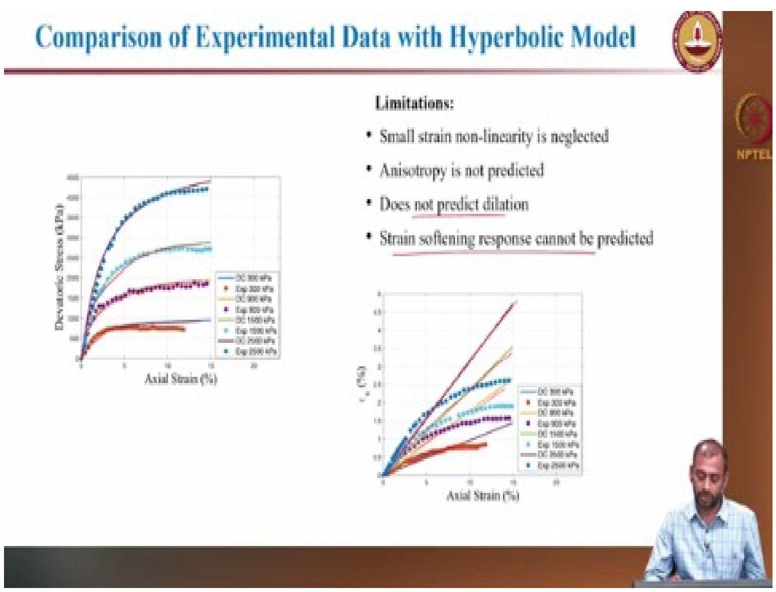

So, this figure actually shows your experimental results are compared with some of the non-linear models. So, as you can see the material sort of shows and hardening response. It keeps hardening and even though this is a simple linear elastic model with the pressure dependency it can match some of the experimental results. So, in case in certain material system for example if you have a soil which is containing some sort of reinforcement or some sort of a waste materials, it sorts of hardens the material sorts of hardens with strain.

So, those sorts of response can be modelled using this simple hyperbolic model but it has deficiency. So, there is no a term called yield so and also the softening response cannot be modelled or predicted. So, material usually has dilation this particular model does not predict the dilation response. And obviously the anisotropic and small strain non-linearity is usually neglected using this hyperbolic model.

As you can see the stress strain response is modelled very well however the volume change response. As you can see the volume change response is not matching well with the model predictions. It is showing a contractive response and it is almost linear. But later this is your original Duncan and Chang model but later actually there are modifications to this original Duncan and Chang hyperbolic model.

And they try to predict the volume change response also fairly accurately. But still, it does not predict the dilation characteristics because of its simplicity. So, this is another simple pressure dependent non-linear hyperbolic model.

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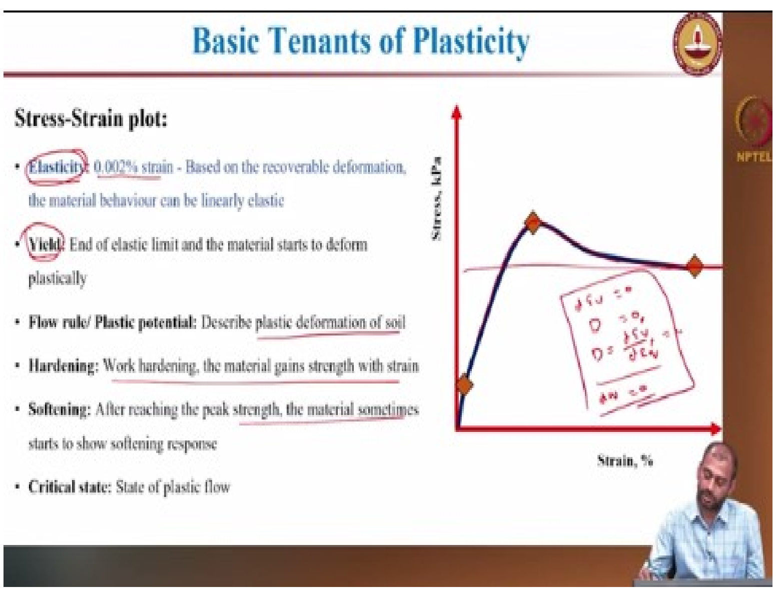
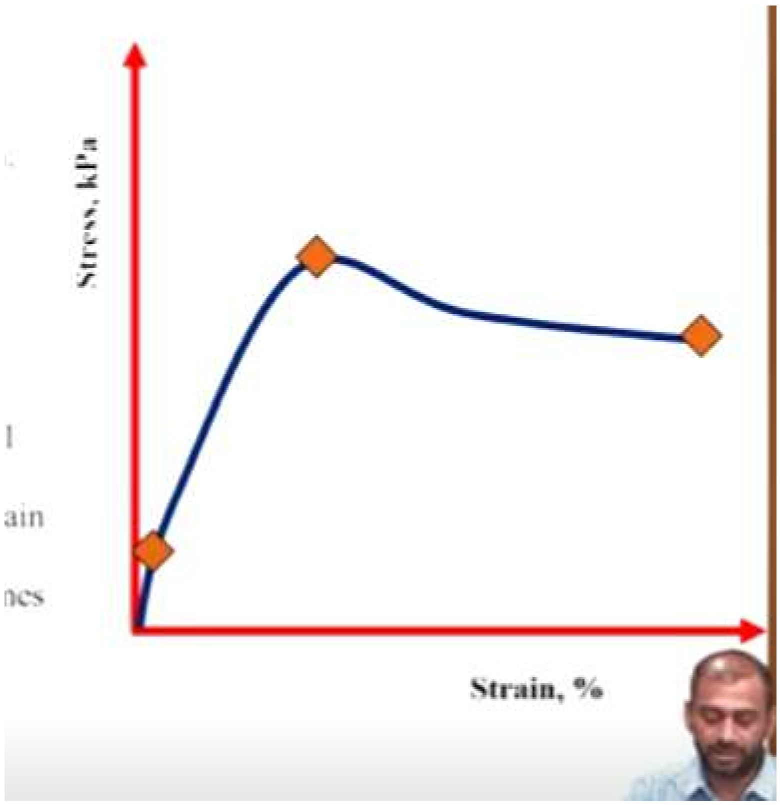


