Finite Element Analysis and Constitutive Modelling in Geomechanics
Prof: K. Rajagopal
Department of Civil Engineering
Indian Institute of Technology – Madras

Lecture - 28
Nonlinear Analysis Technique-2

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Let us continue our non-linear analysis techniques that we started in the previous class and we looked at the linear and non-linear problems in the previous class and now let us look at one more class of problems that is in terms of the materials having some limited strength.

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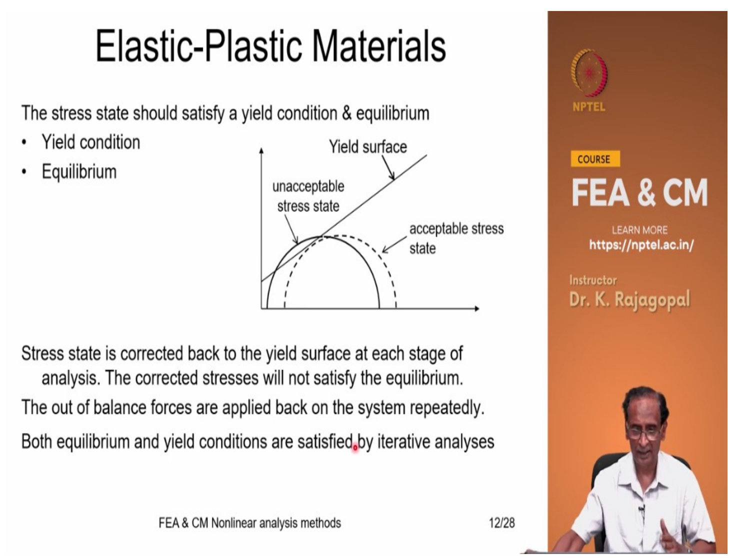
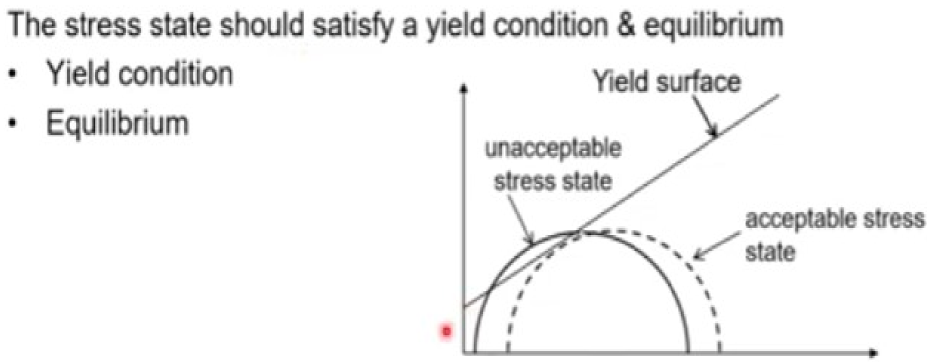


Let us say we have materials with certain limit and like our elastic plastic materials and we already know our more column yield surface in terms of c and phi we can define this yield surface that limits the strength and if any more circle falls within this surface it is it is an acceptable stress state and if it crosses then that means that your stress is more than the strength of the material.



And we have to do something to remove those stresses so that the stress becomes acceptable and so we need to correct the stresses back to the yield surface and so when you correct the stresses back to the yield surface your equilibrium is going to get disturbed. So, in this class of problems we need to satisfy not only the equilibrium, but also certain given yield condition.

The yield condition simplest one is our more coulomb yield condition with this yield surface in terms of the c and phi and actually it is not easy satisfying both the yield that is based on the strength of the material and also the equilibrium. So, we need some type of an iterative analysis so that we can satisfy both.

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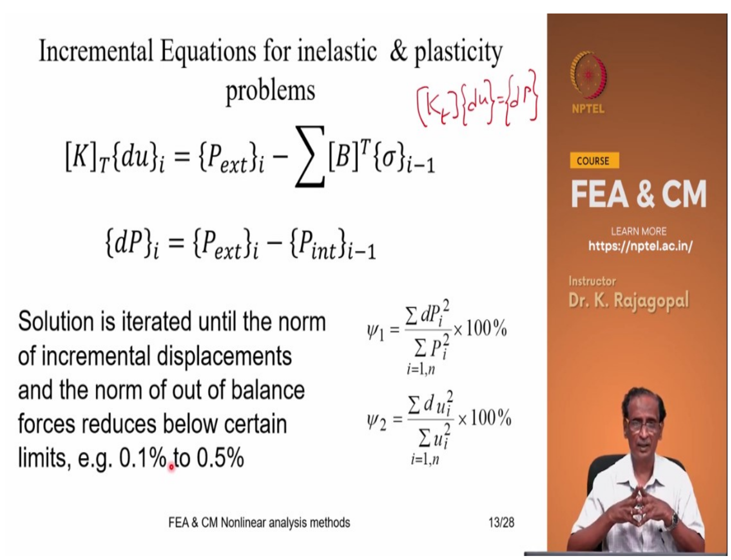

And so we rewrite our equilibrium equations in terms of incremental form, but then our incremental form is written in a slightly different manner K T d u is the applied force P external minus B transpose sigma.

Incremental Equations for inelastic & plasticity problems

$$[K]_T\{du\}_i = \{P_{ext}\}_i - \sum [B]^T\{\sigma\}_{i-1}$$

$$\{dP\}_i = \{P_{ext}\}_i - \{P_{int}\}_{i-1}$$

Previously we had written the equations like this, simply written the incremental stiffness matrix multiplied by d u is d P and that d P is the only thing the applied pressure.

$$[K_t]\{du\}=\{dP\}$$

And we did not bother about the calculating the reaction forces because anywhere they will be equal to the applied forces, but now because of this yield condition the calculated stresses may not satisfy the equilibrium. So, we need to do or form the equations in a slightly different form and we do calculate the reaction forces as B transpose sigma and this summation is for all the elements in the mesh.

