Finite Element Analysis and Constitutive Modelling in Geomechanics
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Lecture - 29
Nonlinear Techniques-3

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I hope you are following my lectures and if not you please send an email to this address and then I can clarify any of your doubts and my suggestion is you please complete reviewing all the previous lectures before you listen to the next lecture so that you can follow the sequence because in this course every lecture is based on what you have learned in the previous lecture. So, my suggestion is please be up to date and work out all the tutorial problems.

And then listen to the lectures before you move on. So, let us continue in the previous class we had discussed about the different non-linear analysis techniques and let me just review once again and then show you an excel spreadsheet program for doing these analysis. I am going to review whatever we had done in the previous class and then show you one excel spreadsheet for you to do the calculations and that will help you in quickly learning these concepts.

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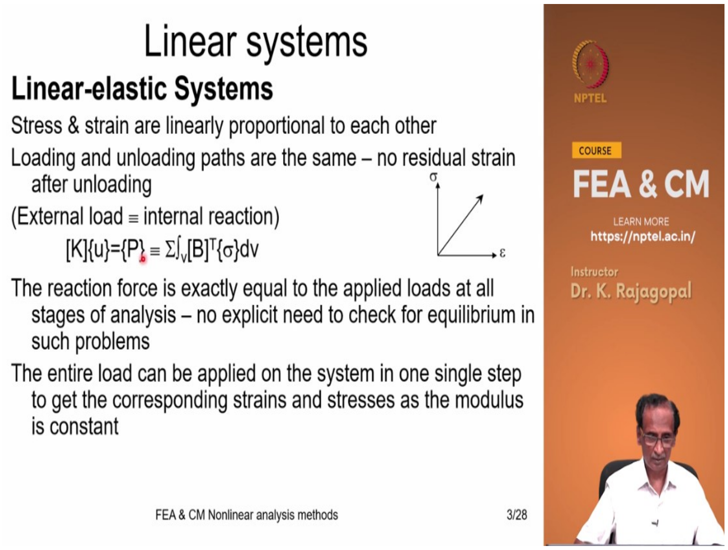
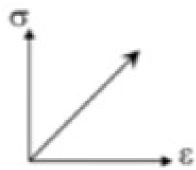


So, depending on the problem that we have we can analyze differently and the simplest system that we can analyze is the linear elastic system where our stress and strain are just linearly proportional to each other both during loading and unloading and in that case we can just write our equilibrium equations and just simply K times u is equal to P where K is our stiffness matrix, u is the vector of displacements and P is the vector of applied forces.



(External load ≡ internal reaction)

$$[K]\{u\}=\{P\} \equiv \Sigma\int_v [B]^T\{\sigma\}dv$$

And in all the finite element analysis you can either apply a force at a degree of freedom or a displacement. You can apply only one of them and although I am saying we apply the force we might as well apply the displacement because as we have seen earlier with the example of the two blocks attached through an inclined joint element where we applied equal vertical displacements at the top and then measure the reaction forces.

So, in this case in the linear elastic systems we do not really need to check for equilibrium it is automatically satisfied because whatever is the developed stress in the elements they will be able to support that much stress because there is no limit on the stresses that you can apply and normally we do not write the right hand side as in an incremental form either as a d P or P - reaction force as we do for other systems.

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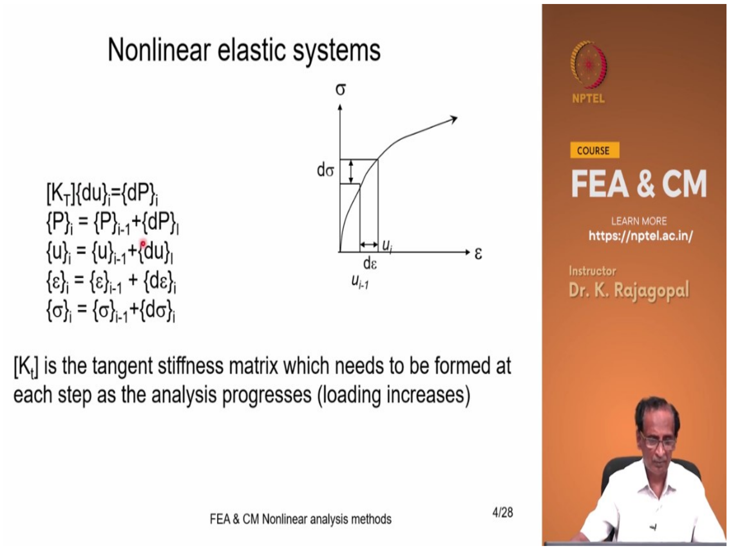
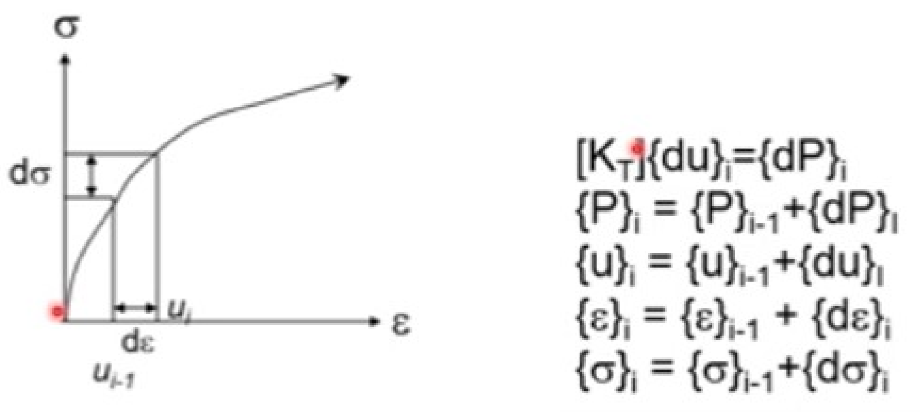


Then slightly more complicated are these non-linear elastic systems and the simplest one is just go in an incremental form, apply the force or displacement in small increments so that you can follow this stress-strain path as closely as possible and our K is now the tangent stiffness matrix because with increasing stress or with increasing strain your slope is changing.