So, before going into some more details on the original cam clay models I just wanted to introduce something on the basic tenets of plasticity. So, till now whatever the linear elastic model does not have any yield or failure does not we do not define any yield of failure. Non-linear hyperbolic model also does not have any yield or failure. So, we use a failure criterion which for simplicity but I think I will discuss more in detail in this particular section on the various standards of plasticity.



Then we will go into the original cam clay and modified cam clay. So, we already discussed about elasticity. So, this is your first tenant which is elasticity and usually for soils the elasticity is very small. So, the elastic nature is actually removed or the elastic nature will not be there when the strain is more than 0.002 percentage. So, again after this the material response is morally mostly the plastic there is some sort of a dissipation that is taking place.

So, once the material starts yielding the point at which the material starts yielding is called as the elastic limits and the behaviour is predominantly plastic after this particular point. So, this is the yield is an important yield or failure is an important tenet of plasticity theory. So, this forms the predictions which we will be carrying it out. Again, the next important tenet of plasticity is the flow roll or the plastic potential which describes the plastic deformation of a soil response.

So, again to get a nice flow rule or a plastic potential you need to understand the volume change response. Again, if the material can be dilated, the material can be contractile. So, you need to understand how the material dilates or contracts using an appropriate plastic potential. Also, the material can harden so the material gain strength with as you start shearing again the material can actually reduces its strength once you start sharing.

So, those are the hardening and softening response that you typically observe. So, once it reaches so the material actually hardens and then maybe it softens and then at large strains it reaches a critical state. So, when it reaches critical state, we can define it as d epsilon v = the volumetric strain incremental volumetric strain is 0 there is no dilation d. Dilation is nothing but d epsilon v by d epsilon q volumetric strain by shear strain.

That is called as ratio of volumetric strain to plastic volumetric strain to plastic shear strain is called as dilatancy that is equal to 0 and the stress d q is also 0. So, these are the condition when the material reaches critical state. So, when once the material reaches critical state then we can choose that to be a state of complete failure.

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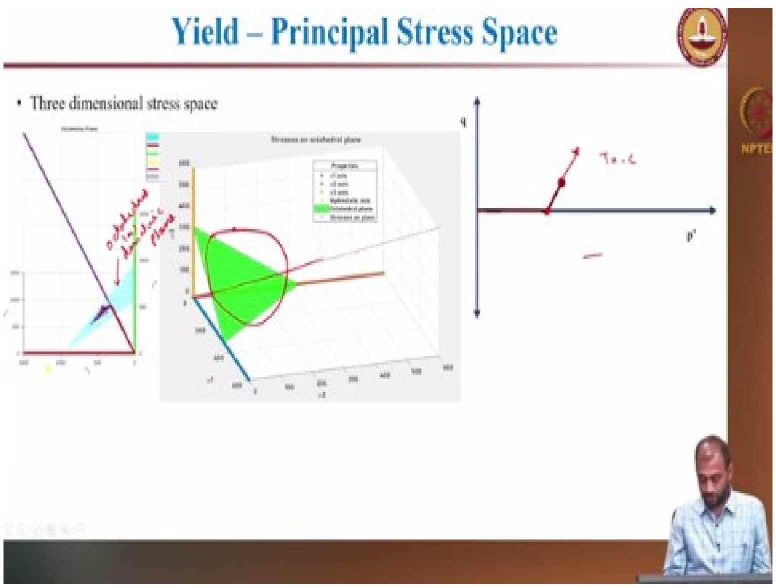

So, how will you represent all these things in a three dimensional space usually you use a principal stress space, an Euclidean space you try to represent the material response. So, you hydrostatically compress the material and then you reach a particular point and then perform your test and then you will try to locate the failure locus on this octahedron or deviatoric plane. So, you have a nice failure locus that is constructed here on the octahedral plane.