Whether it is continuum or a joint element or a bar element or anything we calculate the reaction force and then sum them up and the P external is the applied load that is applied on the system and then this is the reaction force and the difference is our incremental load. So, our d P that we defined earlier is now the difference between the external applied load and then the internal reaction P external - P internal corresponding to the i – 1.

So, the i is the current load step or the current iteration i - 1 refers to the previous load step or the previous iteration based on the stresses that we calculated until the previous analysis or the previous step or the previous iteration. We will give us this reaction for P internal and we need to repeat the solution several times until our P external and P internal are nearly the same and we monitor that in several ways.

One is by monitoring the out of balance force or the d P we can call it as the out of balance force that is the difference between the external force and then the internal force and we are not going to do this at each and every degree of freedom. We can do it in a global sense by calculating some norm this could be; it is a ratio between the squares of all the unbalanced forces divided by the squares of all the applied forces.

And this is the numerator we have sigma for dPi square divided by the applied force square. We take a square so that we have everything as positive like because our forces could be negative forces. The reaction also could be negative, but if you take everything as positive we will end up with only a positive convergence number psi 1 this multiplied by 100 percent and acceptable convergence norm is this the norm of the out of balance force should be about 0.1 to 0.5 percent.

$$\psi_1 = \frac{\sum dP_i^2}{\sum\limits_{i=1,n} P_i^2} \times 100\%$$

$$\psi_2 = \frac{\sum d\,u_i^2}{\sum\limits_{i=1,n} u_i^2} \times 100\%$$

And in some highly non-linear problems we may not be able to reach this level of convergence we might increase it to 1 percent tolerance or something then the other way of monitoring the solution is through the incremental displacements. So, every time you iterate because of your d P you will have some incremental displacements. So, if d P is very, very small your incremental displacements will be very small.

And so our psi 2 is the norm of the incremental displacements is the sum total of the ratio between the sum total of the squares of the incremental displacements divided by the sum total of the squares of the total displacements and this multiplied by 100 percent and once again we aim for about 0.1 to 0.5 percent so that our, solution is as accurate as possible and how do we achieve this type of equilibrium.

And actually it is very, very expensive process because we need to repeat the analysis several times and we may have to also iterate several times to maintain the equilibrium.

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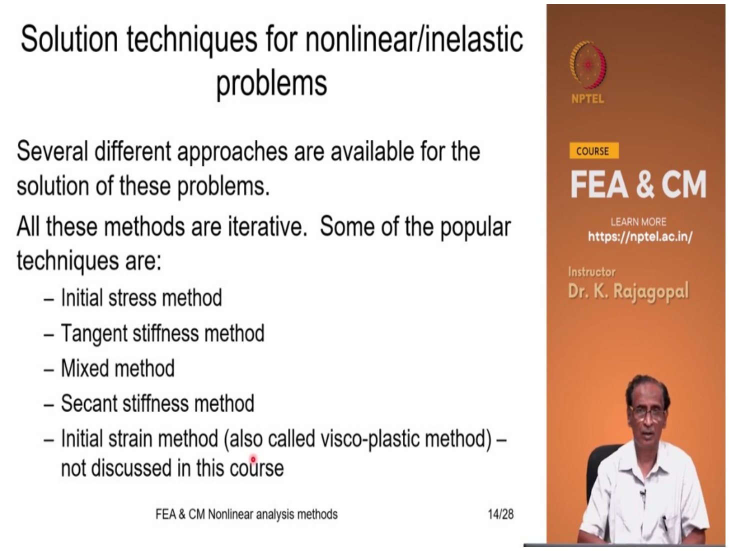

And there are different methods called as initial stress method, tangent stiffness method or mixed method a combination of initial and tangent and then a secant stiffness method then there is another method called as the initial strain method also called as the visco plastic method, but this is not discussed in this course. Actually in the initial strain method what we do is we temporarily allow the stresses to exceed our yield limit.

And then gradually dissipate them using some visco plastic algorithm and then that excess stress is leading to this to the creep strains. Actually in the case of elastic plastic we deal with the plastic strains whereas in the case of visco elastic or visco plastic we have the time dependent strains like the creep strains and this initial strain method is not discussed in this course because we are not dealing with any creep or viscous response.

I will explain these different methods the initial stress method tangent stiffness and then mixed method is of course a combination of these two and then the secant stiffness method.

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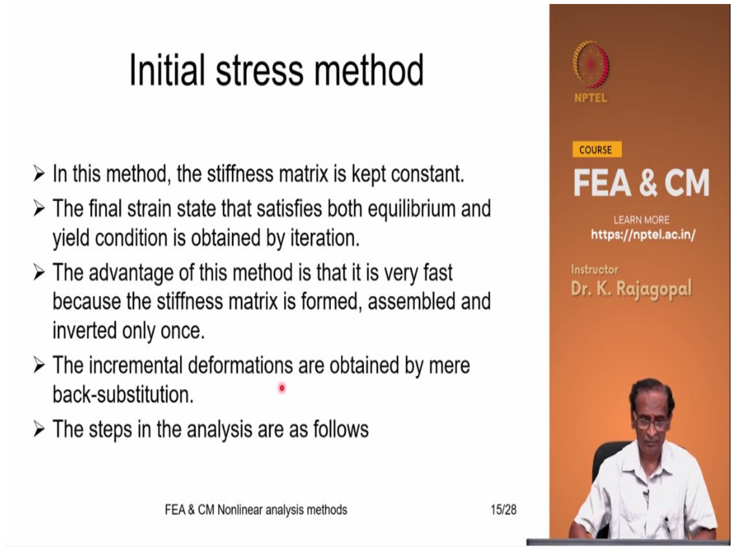

So, among these methods the initial stiffness method is the fastest, in this method the stiffness matrix is kept constant and the final strain state that satisfies both the equilibrium and the yield condition is obtained by iteration. We perform several iterations until we are able to satisfy the equilibrium and also the yield condition. So, the advantage of this method is very fast because we form the stiffness matrix only once and assembled and inverted only once.