So, we update our constitutive matrix and then update our stiffness matrix multiplied by incremental displacement is equal to incremental load corresponding to some incremental stress and then as we are doing we have to go on accumulating the displacements strains stresses and so on. So, our P i is P i - 1 + d P i where i - 1 corresponds to the previous step and i corresponds to the current step.

And so the displacement once you calculate the incremental displacements we can update our displacement vector and then you can calculate your strains as B times d u and then update our strains as the epsilon i is epsilon i - 1 + d epsilon and then we can calculate the stress increment as d times d epsilon and then update our stress vector sigma i is sigma i - 1 + d sigma i.

And all these we are going to do at each and every integration point within the element we will have number of integration points depending on the order of element and so on and in all these problems our stiffness matrix needs to be updated every iteration then once we update we have to assemble and then we have to triangle it like we have to make it as an upper triangular matrix and so on.

So, these analysis they take lot of time and actually this is one simple method, but we can actually improvise this and that is shown here.

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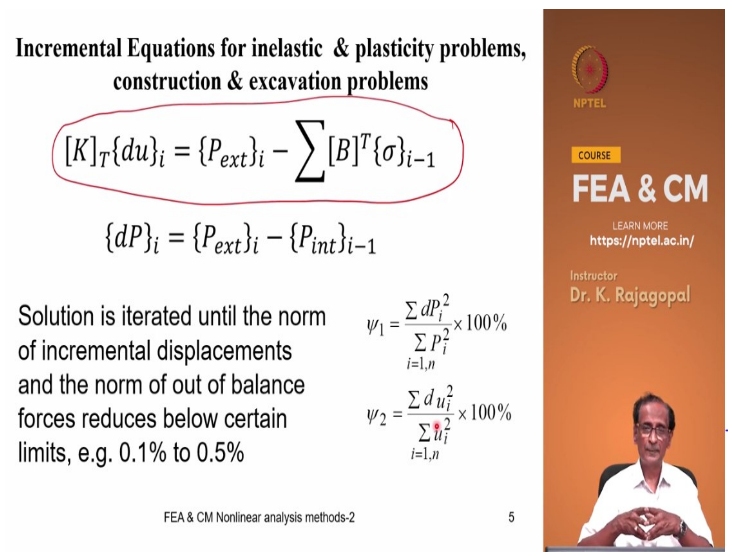

And this equation that you are seeing this equilibrium equation it is valid for all types of non-linear analysis problems are inelastic, elastic plastic or construction and excavation problems whatever may be this equation can take care. Once again our tangent stiffness matrix multiplied by d u i is a P external - B transpose sigma i - 1 actually this is the current applied force.

$$[K]_T\{du\}_i = \{P_{ext}\}_i - \sum [B]^T\{\sigma\}_{i-1}$$

$$\{dP\}_i = \{P_{ext}\}_i - \{P_{int}\}_{i-1}$$

And B transpose sigma i - 1 is the reaction force from the previous iteration or previous step and this sigma could be a corrected stress because when we have any limit on the stresses that we can apply then we correct and then the sigma could be a corrected stress. In that case our equilibrium gets disturbed and this particular form of the equilibrium equation it is the most important one in all our non-linear finite element analysis.

And you have to understand this perfectly and once you understand then you appreciate how we can apply this for all types of problems like especially when we have the excavation type problems we need this for calculating our traction forces to make the excavated surfaces as a free of traction and then when we are iterating we have to monitor the incremental displacements.

And then the out of balance force and d P i is the out of balance force or the difference between the applied force and the reaction force and in the ideal case this should be zero and that is what we get in the case of linear elastic problems and in the other cases we satisfy this only in an approximate sense as like we calculate some norm of out of balance force as a sigma of d P i square the sum total of all the squares of the out of balance forces at all the degrees of freedom and divided by square of all the applied loads multiplied by 100 percent.

And we iterate until this value is less than about 0.1 or 0.5 percent or sometimes even 1 percent depending on the type of non-linearity. If it is a highly non-linear system or the strength is limited then we may tolerate a slightly higher address like 1 percent, 2 percent so that we can complete the analysis quicker and what if we do not apply any force. See the right hand side P i is 0.

$$\psi_1 = \frac{\sum dP_i^2}{\underset{i=1,n}{\sum P_i^2}} \times 100\%$$

$$\psi_2 = \frac{\sum d\,u_i^2}{\underset{i=1,n}{\sum u_i^2}} \times 100\%$$

When you do not apply say if you have a displacement control analysis. In that case we just take it as some value 1 so that your denominator is not 0 then the other num that we monitor is the norm of incremental displacements this is the square of incremental displacements the sum total divided by the sum total of the squares of all the displacements and once again this should be about 0.1 to 0.5 percent.

