Finite Element Analysis and Constitutive Modelling in Geomechanics
Prof: K. Rajagopal
Department of Civil Engineering
Indian Institute of Technology – Madras

Lecture - 27
Nonlinear Analysis Technique-1

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Now let us say in this class let us consider how we can change our analysis procedures to incorporate the non-linearity in our stress strain response. So, when we deal with soils we will not have a straight line a stress strain relation that is the direct relation between the stress and strain we will have some type of non-linearity then when we are removing the load we may not follow the same stress path, we might follow some other stress path. So, we need to adapt our approach for the solutions to incorporate this type of behaviour.

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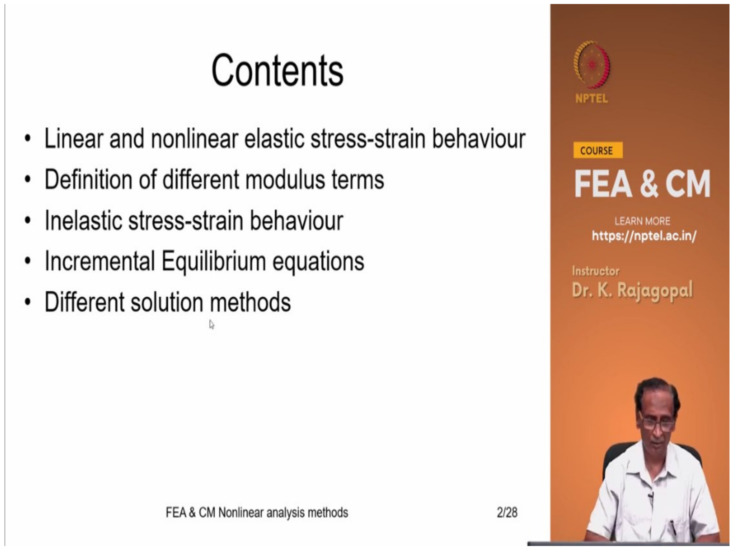

And let us look at the linear and non-linear terms and then the definition for the different modulus and so on in this these two lectures. And most importantly we are going to rewrite our equilibrium equations and in incremental form not in terms of the total stress form, but in an incremental form and we have to write them such that we can maintain the equilibrium between the applied forces.

And then the reactions under all types of loading conditions whether it is a construction induced loading or equation induced loading and so on and then we will also look at the different solution methods. Some may be very expensive, some may be very fast and so on.

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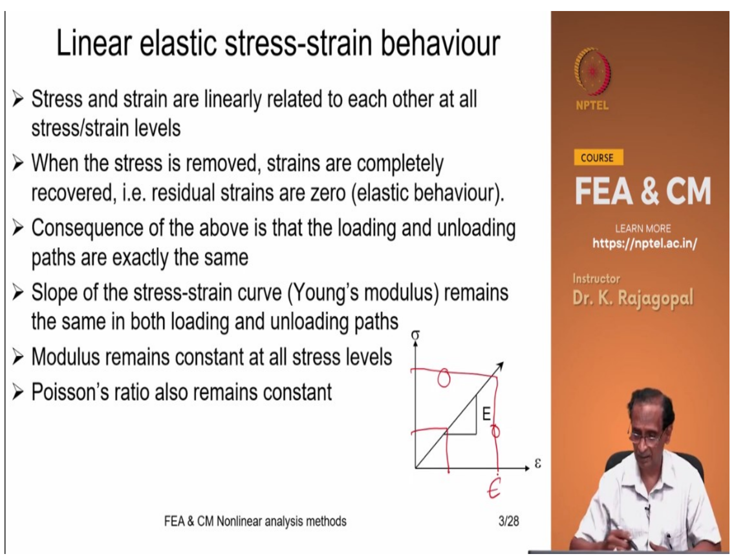
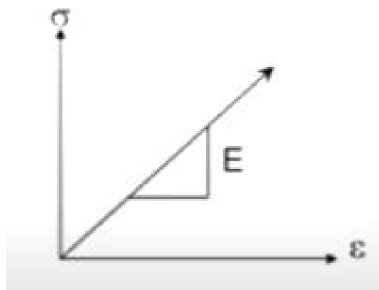


And if you categorize the stress-strain relations. The simplest one to analyze is our linear elastic stress-strain behaviour. So, the stress and strain are linearly related to each other at all the stress and strain levels like, for example, if you plot a graph between the strain and the corresponding stress you will have a straight line and the slope of this is constant that we can call it as some modulus Young's modulus and when the stress is removed the strains are completely recovered.