So, the entire analysis the stiffness matrix calculations for all the elements and their assembly and then after we assemble the stiffness matrix we make it as an upper triangular matrix that is the inversion we do it only once and in the iterations we only do the back substitution which are very, very fast those calculations are very fast and as we iterate we get the incremental deformations just by mere back substitution and the steps are like this.

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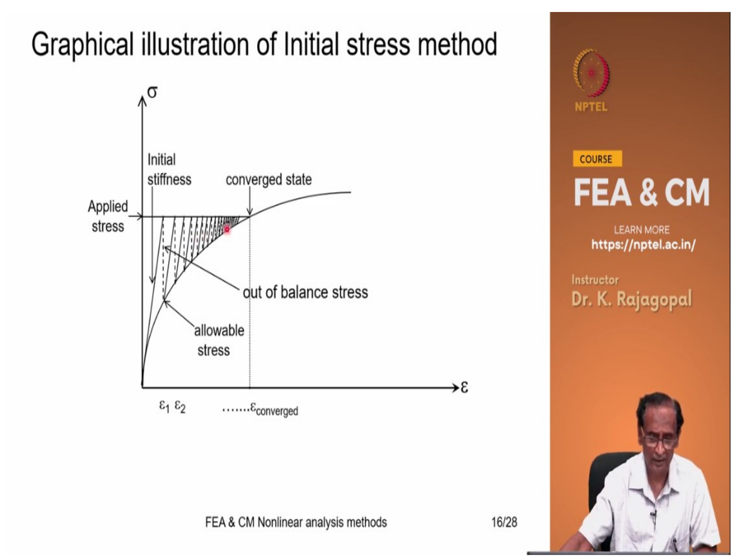

Let us say your stress-strain curve is something like this it is a highly non-linear stress-strain curve and then we have applied a stress of this much and this is the converged state where you are exactly matching with the stress-strain equation and this is your strain at this stress and at our disposal we have only the initial modulus, initial stiffness or initial modulus and the corresponding to that is our initial stiffness equation.

So, if you apply this stress and calculate the displacement or the strain we get an epsilon 1, but then corresponding to epsilon 1 the acceptable stress is only this much and this is the unacceptable stress. So, the material can only support this much stress at this strain level and so this is the out of balance stress and corresponding to this we need to calculate a reaction force.

And then repeat the analysis and if you apply the force corresponding to this unacceptable stress will get some incremental displacement epsilon 2 and then we continue and in all the iterations our stiffness matrix is going to remain the same. So, the slope of this line is remaining the same and as we proceed along your out of balance stress will go on decreasing and then at some stage we will have a converged state.

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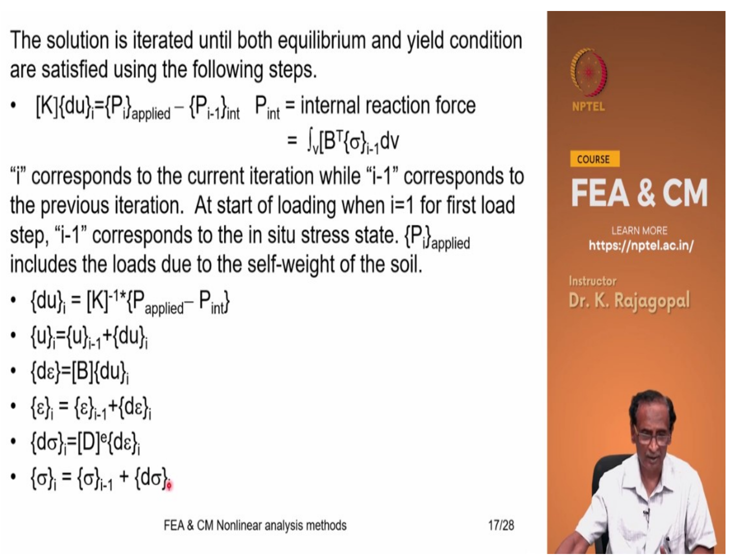

So, the analysis proceeds like this K multiplied by d u i is P i that is the applied - P i - 1 internal and P i is the applied force like, for example, our stress that we applied and the internal reaction that is because of the stresses that are allowable within the material and the integral B transpose sigma i - 1 is your P i - 1 and so as we repeat the analysis our out of balance stress will go on decreasing.

$$[K]\{du\}_i = \{P_i\}_{applied} - \{P_{i-1}\}_{int} \quad P_{int} = \text{internal reaction force}$$
$$= \int_v [B^T\{\sigma\}_{i-1} dv$$

And your incremental displacement will also go on decreasing or at some stage we might have convergence d u i is K inverse times P applied - P internal and u i is u i - 1 + d u and d epsilon is B times d u and epsilon i that is the total strain in the current iteration is the strain in the previous iteration epsilon i - 1 + the incremental strain d epsilon and our total stress sorry the incremental stress d sigma is d elastic multiplied by d epsilon.

$$\{du\}_i = [K]^{-1} * \{P_{applied} - P_{int}\}$$
$$\{u\}_i = \{u\}_{i-1} + \{du\}_i$$
$$\{d\varepsilon\} = [B]\{du\}_i$$
$$\{\varepsilon\}_i = \{\varepsilon\}_{i-1} + \{d\varepsilon\}_i$$
$$\{d\sigma\}_i = [D]^e\{d\varepsilon\}_i$$
$$\{\sigma\}_i = \{\sigma\}_{i-1} + \{d\sigma\}_i$$

And our sigma i that is the stress at the end of this iteration is the stress in the previous iteration d sigma i - 1 plus the stress increment that is produced in this increment d sigma i.