Solution is iterated until the norm of incremental displacements and the norm of out of balance forces reduces below certain limits, e.g. 0.1% to 0.5%

Then there is another force norm that is the incremental work done that is the d P multiplied by d u we can monitor that also and I will show you that quantity later when we deal with finite element analysis.

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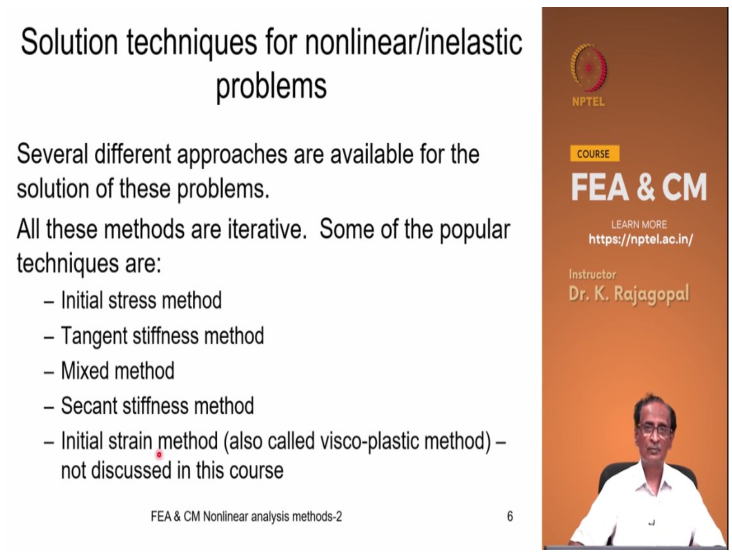

And there are different solution methods that we can adapt for different problems. The simplest one is the initial stress method then the tangent stiffness and then the mixed method the combination of initial stress and tangent stiffness and then secant stiffness method and there is another method called as the initial strain method that is more applicable for visco plastic or visco elastic type problems and I am not going to discuss this method in this course.

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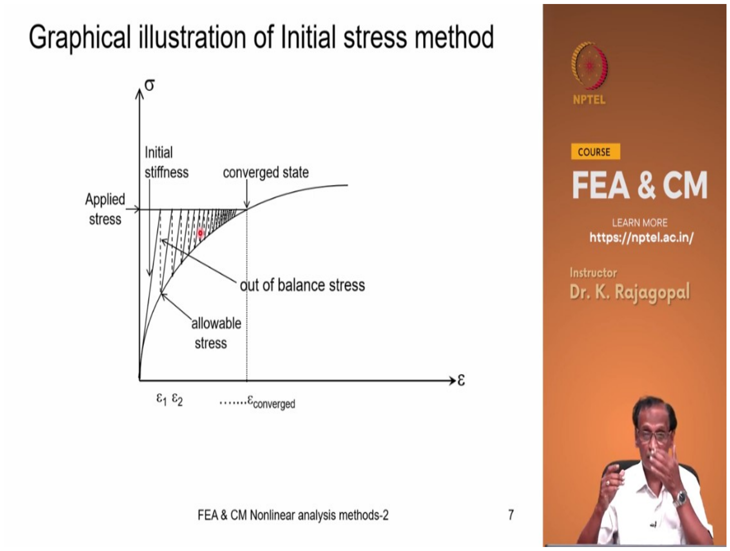

In the initial stress method our stiffness is kept constant and then we apply some stress and then at each stage we see what is the out of balance force or what is the difference between the stress applied and then the actual stress that the material can support corresponding to that particular strain level and then we apply this back on the system and then get your incremental displacements, incremental strains and other things.

And then we repeat this process until your out of balance force is negligible. In the process we are not going to change the stiffness matrix it remains constant that is why it is called as the initial stiffness or initial stress method and the slopes are all constant and it goes very fast because we are not going to reformulate our stiffness matrix we formulate only once. Then subsequently we do only the back substitution for determining the displacement increments.

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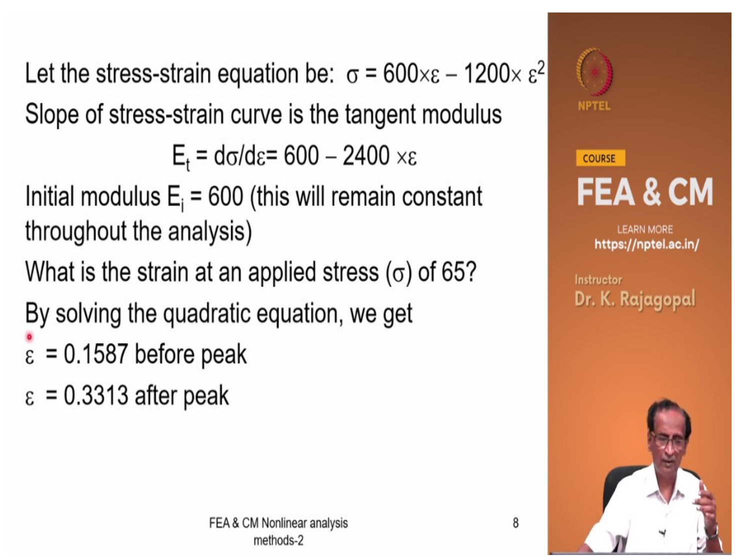

And yesterday I have shown you one small example of this non-linear equation sigma is 600 epsilon - 1200 epsilon square and in this case our tangent modulus E t is d sigma by d epsilon and this is actually it is a one dimensional problem just for hand calculation so that we can illustrate the procedures more easily. The tangent modulus is a d sigma by d epsilon that is 600 - 2400 epsilon and the initial modulus is 600 that is when epsilon is 0.