And this will and this failure locus will be helpful in finding out an appropriate yield equation. Similarly, as you see here when you are performing your hydrostatically compressing and then you are actually going on this line and then reach the particular failure locus. This failure locus is how you try to get this by performing a triaxial compression test. So, you represent the yield on a principled stress space and try to get the failure equation the yield equation.

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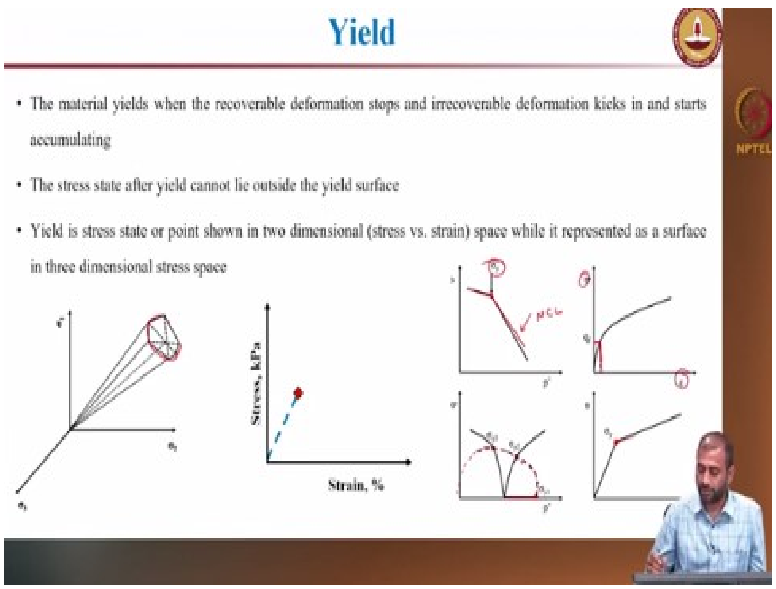
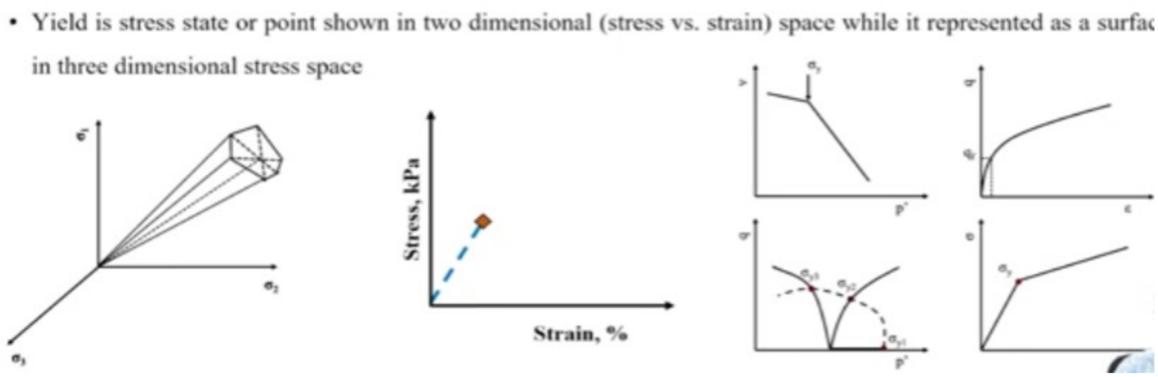


So, what is yield? The material once the material yields it starts its the recoverable deformation stops and they recall deformation kicks in and then it starts accumulating and there cannot be any stress state that lies outside the yield surface. So, the yield is a stress state or a point that is shown here on a two-dimensional space. So, this is considered as an yield point. But in 3D it is like a surface you are trying to see a failure locus on the three-dimensional space sigma 1, sigma 2, sigma 3 space.



So, how to identify this yield? For example, if you perform an odometer test you can see there is a change in slope. So, this is your NCL and this change in slope will actually identify your yield point. Similarly, you can do an isotropic compression you get an yield point. Then finally you can join all the yield point to construct the failure locus. See typically it is easy to locate as the slope changes in your stress strain plot as your yield point but it is not that easy in case of a geo material.

So, you will not see this sort of a nice bilinear plots in case of a geo material. So, you have you need to identify by placing different plots. For example, p prime is a specific volume plot or deviated stress versus strain plot from here you can identify the failure point of the yield point.