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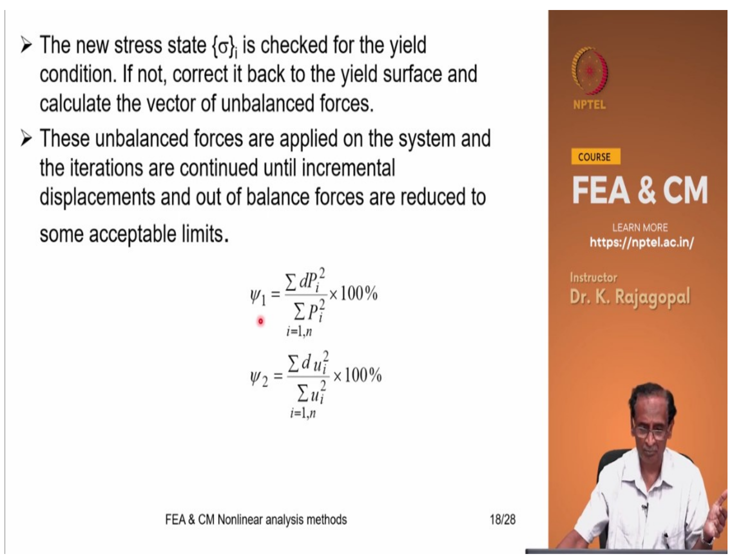

And then once you get this new stress state sigma i we check for the yield condition and if it is not satisfied then we correct it back to the yield surface and then calculate the vector of unbalanced forces and then repeat these calculations until our norm of out of balance forces then the norm of incremental displacements they reach some constant value.

$$\psi_1 = \frac{\sum dP_i^2}{\sum P_i^2} \times 100\% \quad i=1,n$$

$$\psi_2 = \frac{\sum d\,u_i^2}{\sum u_i^2} \times 100\% \quad i=1,n$$

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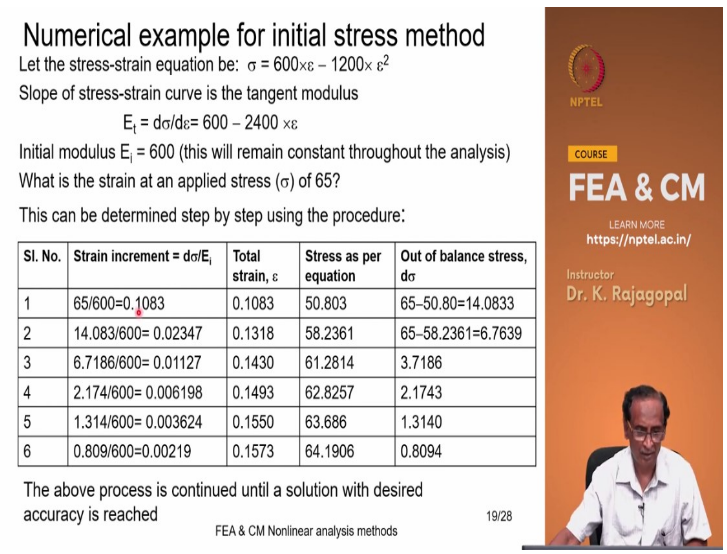

So, now let us apply this to a simple equation because I am going to demonstrate all these through simple hand calculations because if you do through finite element programs we will not really know what is happening and let us take a stress-strain equation like this sigma is 600 epsilon - 1200 epsilon square and this is our constitutive equation and the slope of the stress-strain curve is E tangent d sigma by d epsilon that is 600 - 2400 epsilon.

$$\sigma = 600\times\varepsilon - 1200\times \varepsilon^2$$

$$E_t = d\sigma/d\varepsilon = 600 - 2400 \times\varepsilon$$

E_i = 600 (this will remain constant throughout the analysis)

And the initial modulus E i is 600 because if you substitute epsilon of 0, E tangent is 600. So, E i is 600 that is going to remain constant throughout the analysis and now we are interested in knowing the strain corresponding to a stress of 65 and what we have is we know that the initial modulus is 600 and your stress-strain equation is sigma is a 600 times epsilon - 1200 epsilon square and then the tangent modulus is a 600 - 2400 epsilon.

And let us say we apply this the first time and the 65 by 600 and we get a strain increment of 0.1083 and our strain is 0.1083. So, if you substitute this back in our stress-strain equation we get a stress of 50.803, but then we have applied a stress of 65 so the out of balance stress is 65 - 50.803 that is 14 and you apply this back on the system so 14 divided by E i that is 600 that is 0.02347.

Sl. No.	Strain increment = $d\sigma/E_i$	Total strain, ε	Stress as per equation	Out of balance stress, $d\sigma$
1	65/600=0.1083	0.1083	50.803	65–50.80=14.0833
2	14.083/600= 0.02347	0.1318	58.2361	65–58.2361=6.7639
3	6.7186/600= 0.01127	0.1430	61.2814	3.7186
4	2.174/600= 0.006198	0.1493	62.8257	2.1743
5	1.314/600= 0.003624	0.1550	63.686	1.3140
6	0.809/600=0.00219	0.1573	64.1906	0.8094

And the total strain is 0.1083 + this value 0.13 and the stress as per whether equation is 58 and the out of balance stress is 6.76 and so you go on repeating this process and each time we are maintaining the same Young's modulus of 600 then we are calculating incremental strains and then the total strains and then the stress as per our equation that is given and you see as the number of iterations are increasing your out of balance stress is decreasing.

And your total strain is also converging and we continue until we read some convergence like let us say we want the out of balance force to be within about 0.5 percent accuracy or 1 percent accuracy and so on. So, this procedure will take a very large number of iterations because we are maintaining a constant Young's modulus the Young's modulus is kept constant at 600.