Let the stress-strain equation be: $\sigma = 600\times\varepsilon - 1200\times \varepsilon^2$

Slope of stress-strain curve is the tangent modulus

$$E_t = d\sigma/d\varepsilon = 600 - 2400 \times\varepsilon$$

Initial modulus E_i = 600 (this will remain constant throughout the analysis)

What is the strain at an applied stress (σ) of 65?

By solving the quadratic equation, we get

ε = 0.1587 before peak

ε = 0.3313 after peak

And then let us say that we are interested in finding what is this the strain corresponding to an applied stress of 65 and by solving this quadratic equation we get two values for a strain that is 0.1587 before the peak that is during the strain hardening path then beyond the peak 0.33, but in this course we are going to do deal with only the strain hardening path before the peak and beyond the peak also like that is corresponding to the strain softening path and that will come later.

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Numerical example for initial stress method

Let the stress-strain equation be: $\sigma = 600\times\varepsilon - 1200\times \varepsilon^2$

Slope of stress-strain curve is the tangent modulus

$$E_t = d\sigma/d\varepsilon = 600 - 2400 \times\varepsilon$$

Initial modulus E_i = 600 (this will remain constant throughout the analysis)

What is the strain at an applied stress (σ) of 65?

This can be determined step by step using the procedure:

Sl. No.	Strain increment = $d\sigma/E_i$	Total strain, ε	Stress as per equation	Out of balance stress, $d\sigma$
1	65/600=0.1083	0.1083	50.803	65–50.80=14.0833
2	14.083/600= 0.02347	0.1318	58.2361	65–58.2361=6.7639
3	6.7186/600= 0.01127	0.1430	61.2814	3.7186
4	2.174/600= 0.006198	0.1493	62.8257	2.1743
5	1.314/600= 0.003624	0.1550	63.686	1.3140
6	0.809/600=0.00219	0.1573	64.1906	0.8094

The above process is continued until a solution with desired accuracy is reached

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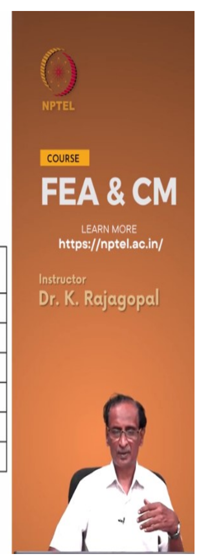

And this is the procedure for initial stress method this is the step-by-step method. So, our applied, stress is 65 and our initial modulus is 600 that is going to remain constant. So, we apply at the 65 our initial guess on the strain is a 0.1083 and the corresponding stress at strain of 1.1083 is a 50.803 and then the difference between the applied stress and then the stress corresponding to this strain of 0.1083 is 14.

Numerical example for initial stress method

Let the stress-strain equation be: $\sigma = 600\times\varepsilon - 1200\times \varepsilon^2$

Slope of stress-strain curve is the tangent modulus

$$E_t = d\sigma/d\varepsilon = 600 - 2400 \times\varepsilon$$

Initial modulus E_i = 600 (this will remain constant throughout the analysis)

What is the strain at an applied stress (σ) of 65?

We apply this back on the system and then get an incremental strain and our total strain and then our stress corresponding to this strain and then we calculate once again the out of balance force and then repeat the process until we are happy with the solution that we get like we compare our out of balance force to the applied stress and then within about 1 percent or 0.1 percent we can go on iterating.

Sl. No.	Strain increment = $d\sigma/E_i$	Total strain, ε	Stress as per equation	Out of balance stress, $d\sigma$
1	65/600=0.1083	0.1083	50.803	65–50.80=14.0833
2	14.083/600= 0.02347	0.1318	58.2361	65–58.2361=6.7639
3	6.7186/600= 0.01127	0.1430	61.2814	3.7186
4	2.174/600= 0.006198	0.1493	62.8257	2.1743
5	1.314/600= 0.003624	0.1550	63.686	1.3140
6	0.809/600=0.00219	0.1573	64.1906	0.8094

And then once you are happy with the with the out of balance force that you have you can stop and in this case within about 6 iterations our total strain obtained is 0.1573 which is very close to this value of 0.1587. Well actually it is a very simple problem and also the non-linearity is not very high at 65 when we go to the excel spreadsheet I will illustrate this a bit more.