So, if you remove the stress you follow the same path and then your strain becomes zero and that is we call as the strain recovery whatever stress that you induce under loading are removed when you are unloading and so the residual strain could be zero at the end of the removal of the total stress and this is the classic elastic behaviour. Elastic behaviour is the stress and strain or the loading and unloading path are the same.

And the consequence of this is that the loading and unloading paths are exactly the same that is; so we have a linear elastic. So, linear means it is a straight line relation elastic means we are following the same path and the slope of the stress strain curve and that is called as the Young's modulus remains the same in both the loading and unloading. Basically E is defined as D sigma by D epsilon.

This is for the one dimensional case and the Young's modulus is the same during both the loading and also the unloading and the modulus remains constant at all the stress levels and the Poisson’s ratio also remains constant because there is nothing like a critical state where your volume remains constant. So, we can say that the Poisson’s ratio simply remains constant and actually here the stress and strain are linearly related to each other.

So, that means that whatever loading that you have you can just apply in one go and determine the corresponding displacements and then the strains and then of course the stresses without bothering to go through steps because the Young's modulus is constant. So, we can just simply apply the full loading and then get our corresponding displacements and then the strains like whatever maybe we can just apply.

And then get your displacements and then the strains corresponding to this to this applied stress.

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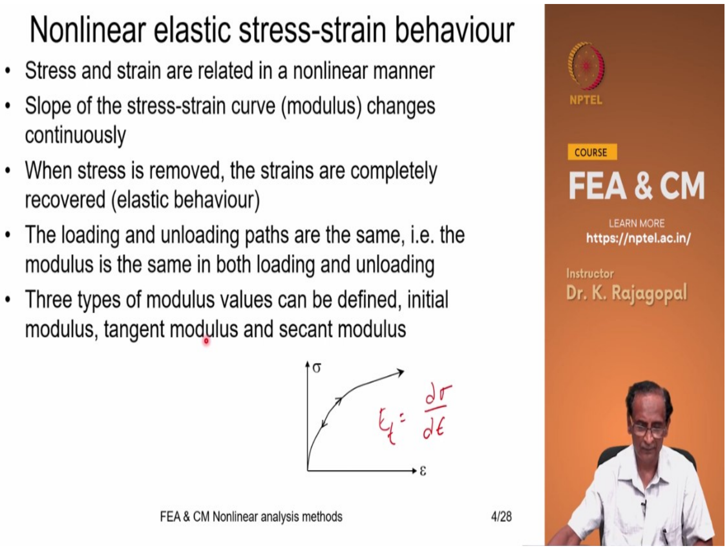
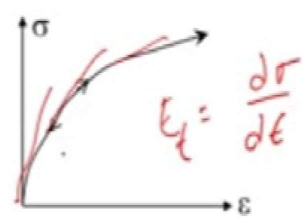


And just a bit more complicated is the non-linear elastic stress strain behaviour. It is non-linear so our sigma and epsilon are not linearly related to each other and the slope of the stress strain curve goes on changing. See the slope is going to change; so our E tangent is our Young's modulus and so if you calculate the tangent of this the sigma by epsilon at a low strain level you will get some value and this we could call it as an initial modulus.



And then this anything beyond that we can call it as a tangent modulus and this tangent modulus will go on changing with the strain and the stress level and the slope of the stress strain curve changes that means our modulus is changing. So, when the stress is removed the strains are completely recovered that both the loading path and the unloading path are the same. So, the loading and unloading of the paths are the same.