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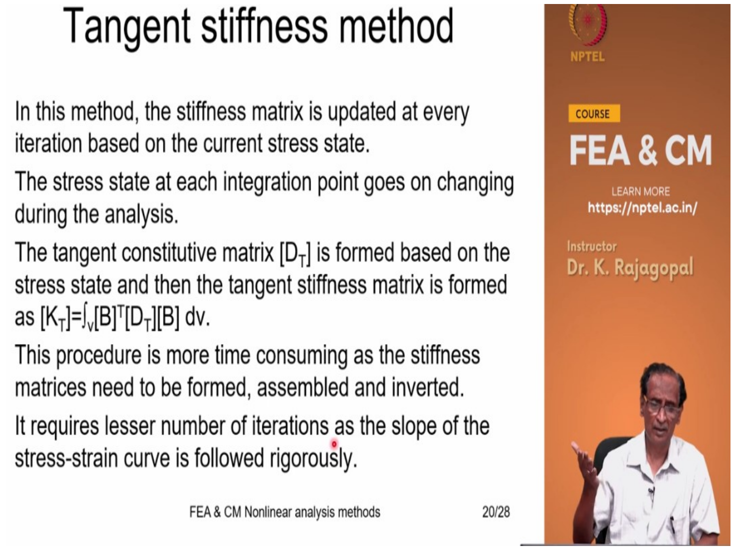

We can improvise this by using the tangent stiffness method. So, in this method the stiffness matrix is updated at every iteration based on the current stress state and the stress state at each integration point goes on changing during the analysis because as you apply the loading you will have some deformations and these are different at different elements. So, at each integration point we calculate the stresses.

$$[K_T]=\int_v [B]^T[D_T][B]\, dv.$$

And then form our constitute matrix D T and our tangent stiffness matrix K T integral B transpose D T B and so this tangent stiffness matrix we have to formulate at every iteration and then we have to make it as an upper diagonal matrix and then back substitute to get our displacements because the right hand side we have the out of balance force the P t minus R t or B transpose sigma and so this is actually it is a very highly involved process.

And we need lot of time and effort and this tangent stiffness method requires lesser number of iterations because at each step we are estimating the new slope of the stress-strain curve.

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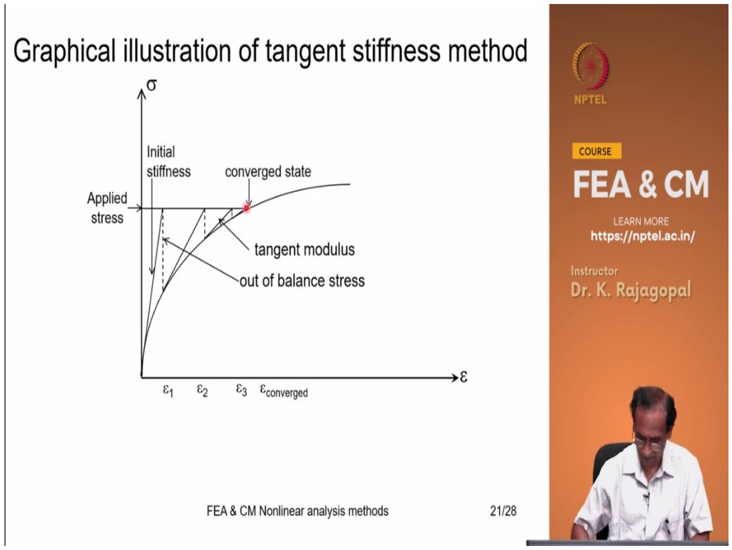
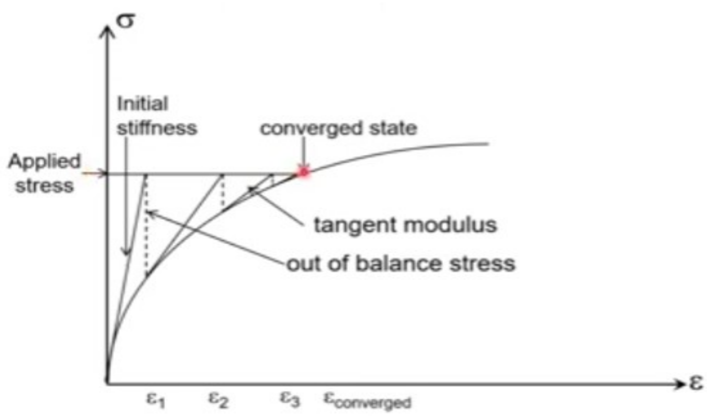


So, let me explain that. So, here we have this non-linear stress-strain equation or the stress-strain curve we applied some stress and we want to know what is the converged strain and so for that we initially apply with the initial modulus. We get some out of balance stress and then we update our stresses and then we get a tangent modulus like this and then once again the tangent modulus at this step this step and so on.



So, within a few steps we will see that we reached the converged state as compared to initial stress method.