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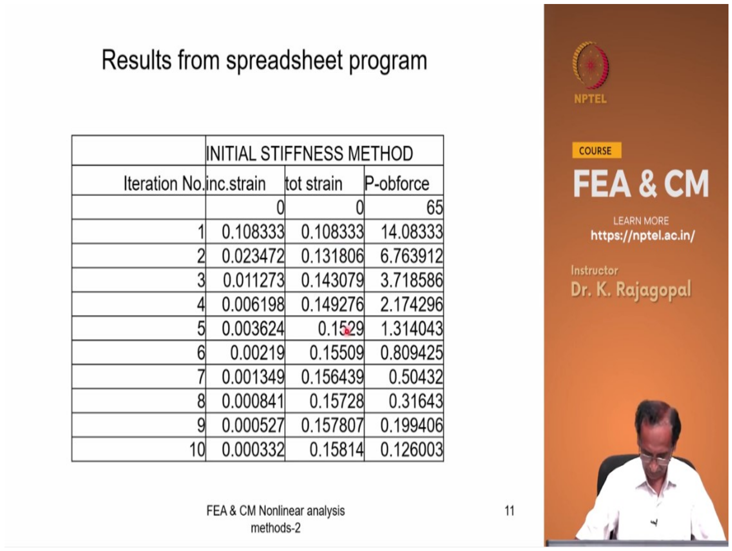

Results from spreadsheet program

	INITIAL STIFFNESS METHOD		
Iteration No.	inc.strain	tot strain	P-obforce
	0	0	65
1	0.108333	0.108333	14.08333
2	0.023472	0.131806	6.763912
3	0.011273	0.143079	3.718586
4	0.006198	0.149276	2.174296
5	0.003624	0.1529	1.314043
6	0.00219	0.15509	0.809425
7	0.001349	0.156439	0.50432
8	0.000841	0.15728	0.31643
9	0.000527	0.157807	0.199406
10	0.000332	0.15814	0.126003

And this is the result from our spreadsheet program I will explain it when we go to the spreadsheet.

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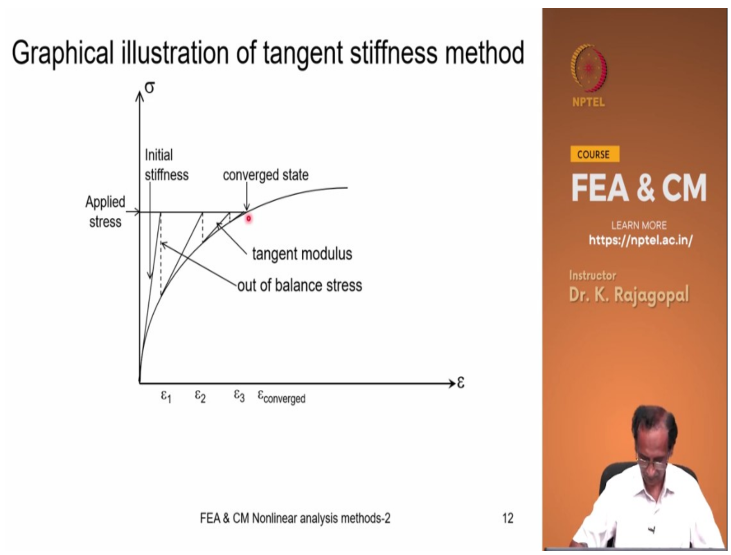

And then the other one is the tangent stiffness method where at every stage we calculate the tangent stiffness and the estimate our new strain increments. So, the first time we are going to apply with initial modulus of 600 for this particular case then we get this out of balance force and then calculate our strain increment in turn as the ratio between the out of balance stress divided by the tangent modulus at this strain level.

And then we repeat this and as we proceed your tangent modulus is going to reduce and your strain increment is going to be very large compared to what we had earlier then we can reach the convergence very fast in this tangent stiffness method because we are following the curve as much as possible.

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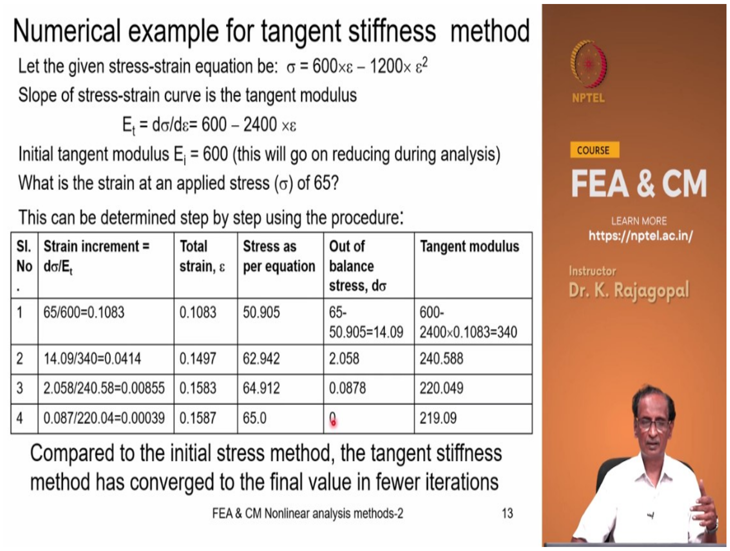
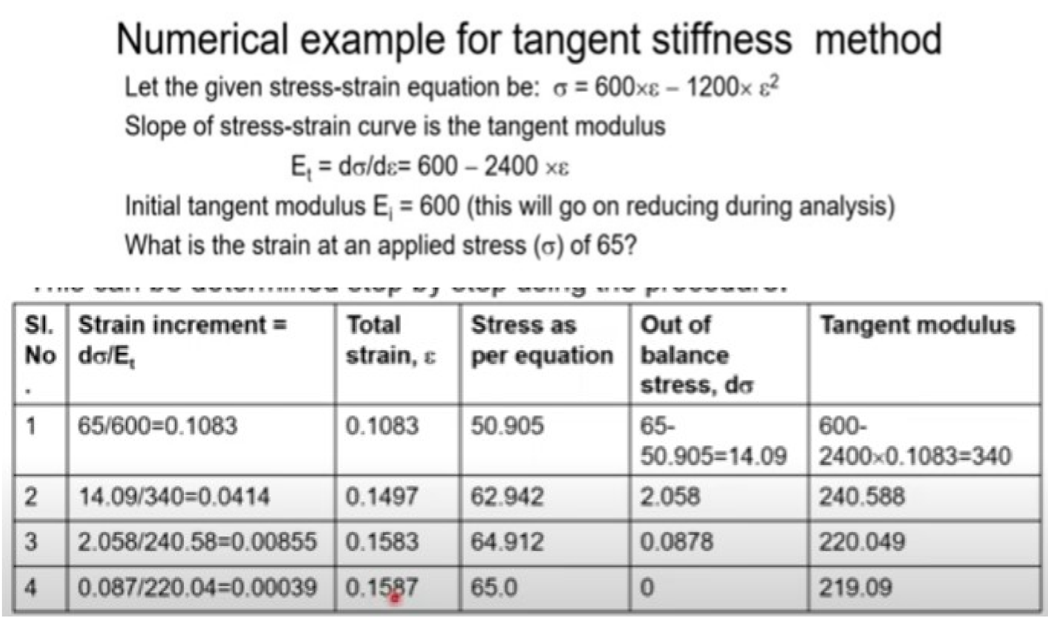