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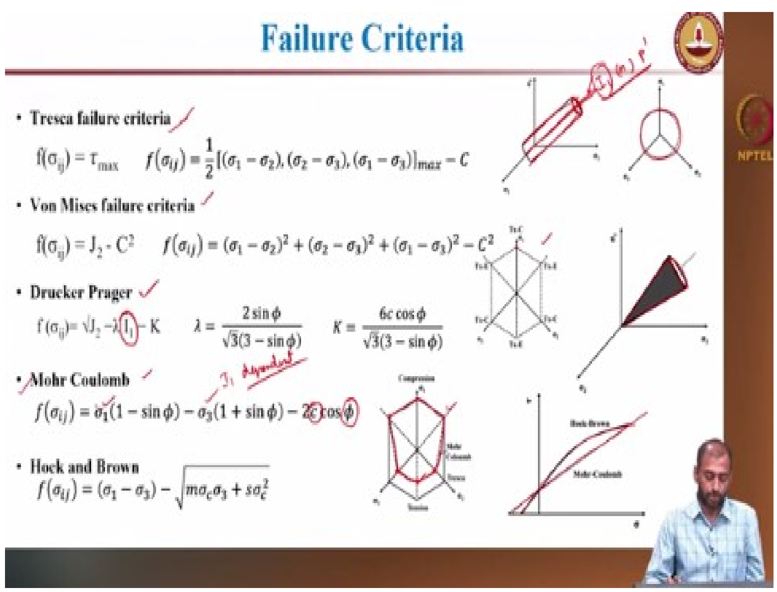
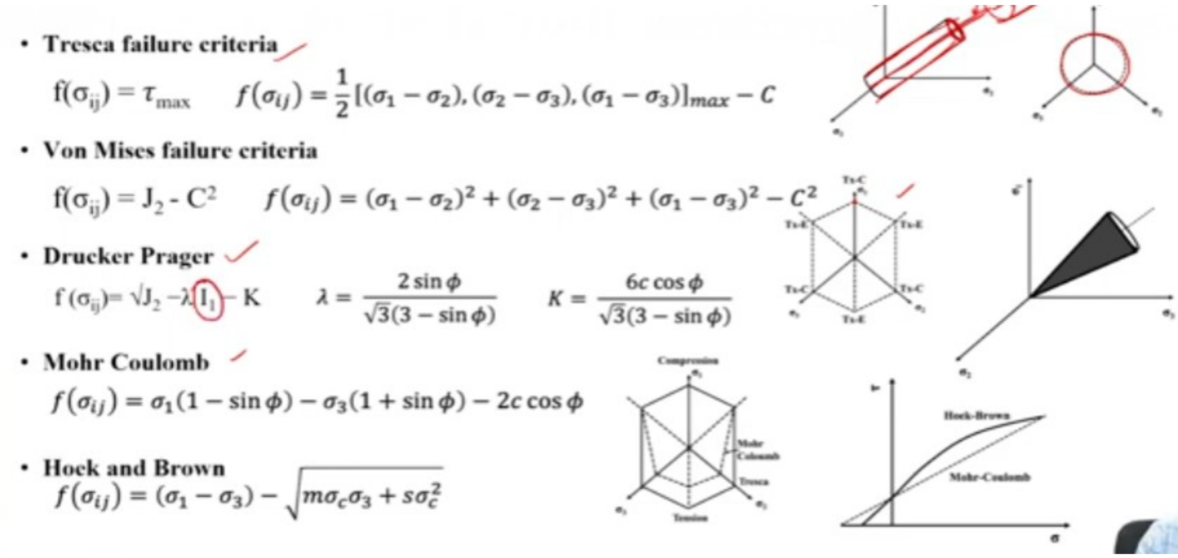


So, what you are seeing on the screen are some of the classical failure criteria's that has been given in past several centuries in the past couple of centuries. For example, Tresca was the first one to come up with this failure criteria based on the maximum shear stress theory and Von Mises also provided a failure criteria specifically for metals. So, if you see a Trescas failure locus it is not pressure dependent.



Of course, the Tresca and Von Mises a failure criteria or not pressure dependent that is why you can see a cylindrical failure locus in the principal stress space. So, it is definitely not pressure level it is not I 1 dependent. So, I 1 or p prime so these are all invariants and it is not pressure dependent. So, these two failure criteria is generally applicable for pressure independent materials. So, this is the pressure dependency is not there for these two material systems.

And for Von Mises failure criteria you see a circular failure locus. Whereas in case of a Tresca you see a nice hexagonal failure locus on the octahedral plane. So, these two are pressure independent failure criteria whereas the next two failure criteria what you are going to see the Mohr Coulomb and the Drucker Prager failure criteria are pressure dependent failure. Because you can see in the failure equation you have a pressure dependent I 1.