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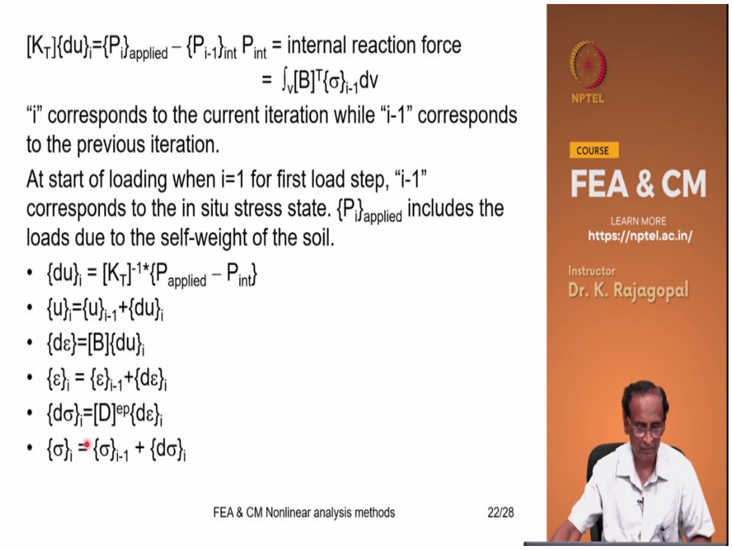
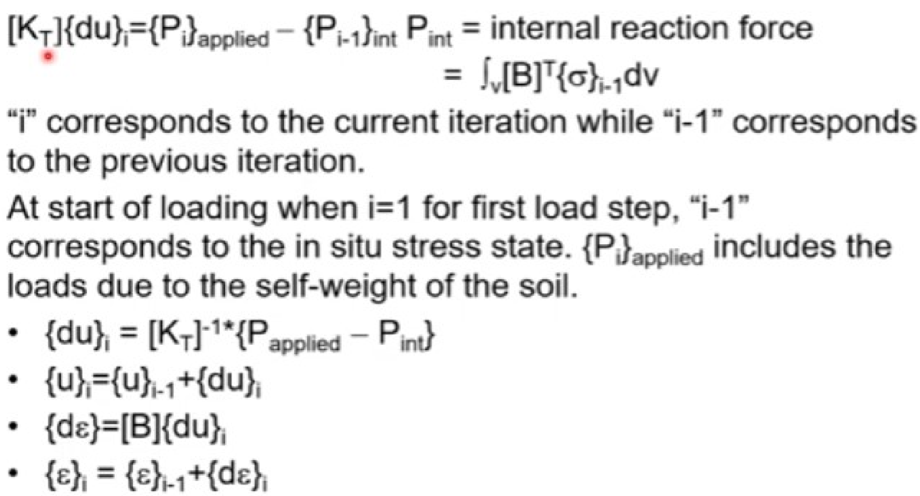


So, the only difference here is we are using this K T instead of K in the initial stress method that K was remaining constant whereas here we are updating it every iteration and every step.



So, our stresses are computed finally and then we check for whether this particular stress state is within the yield limit or not if not we do the corrections and then repeat the process.

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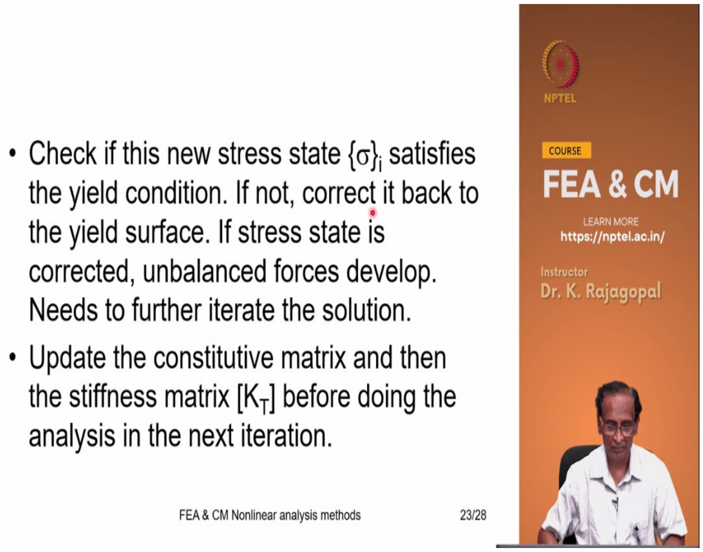

And then we check if the new stress state sigma i satisfy the yield condition if not correct it back to the yield surface and then we develop some unbalanced force and then we apply this back on the system and repeat the analysis.

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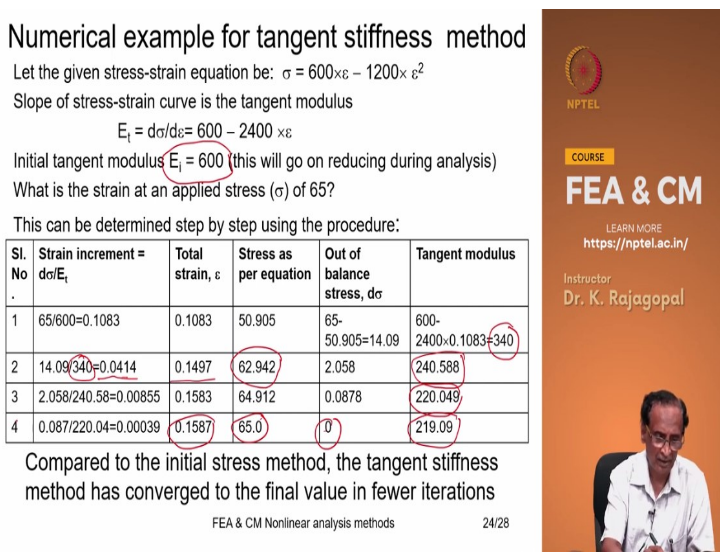

And so this is actually it is a slightly faster method or it takes lesser number of iterations compared to the initial stress method because we are updating the stiffness matrix and once again let us do one hand calculation to illustrate this and our E that we are calculating here is similar to our stiffness matrix. We say this is actually it is a single degree of freedom system. So, for the single degree of freedom system your modulus itself is your stiffness matrix because you can get your deformations.

Let the given stress-strain equation be: $\sigma = 600 \times \varepsilon - 1200 \times \varepsilon^2$

Slope of stress-strain curve is the tangent modulus

$$E_t = d\sigma/d\varepsilon = 600 - 2400 \times \varepsilon$$

Initial tangent modulus $E_i = 600$ (this will go on reducing during analysis)

What is the strain at an applied stress (σ) of 65?