See within four steps for the same problem our out of balance stress is nearly zero whereas with the initial stress method even after 6 iterations there were some out of balance stress and our total strain calculated one is 0.1587 which is exactly equal to the to the solution of that equation and the problem with the tangent stiffness method is the slope will go on reducing as you are nearing the peak and at the peak your slope will be zero.



Sl. No.	Strain increment = dσ/E_t	Total strain, ε	Stress as per equation	Out of balance stress, dσ	Tangent modulus
1	65/600=0.1083	0.1083	50.905	65-50.905=14.09	600-2400×0.1083=340
2	14.09/340=0.0414	0.1497	62.942	2.058	240.588
3	2.058/240.58=0.00855	0.1583	64.912	0.0878	220.049
4	0.087/220.04=0.00039	0.1587	65.0	0	219.09

And then your modulus becomes negligible and then you might start having numerical issues and especially when you go into the strain softening part your modulus becomes negative and if too many elements have a negative modulus then you cannot work with that stiffness matrix and you will get numerical problems or the program will just simply stop saying that one of the diagonal elements is negative.

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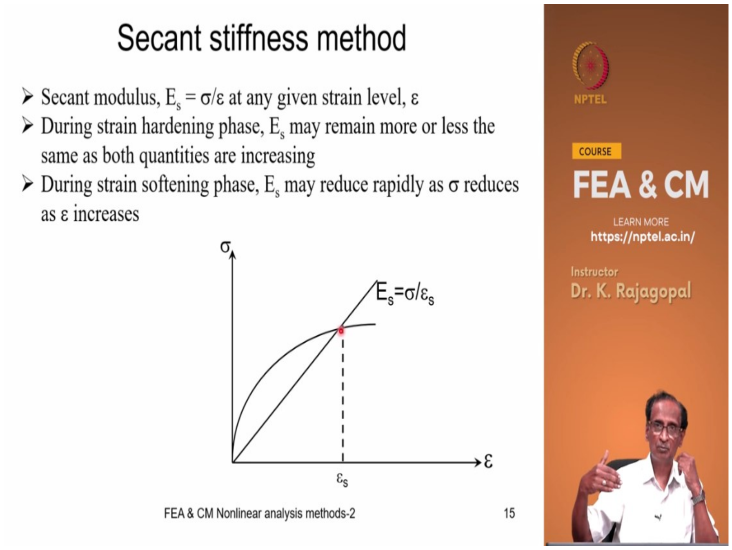

In that case we can go in for the secant stiffness method where the secant stiffness is the stress divided by the corresponding strain and in this case our secant stiffness will always be positive and it goes on reducing because as your strain is increasing stress will reach some peak value and the rate of increase of stress may be lower compared to the rate of increase of the strain and as you are reaching the peak your secant modulus will reduce.

And especially in the strain softening path your stress is decreasing while your strain is increasing. So, in that case our tangent modulus or the secant modulus will fall rapidly and in a way the secant modulus method is more stable for strain softening type problems.

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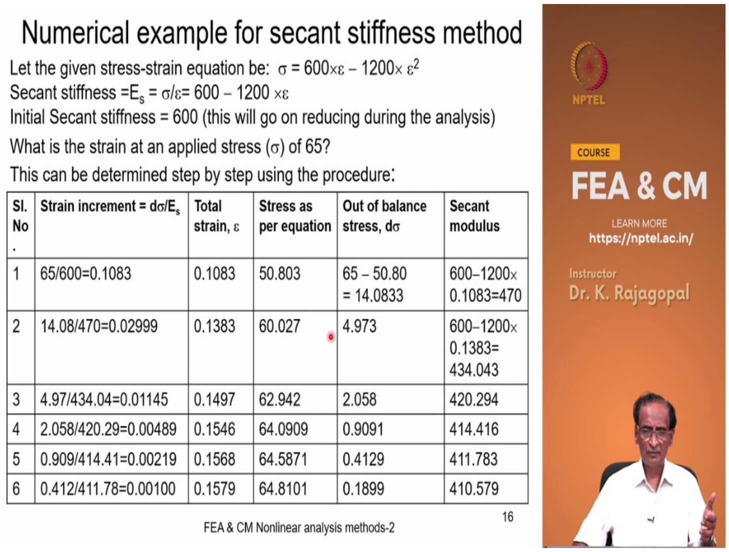

Numerical example for secant stiffness method

Let the given stress-strain equation be: $\sigma = 600\times\varepsilon - 1200\times \varepsilon^2$

Secant stiffness $=E_s = \sigma/\varepsilon = 600 - 1200 \times\varepsilon$

Initial Secant stiffness = 600 (this will go on reducing during the analysis)

What is the strain at an applied stress (σ) of 65?