And here it is again sigma 1 and sigma 3 you can just make it as a mean effective stress. So, this is again I 1 dependent. So, the difference between Drucker Prager and Mohr Coulomb failure criteria again you can see since it is pressured dependent you can see a conical failure locus on the principle three-dimensional failure of stress space and you can see a irregular hexagon in the octahedral plane for a Mohr Coulomb failure also.

And you can also have you can see the similarity between Von Mises and Drucker Prager. So, the failure locus is circular on octahedral plane in both Von Mises and Drucker Prager criteria. Whereas the failure locus is hexagonal in case of Tresca failure criteria and you see a irregular hexagon in case of a Mohr Coulomb field criteria. This irregular hexagon is predominantly due to the material response again the soil behaves differently in extension.

So, that is why it is not an hexagon it is an irregular. And you can see the strength in the extension part or the tension part is lower than in the compression part. So, again this Mohr Coulomb failure criteria will have a linear failure envelope in a sigma tau space but some material does not have this sort of a linear failure envelope. In that case you try to use other models which can model this non-linear part of the failure envelope.

So, these are all classical failure criteria's and you can use this for modelling geo material. So, again you all know that the Mohr Coulomb failure criteria is a commonly used extensively used in many solving many boundary value problems because of its simplicity you need to know only two parameters c and phi in order to model this failure response.

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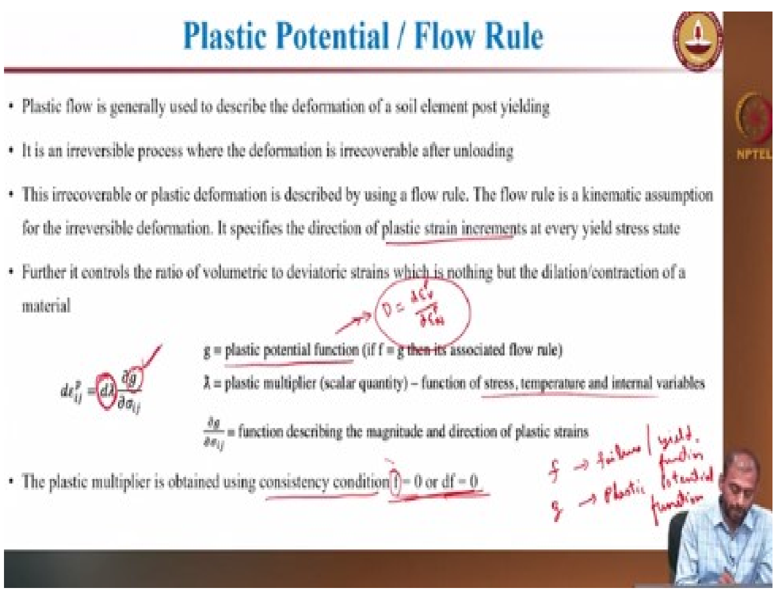

So, the next important tenet of plasticity theory is the plastic potential or the flow rule. So, the plastic flow usually describes the deformation of soil post yielding and again as you know it is a reversible process. Since this irrecoverable plastic deformation is described using the flow rule and as you know this is a kinematic assumption of this irrecoverable deformation. And you need to know the plastic strain increments for every stress increment.

So, the direction which specifies the direction of the plastic strain increment is generally specified on the yield locus. And if the plastic strain increment coincides with the stress increment, then the normality condition is satisfied. We will discuss about the normality condition and the associated or not associated flow rule in the next slide. But before that to have this flow rule you need to know what is a plastic potential function.

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial g}{\partial \sigma_{ij}}$$

g = plastic potential function (if f = g then its associated flow rule)

λ = plastic multiplier (scalar quantity) – function of stress, temperature and internal variables

$\frac{\partial g}{\partial \sigma_{ij}}$ = function describing the magnitude and direction of plastic strains

So, this plastic potential function if you want to get an appropriate plastic potential function you need to have a good idea about the volume change response of the material. So, this is directly determined from the stress dilatancy relationship. For example, D which is the ratio of plastic volumetric strain by the ratio of the deviatoric strain plastic deviatoric strain. So, if you know the response of a material especially the dilatancy characteristics you can come up with the proper plastic potential equation.

The plastic multiplier is obtained using consistency condition f = 0 or df = 0

If you know the proper plastic potential equation then using consistency condition you can determine this particular plastic multiplier rate. It is a scalar quantity which is a function of stress, temperature and other internal variables. So, what is consistency conditions? So, consistency condition actually describes that for any plastic deformation the stress state is always on the yield surface.