So, if you completely remove the stress that you applied you will recover all the strains that were caused and we can actually define three different modulus values.

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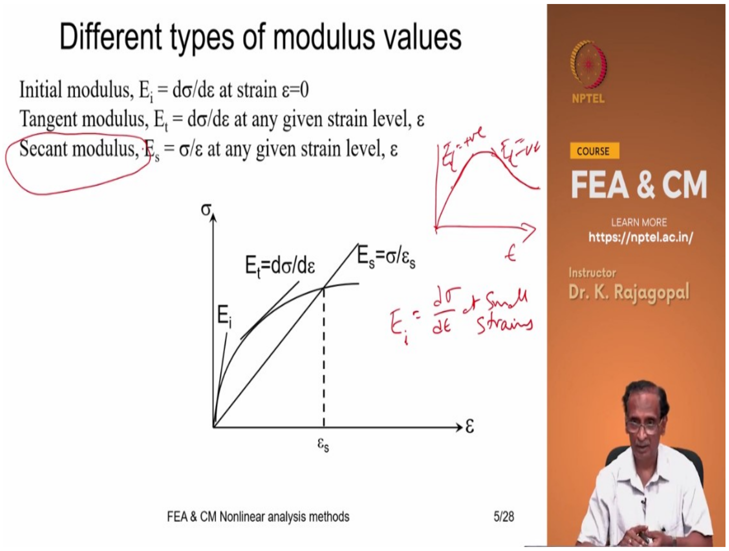
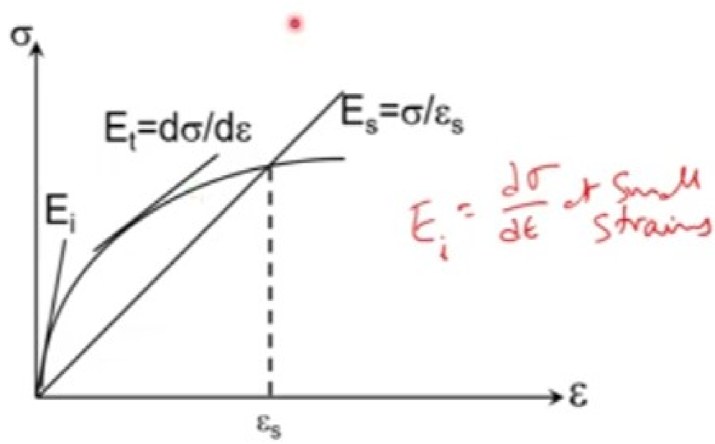


For this type of behaviour. One is a Young's modulus sorry the initial modulus the slope of this stress-strain curve at a very low strain level or the initial modulus. So, we can also call it as a small strain modulus whereas E tangent E tangent is our tangent modulus and then we can also define another quantity called as the secant modulus the stress divided by the corresponding strain.



And the initial modulus E i is a d sigma by d epsilon at beginning at epsilon is zero and tangent modulus E tangent is d sigma by d epsilon at any given strain level epsilon; at any strain level we can find the tangent and then that could be your tangent modulus and then the secant modulus E s is sigma by epsilon at any given strain level epsilon and you might wonder why we need this secant modulus that is because let us say we are dealing with the strain softening problem like this.

Your E t is positive in this strain hardening part, but here it is a negative in the in the strain softening part and if you try to apply the tangent modulus method for strain softening part you might have negative modulus. So, negative stiffness coefficients and then you may not be able to solve the system of equations. So, in that case we can go in for secant modulus because the secant modulus.

It will always remain positive definite because its sigma by epsilon and so during the strain softening part; see during the strain softening part your strain is increasing, but your stress is decreasing. So, as the strain is increasing your secant modulus will go on decreasing. So, it is in a way it is going to simulate the stress reduction with the strain and so in some problems the secant modulus may be more convenient to use compared to tangent modulus because we can get a numerical stable solutions.