So, our E tangent in this case is a d sigma by d epsilon that is at 600 - 2400 epsilon and then our initial modulus is a 600 and the applied stresses is 65. So, at the end of first titration 65 by 600 that is 0.1083 and our strain is 0.1083 and the stress is 50.905 and then the out of balance stresses are 65 - 50.905 that is 14.09 and now we calculate the tangent modulus in terms of this strain 0.1083.

So, the equation for tangent modulus is 600 - 2400 epsilon. So, our tangent modulus is now 340. So, to start with we started with an initial modulus of 600 then at the end of the first iteration it has come down to 340. So, now we calculate the strain increment in terms of this updated tangent modulus. So, 14 is the out of balance stress divided by 340 previously we had only 14 by 600. Now we have 14 by 340 that is 0.0414 the total strain is a 0.1497.

Sl. No.	Strain increment = $d\sigma/E_t$	Total strain, ε	Stress as per equation	Out of balance stress, $d\sigma$	Tangent modulus
1	65/600=0.1083	0.1083	50.905	65-50.905=14.09	600-2400×0.1083=340
2	14.09/340=0.0414	0.1497	62.942	2.058	240.588
3	2.058/240.58=0.00855	0.1583	64.912	0.0878	220.049
4	0.087/220.04=0.00039	0.1587	65.0	0	219.09

And then the stress is 62.94 and then the out of balance stress is only 2.05 and then we calculate our updated modulus. So, our updated modulus at the end of second iteration is a 240. Next it has come to 220 and then it has come to 219 you see that it is getting closer and closer. So, in the second step instead of using 600 we use only 340 for the modulus. So, our strain increment is this and our total strain is this.

And our stress is 62.94 and the out of balance stress is 2 and you see at the end of the fourth iteration we are able to reach a stress of 65 and then converged strain state of 0.1587 and once you see that there is no out of balance force we can just simply stop and this particular tangent stiffness method it is converged within four steps whereas let us look at the initial stress method.

So, in the initial stress method even after 6 iterations we still have some out of balance force 0.80 it has not come down to 0 whereas in the tangent stiffness method it has come down to an out of balance stress of 0 basic practically 0.

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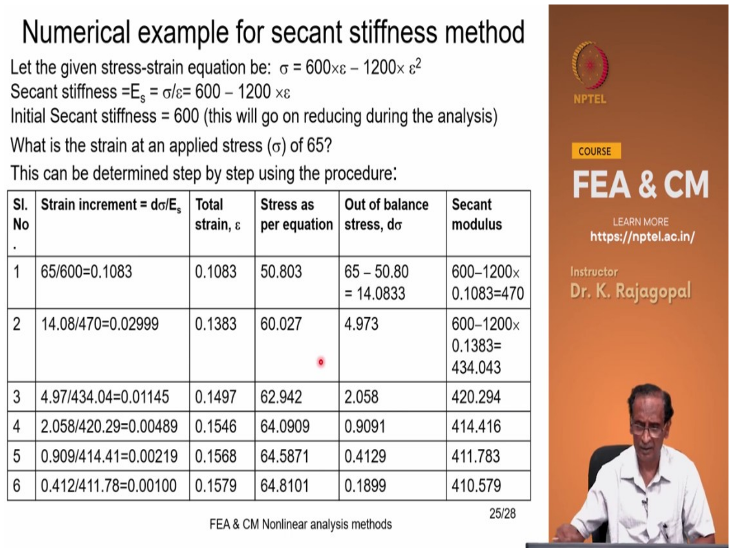
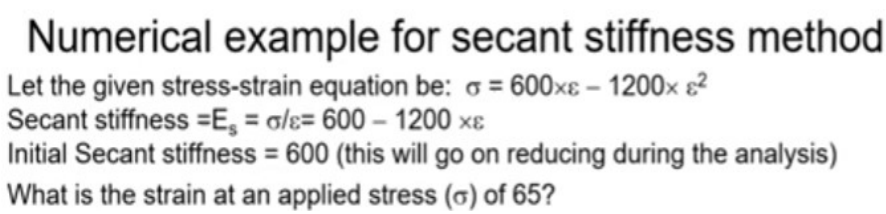


Sl. No.	Strain increment = $d\sigma/E_s$	Total strain, ε	Stress as per equation	Out of balance stress, $d\sigma$	Secant modulus
1	65/600=0.1083	0.1083	50.803	65 – 50.80 = 14.0833	600–1200× 0.1083=470
2	14.08/470=0.02999	0.1383	60.027	4.973	600–1200× 0.1383= 434.043
3	4.97/434.04=0.01145	0.1497	62.942	2.058	420.294
4	2.058/420.29=0.00489	0.1546	64.0909	0.9091	414.416
5	0.909/414.41=0.00219	0.1568	64.5871	0.4129	411.783
6	0.412/411.78=0.00100	0.1579	64.8101	0.1899	410.579

And there is another method called as the secant stiffness method wherein our modulus is the secant modulus is calculated as sigma by epsilon. So, our sigma equation for sigma is 600 epsilon - 1200 epsilon square. So, our secant modulus is sigma by epsilon so that comes to 600 - 1200 epsilon and once again the initial tangent modulus is 600. So, our initial strain increment is 0.1083.



And then the stress if you substitute this back in our equation for stress we get this and the out of balance stress is 14 and our secant modulus is 600 - 1200 times 0.1083 that is 470. So, in the next increment when you calculate the incremental strain the out of balance stress 14.08 divided by 470; 470 is secant modulus. So, 0.0299 total strain is 0.1383 and the stress corresponding to this strain is a 60.02.