So, that is why you are using df = f this is the failure equation f is generally defined as failure or yield function and g is called as the plastic potential function. So, we try to use this consistency condition. So, as there is a plastic deformation that is happening the stress state is always on the failure locus so there cannot be a stress state beyond the failure locus. So, as plastic deformation as strain that is getting accumulated increases the failure locus actually expands.

So, using this consistency condition you can determine this plastic multiplier and if you know an appropriate plastic potential you can determine the plastic strain increment. So, this is key and, in some cases, you assume the plastic potential to be equal to the yield locus.

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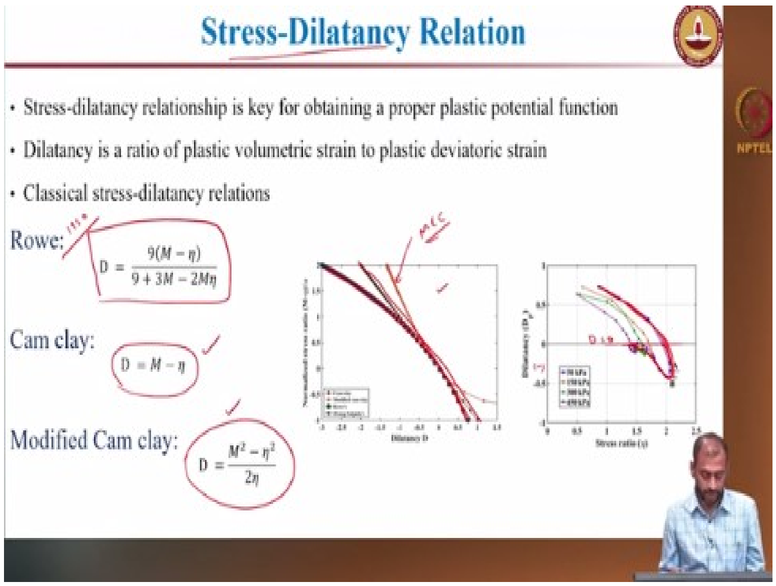
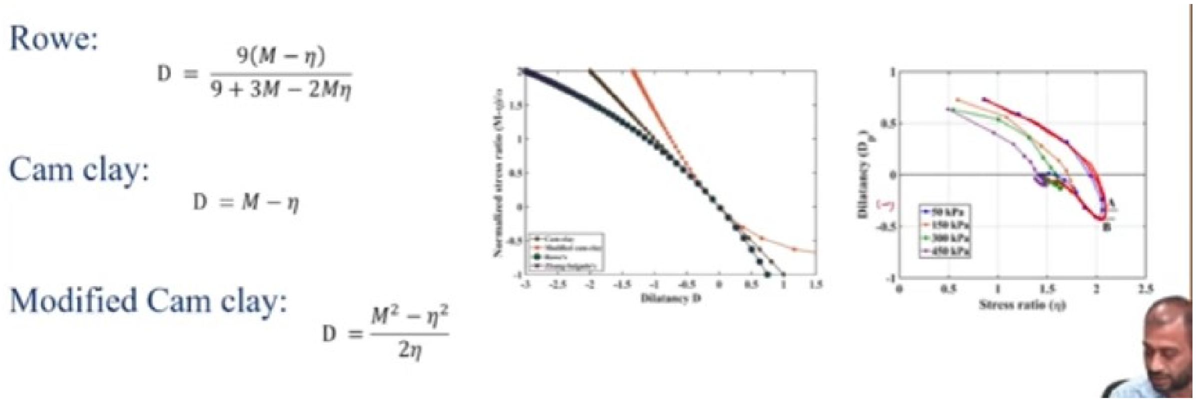


So, we will see when that is being assumed. But before that as I said if you want to get a proper plastic potential you need to know the stress dilatancy characteristics. So, as you can see a typical stress dilatancy characteristics of a geomaterial you can see as the stress ratio increases the material contracts up to a point after that it starts dilating. It goes so the negative is the dilation part so it starts dilating and then reaches the critical state, where the dilatancy = 0.

So, this is your typical response and on the left you can see classical stress dilatancy equation that has been proposed by several researchers. So, the most famous stress dilatancy relationship which was provided by Rowe in 1950s. So, this is the stress dilatancy relation given by Rowe and you can see how this predicts the material response. So, again the prediction is shown here. So, it is non-linear and this response is similar to you can see the actual experimental response.

So, this response stress dilatancy you can see response is somewhat close to this. We are also trying to compare it with the stress dilatancy relation given from cam clay and the modified cam clay. So, you can see a linear relationship is obtained in case of cam clay whereas a non-linear relationship or non-linear curve is obtained from when you are using the stress relative relation of modified cam clay.



So, we will see how to derive these things in the next few slides. So, this is on cam clay and modified cam clay. So, this is the stress dilatancy relationship and using this you can get a plastic potential function.