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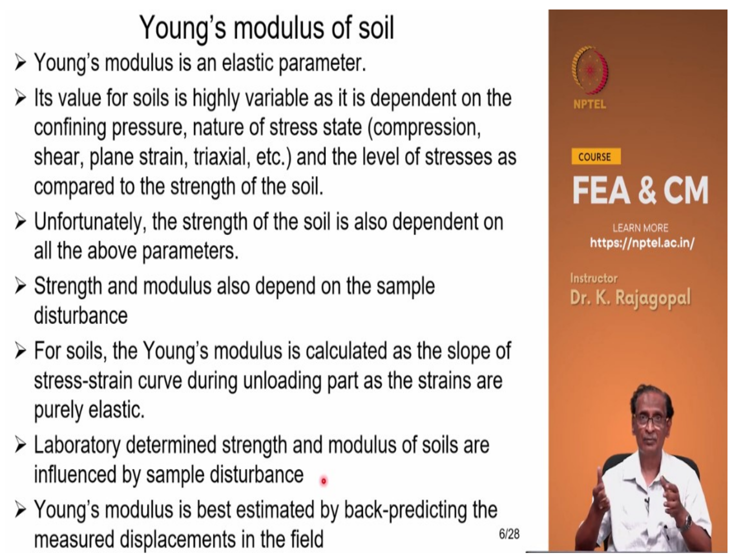

And how do we determine the Young's modulus because the Young's modulus is actually an elastic parameter. It is related only to elastic stresses and not to plastic stresses. So, its value for the soils is highly variable as it is dependent on the confining pressure the nature of stress state whether it is a compression or the shear or whether it is plane strain or triaxial iodometric and so on and then also the level of stresses compared to the strength of the soil.

So, if the shear strength of the soil is 300 and if you apply a stress very close to 300 let us say 290. You are almost at the verge of limit state so your modulus could be very, very low whereas if your stress is only say some 25 or 30 you are still in the initial state. So, your modulus could be very high. So, it is for the soils, it is not easy to define Young's modulus there is no one single parameter.

It varies with stress level and the confining pressure and then the type of stress condition whether it is a plane strain or axis symmetric or whether it is compression or pure shear and so on and the strength of the soil is also dependent on all the above parameters if you are confining pressure is higher your strength is also going to be higher or the strength could be different in the plane strain case compared to the triaxial the axis symmetric case or in the three dimensional case and the strength and modulus also depend on the sample disturbance.

So, we cannot really depend on the test that we perform on the samples that are collected from the field because sometimes our samples might be so disturbed that if you test them in the lab you might estimate very low strength or very low modulus and in general the Young's modulus is calculated as the slope of the stress strain curve during the unloading part because the unloading part is truly elastic compared to the loading part.

The loading part might produce both elastic strains and also the plastic strains. So, all our calculations they depend only on the unloading response and so the laboratory determine the strength and modulus parameters are dependent on the sample disturbance and so the best estimate for the Young's modulus is by back predicting the field measured the responses. Let us say construct a building and you know the settlements at some neighboring point.

And then you can back predict that settlement under the building induced pressure through some finite element analysis and then you can calibrate your Young's modulus until you get that settlement value and that Young's modulus that you get may be more reliable because it is by correlation to field measured values because when you collect a soil sample the cementation in the soil in the ground you may get disturbed because we do disturb the soil when we are collecting unless your soil is a relatively soft like a soft clay.

In that case you can push a sampling tube and then retrieve relatively undisturbed sample, but then if you have any medium dense sand or medium dense clay it is impossible to get an undisturbed sample.