Sl. No.	Strain increment = $d\sigma/E_s$	Total strain, ε	Stress as per equation	Out of balance stress, $d\sigma$	Secant modulus
1	65/600=0.1083	0.1083	50.803	65 – 50.80 = 14.0833	600–1200× 0.1083=470
2	14.08/470=0.02999	0.1383	60.027	4.973	600–1200× 0.1383= 434.043
3	4.97/434.04=0.01145	0.1497	62.942	2.058	420.294
4	2.058/420.29=0.00489	0.1546	64.0909	0.9091	414.416
5	0.909/414.41=0.00219	0.1568	64.5871	0.4129	411.783
6	0.412/411.78=0.00100	0.1579	64.8101	0.1899	410.579

And in this case the secant stiffness is a sigma by epsilon and that is 600 - 1200 epsilon and then these are the results with secant stiffness method after 6 iterations our total strain is still 0.1579 whereas even with the initial stress method we got about 0.1583 and our out of balance stress was slightly smaller in that case.

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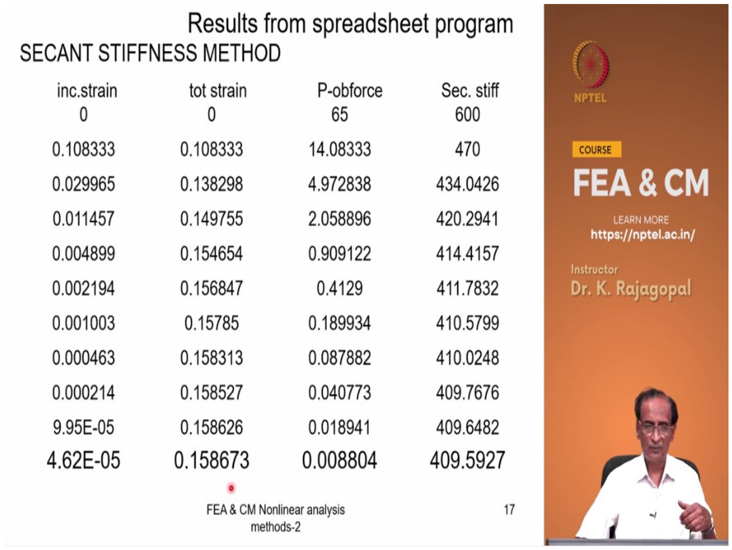
Results from spreadsheet program

SECANT STIFFNESS METHOD

inc.strain	tot strain	P-obforce	Sec. stiff
0	0	65	600
0.108333	0.108333	14.08333	470
0.029965	0.138298	4.972838	434.0426
0.011457	0.149755	2.058896	420.2941
0.004899	0.154654	0.909122	414.4157
0.002194	0.156847	0.4129	411.7832
0.001003	0.15785	0.189934	410.5799
0.000463	0.158313	0.087882	410.0248
0.000214	0.158527	0.040773	409.7676
9.95E-05	0.158626	0.018941	409.6482
4.62E-05	0.158673	0.008804	409.5927

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These are the results from the excel spreadsheet program. **(Video Starts: 20:01).** And now let me illustrate this you will find this excel spreadsheet program in your course material and it is called as initial tangent and the secant stiffness and this particular equation that we are solving is A epsilon - B epsilon square and A is 600 B is 1200 the way we have seen and the maximum stress that you can apply for these properties is 75.

And that depends on this A and B values like let us say if your B is 1400 your maximum stress that you can apply is a 64 and so I am setting it back to 600 and 1200 and then the maximum stress that you can apply 75 and the strains in the hardening part is 0.1587 and the strain during the softening part is 0.34 and we are interested in determining the strain corresponding to an applied stress of 65.

And for now we will only be looking at the strain hardening part we are not going to go into the strain softening part. So, initially our applied stress is 65 and so we can call that as in the out of balance force and the strain increment is 65 divided by B4 I hope you know the excel spreadsheet programming if you indicate anything with a dollar sign that will refer to one particular cell or one particular column dollar B dollar 4 means column B and row 4.

So, that is fixed and then if I say D 11 it is initially it is D 11 and then next one it will be D 12 and so on. So, here this 0.1083 is the out of balance stress in the previous step divided by 600. So, that B4 is our 600 and then our total strain at this stage is 0.1083 and our out of balance stress is the applied stress of 65 - the reaction that we get from our stress equation A epsilon - B epsilon square.

And then we take this new out of balance stress of 14 that divided by 600 is your strain increment this is our total strain and then now the out of balance stress is a 65 - this stress corresponding to the strain of 0.1318 and then we calculate the new strain increment and then the total strain and then out of balance stress and so on like we go on continuing until we are happy with the solution that we get.