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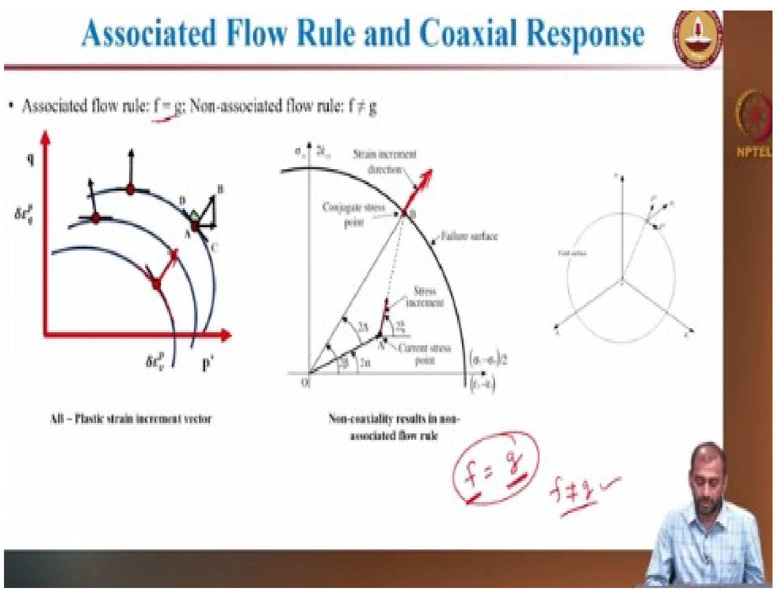

So, as I discussed in some cases you can use f = g, your yield function is equal to the plastic potential function. So, when you assume f = g this is called an associated flow rule. If f is not equal to g then you have a separate yield function and you also have separate plastic potential function and then you use it in the framework and then solve it. So, you have so that is called a non-associated flow rule based models.

So, this figure what you see on the screen actually represents the plastic strain increment which is plotted on the yield locus. So, if this plastic strain increment is normal to the yield locus, then it follows the normality condition. And if this plastic strain increment and the stressing stress is in the same direction. So, this is the current stress point and this current stress state and the strain increment are aligning with each other then the material response is said to be coaxial.

But as you know the soil response is complex so your stress strain increment direction does not tend to coincide with your stress increment direction. So, it exhibits a non-coaxial response which has been well established by many researchers. So, the soil exhibits are non-coaxial response. However, when the material reaches critical state so the stress increment and the strain increments is almost following the same direction.

So, it becomes a coaxial response at very large when it is under plastic flow state. So, this is again about the associated and the coaxial response that you would generally observe.

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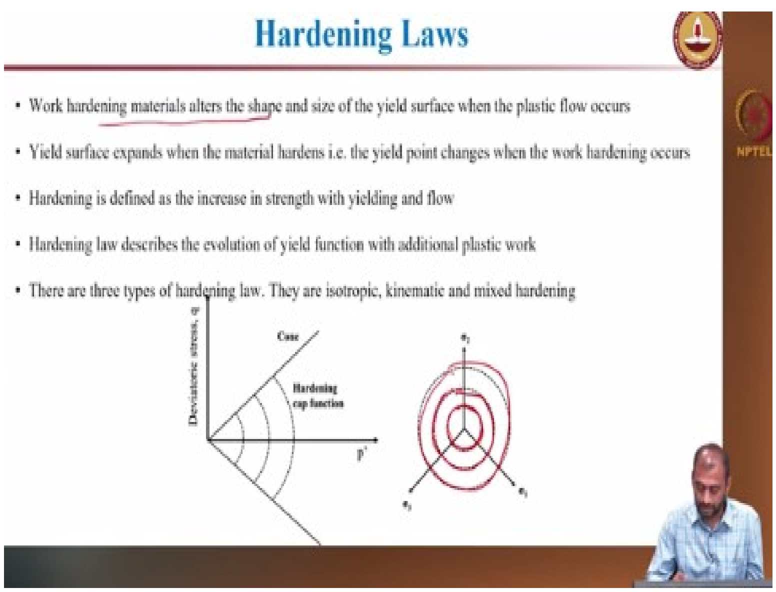

The next important tenet is the hardening laws. As you see when there is a work hardening material the size of the yield locus expands as you start sharing. So, the material gains in strength. So, if this is your initial failure locus the failure locus keeps on expanding. So, you can have a separate hardening equation or you can have an yield equation which takes into account the effect of hardening.

So, again this hardening is defined as an increase in strength after yielding or when there is a flow that is happening. So, this is an additional plastic work that has been done and this hardening is again classified into different types. One is the isotropic running where you can see the failure locus actually expands in size there is no translation there is only expansion of the failure locus then there is called something called as kinematic hardening where there is a translation of the yield locus that is happening.