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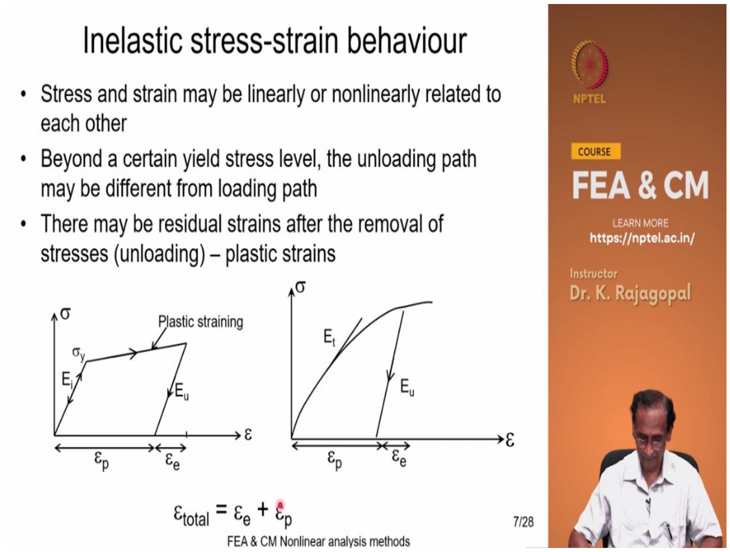
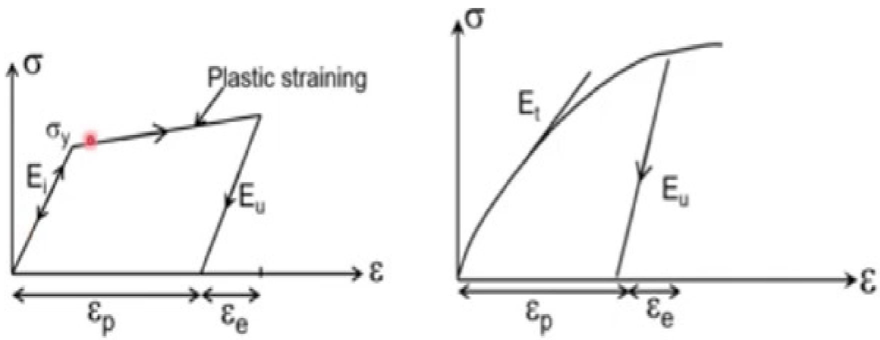


Similar to your linear and non-linear responses we could also have inelastic stress strain behaviour inelastic that is not elastic and the stress and strain may be linearly or non-linearly related to each other and beyond a certain stress level the unloading path may be different from the loading path and there may be some residual strains after the removal of the stresses that is unloading and we call these permanent strains as the plastic strains.

Let us say your stress strained behaviour is something like this initially your stress is directly proportional to strain in some manner and then beyond some yield limit your stresses may increase at a lower rate that because the because of the loss of strength or because you have entered the plastic limit of the soil and this is called as the strain hardening path and so if you unload before reaching the yield stress.

You will have an elastic response your all the strains that have induced because of this loading are recovered, but beyond this yield stress when you enter the plastic straining or the plastic state when you unload you are going to leave behind some permanent strains that is called as the plastic strain epsilon p and the path of the strain that is recovered is called as the elastic strain.



So, let us say we have loaded up to this point and your strain at the end of the unloading is epsilon p that is the permanent strain or plastic strain and then the epsilon e that is the recovered strain and below the yield stress sigma y we get back our complete strain that we have caused because of the stress and that is our pure elastic response and these lines may not be straight they could be curvilinear like non-linear response like this.

And so this could be your initial modulus this could be your tangent modulus and so on and during unloading we will have a pure elastic strains because we are causing only elastic strains during the unloading and this slope is your Young's modulus then after the stress is completely removed we are left with some permanent strains epsilon p and epsilon total is epsilon elastic plus epsilon plastic.

$$\varepsilon_{total} = \varepsilon_e + \varepsilon_p$$

So, our elastic strains are addressed reasonably well by our elastic stress-strain relations, but then the plastic strains we do not have any capacity to address them right now, but we will see later some methods for estimating our plastic strains.