So, here I think in the class example we stopped here and that is after 1, 2, 3, 4, 5, 6 iterations our out of balance stress is 0.809 and the total strain is a 0.155 and we can continue for as long as we want and is actually it is a very simple procedure and let us apply actually let me show you the stress strain curve. See here this is the stress strain curve that we have and we are trying to solve for the strain at a stress of 65.

And then if our applied stress is very close to the peak then we need more number of iterations because you see here there is lot of non-linearity and if it is only 20 we are still within maybe very close to the linear limit so we may not need many iterations. So, let me just do this say if I apply a stress of 20 our strain is 0.0359 and within four steps we get that strain even with the initial stiffness method because the stress is very small.

And we are within the elastic limit and let us see I increase this to 74 very close to the ultimate stress. In this case we will our strain is 0.221 and we see here we need a very large number of iterations like after about maybe 30 iterations we get about 0.22 and the out of balance stress is 0.064 and if you apply a stress of 75 then we will have a problem. In fact the program will get confused because this is the peak point.

So, there is no difference between strain hardening and the strain softening and the program gives the same stress value because it cannot distinguish because that is the peak point 0.25 and if you see here it is going on and on and on and 0.25 we are not able to reach. We are reaching that after almost maybe more than 200 it is actually even after 400 cycles our total strain is a 0.2488.

So, depending on the problem that we have your number of iterations might change. Let me just set it back to 65 and within six cycles our total strain with the initial stiffness method or initial stress method is a 0.155 against the calculated strain of 0.1587 and now let us look at the tangent stiffness method where we update our stiffness matrix or our stiffness we are doing the same thing in the first iteration.

Our out of balance stress is 14 and then we calculate the stiffness at this strain of 0.108 and if you see here the tangent stiffness is A – 2 B epsilon. So, the tangent stiffness becomes 340 and our incremental strain is the out of balance stress divided by 340 instead of 600. So, our total strain has increased to 0.149 and then our out of balance stress is reduced and our tangent stiffness has reduced.

And within about 5 cycles 1, 2, 3, 4, 5 cycles our incremental strain is negligible and out of balance force is also negligible and whereas after five cycles 1, 2, 3, 4, 5 cycles the total strain with the initial stress method was 0.1529 whereas here we are able to get our value exact value of 0.1587 and let me increase this stress 74 and our strain is 0.221 and even with the tangent stiffness method within about five or six steps we are able to get to that strain level.

Whereas with the initial stress method it is taking very large number of iterations 221 we are able to get at about after about 50 cycles and let me just get the get back to 65 and let us see what happens with the secant stiffness method. Actually the illustration is also shown here and with the secant stiffness method our secant stiffness is sigma by epsilon that is A - B epsilon.

And our initial out of balance stress is 65 divided by our initial modulus of 600 is 0. 108 and our out of balance stress is 14 and our secant stiffness at this strain level of 0.1083 is this A - B epsilon that is 470 and this is our incremental strain and our total strain and then the out of balance stress that is the applied stress of 65 minus the reaction stress and then the secant stiffness.

So, you see here as this strain is increasing your secant stiffness is reducing and the same thing with the tangent stiffness and our exact result is 0.1587 and in this case after about maybe 20 iterations we are able to get that exact value where our out of balance force is negligible and this method is a quite easy to implement the secant stiffness and because our stiffness values are always positive definite we will not have any problem.

Let me just apply stress of 75 and see what happens our strain corresponding to a stress of 75 is 0.25 and our tangent stiffness method is able to predict the stress after a large number of iterations, but then this being a small problem it is not an issue because your tangent stiffness is very low 0.009, but if it were a large finite element program these small values will compound with each other.

And you might start forming a mechanism with very, very large displacements, but this being a small problem we are able to converge without any numerical issues. So, here after about 20 cycles 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 iterations we are able to get to the exact value with the tangent stiffness method and with the secant stiffness we see that even after about 35 cycles it does not reach the 0.25.

And with the initial stiffness also it takes a very large number of iterations to get that 0.25. In fact that we have seen earlier we did not reach. So, depending on the situation in some problems we might use initial stress method and the tangent stiffness we should be very careful because in most cases the solutions will simply diverge because our denominator becomes smaller and smaller with the each load increment.

And then at some point the denominator might become so small that your displacements could be very, very large and both secant stiffness and initial stress methods are very stable and the initial stress method is very fast because we are not reformulating the stiffness matrix and the tangent stiffness method is very slow because at each and every step we are updating our stiffness matrix.

And we can combine the initial stiffness and tangent stiffness and maybe we can update our stiffness matrix every say in 10th iteration or something like that and in between we can use the initial stiffness method so that we can economize on the number of computations. So, this is brief introduction to the different analysis methods just to summarize let me show you with this figure we have the initial stress method where our stiffness matrix remains constant.

And then the tangent stiffness method where we follow the stress strain curve as closely as possible by taking a tangent modulus as d sigma by d epsilon then the other one is the secant stiffness where there were modulus is stress divided by strain and that is also very stable because our stiffness values are always positive definite and we can get a reasonably good solutions.

But then the secant stiffness method also takes up a lot of time because every time we have to update our stiffness matrix and do the calculations. So, that is the brief introduction to different non-linear analysis methods. **(Video Ends: 36:25)**. And we will see more of this later on when we are doing the finite element analysis. So, thank you very much we will meet in the next class.