Sl. No.	Strain increment = $d\sigma/E_s$	Total strain, ε	Stress as per equation	Out of balance stress, $d\sigma$	Secant modulus
1	65/600=0.1083	0.1083	50.803	65 – 50.80 = 14.0833	600–1200× 0.1083=470
2	14.08/470=0.02999	0.1383	60.027	4.973	600–1200× 0.1383= 434.043
3	4.97/434.04=0.01145	0.1497	62.942	2.058	420.294
4	2.058/420.29=0.00489	0.1546	64.0909	0.9091	414.416
5	0.909/414.41=0.00219	0.1568	64.5871	0.4129	411.783
6	0.412/411.78=0.00100	0.1579	64.8101	0.1899	410.579

And so on like we can continue until we get a good equilibrium and the advantage with the secant modulus procedure or the secant stiffness procedure is that our secant modulus will never become negative.

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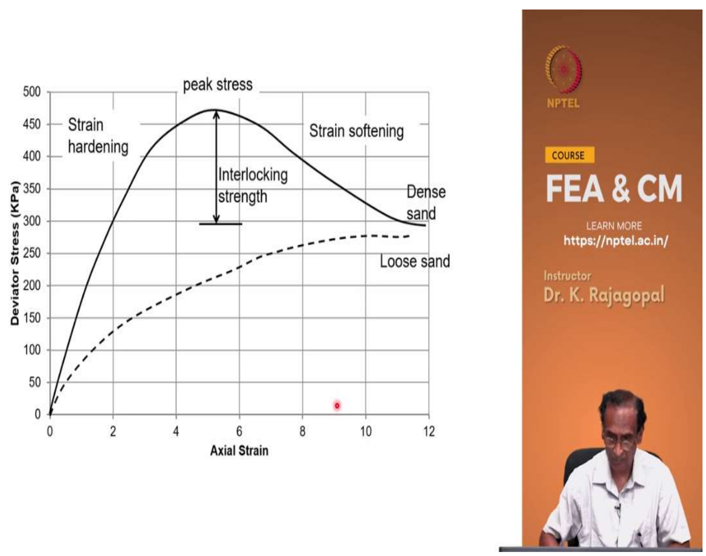

So, even if you are doing this type of strain softening analysis what we do is we calculate at each and every strain softening interval we calculate the secant modulus as the stress divided by the corresponding strain and now let me illustrate this with an excel spreadsheet. I think I forgot to bring my excel file, but I think that we will do in the next class. The first thing that we will do in the next class is I will illustrate the procedure through an excel spreadsheet.

The initial stiffness method and then the tangent stiffness and then the secant stiffness so that all these three procedures are clear for you and then we are going to follow the same procedures even for our finite element analysis.

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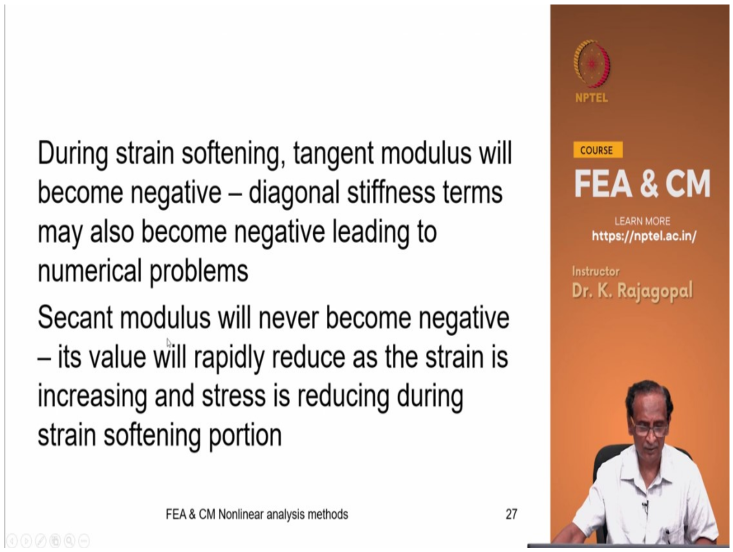

So, the advantage with the secant modulus method is the secant modulus will never become negative. So, we have the ease in the numerical computations.

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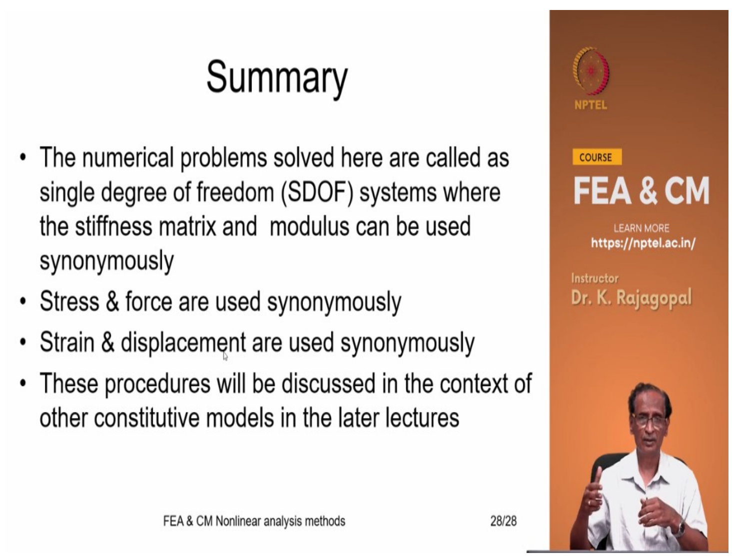

Summary so in this we have explained all these three procedures using the single degree of freedom system where the stiffness matrix and then the modulus are synonymously used because we have only one value so that we can easily do the computations, but similar procedure can be used even in the finite element analysis and I have used the stress and the force synonymously.

And then the strain and displacement are also used synonymously because the displacement is just simply strain multiplied by some length and so I will demonstrate all these three procedures using an excel spreadsheet so that you understand the difference between each of these methods and then you can solve some problems on your own using these spreadsheet programs. Thank you very much.

So, if you have any questions you please send an email to this address profkrg@gmail.com then I will try to respond back to you as fast as possible. So, thank you very much.