You typically observe this sort of a kinematic hardening especially when there is some sort of a cyclic response that is happening. So, when there is some shearing that is happening because of some seismic or earthquakes. So, that particular short-term sharing sometimes tends translates this yield locus in a or translates this yield locus so that you can see or observer kinematic hardness.

Of course, there is some mixed hardening the failure locus expands as well as it translates. So, you see all these different types of hardening under certain conditions.

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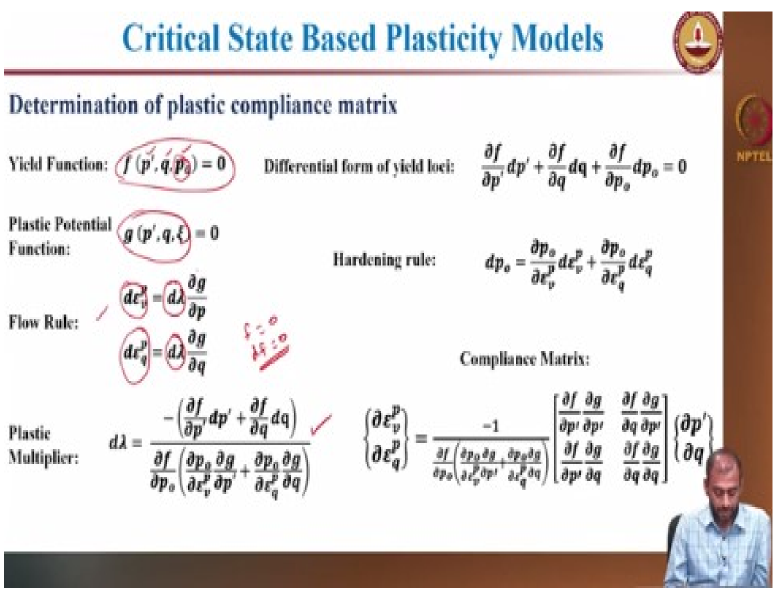

So, from this discussion on the fundamental tenets of plasticity you have an yield equation. In our case it is dependent on p prime q p 0 p 0 is the pre-consolidation pressure and you have a plastic potential function. So, it is a function of internal variables p prime and q and you also have the flow rule. This is on the volumetric strain and this is for the deviatoric strain. So, if you want to find this plastic multiplier rate as I said you have to use this condition f = 0 or d f = 0 which is the plastic consistency condition and try to determine this d lambda.

Determination of plastic compliance matrix

Yield Function: $f(p', q, p_o) = 0$

Differential form of yield loci: $\frac{\partial f}{\partial p'}dp' + \frac{\partial f}{\partial q}dq + \frac{\partial f}{\partial p_o}dp_o = 0$

Plastic Potential Function: $g(p', q, \xi) = 0$

Hardening rule: $dp_o = \frac{\partial p_o}{\partial \varepsilon_v^p}d\varepsilon_v^p + \frac{\partial p_o}{\partial \varepsilon_q^p}d\varepsilon_q^p$

Flow Rule:

$$d\varepsilon_v^p = d\lambda\frac{\partial g}{\partial p}$$

$$d\varepsilon_q^p = d\lambda\frac{\partial g}{\partial q}$$

Plastic Multiplier:

$$d\lambda = \frac{-\left(\frac{\partial f}{\partial p'}dp' + \frac{\partial f}{\partial q}dq\right)}{\frac{\partial f}{\partial p_o}\left(\frac{\partial p_o}{\partial \varepsilon_v^p}\frac{\partial g}{\partial p'} + \frac{\partial p_o}{\partial \varepsilon_q^p}\frac{\partial g}{\partial q}\right)}$$

Compliance Matrix:

$$\begin{Bmatrix} \partial\varepsilon_v^p \\ \partial\varepsilon_q^p \end{Bmatrix} = \frac{-1}{\frac{\partial f}{\partial p_o}\left(\frac{\partial p_o}{\partial \varepsilon_v^p}\frac{\partial g}{\partial p'} + \frac{\partial p_o}{\partial \varepsilon_q^p}\frac{\partial g}{\partial q}\right)} \begin{bmatrix} \frac{\partial f}{\partial p'}\frac{\partial g}{\partial p'} & \frac{\partial f}{\partial q}\frac{\partial g}{\partial p'} \\ \frac{\partial f}{\partial p'}\frac{\partial g}{\partial q} & \frac{\partial f}{\partial q}\frac{\partial g}{\partial q} \end{bmatrix} \begin{Bmatrix} \partial p' \\ \partial q \end{Bmatrix}$$

Once you have all these things you can construct this compliance matrix. So, this is your compliance matrix once you construct these complaints matrix, I think you can if you know the stresses you can get the strains. So, now I will stop with this on the critical state framework. In the next lecture I will be discussing on original cam clay and modified cam clay and its predictions.