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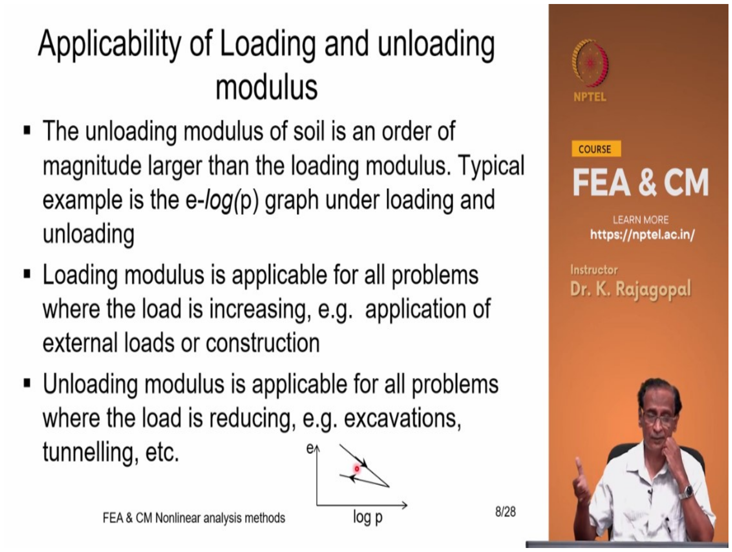
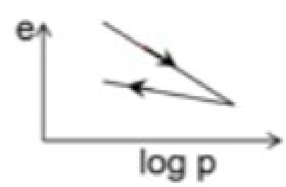


So far we have been seeing about the loading modulus and unloading modulus and so on and this particular response is very clear from our consolidation test. See during the consolidation test we know that as you are increasing the pressure the soil will undergo lot of volume changes so your wide ratio will go on changing, but if you remove the loading you are not going to get back your initial void ratio.

So, if you plot your e log p graph this could be your loading path and during unloading we will not go along this line, but only some small part of the void ratio is recovered. So, we might get a response something like this and this is our unloading response and this is our loading response and so the loading modulus the typical Young's modulus that we define. It is applicable for all the problems where the load is increasing.



Like our construction or gradually placing the soil during the embankment construction that we can simulate with our loading modulus, but then the unloading modulus is the slope of this unloading part of the stress-strain curve or e log p graph that is applicable for all the problems where the load is removed that is x equations and then tunneling and so on. So, if you have any deep x equations we prefer using some other modulus during the unloading part because if you use a normal loading and unloading modulus the same. You might over predict your deformations.

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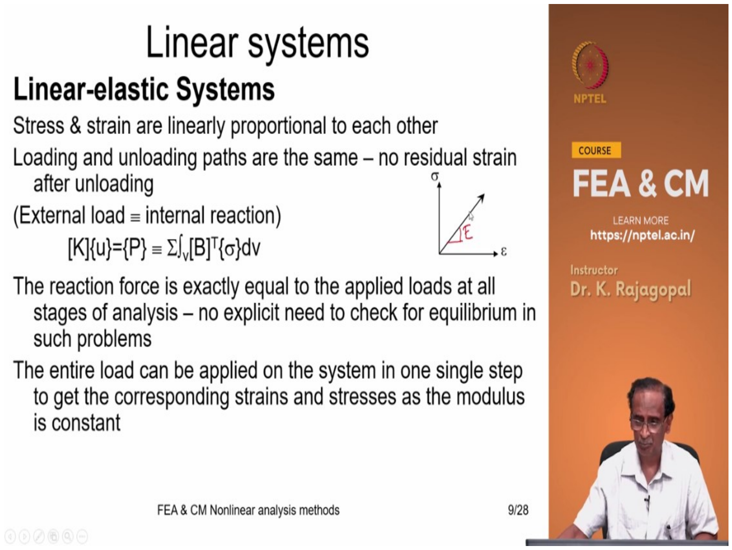

So, the linear elastic systems are those in which the stress and strain are proportional to each other in a linear manner and there is no residual strain and in this case your external load is exactly equal to internal reaction because whatever stress that you apply to the system the system can take without any complaints it will be able to support all the stresses that are applied.

(External load ≡ internal reaction)

$$[K]\{u\}=\{P\} \equiv \Sigma\int_v [B]^T\{\sigma\}dv$$

And so our K u is equal to P and this P is the applied force and that will be exactly equal to B transpose sigma dv and actually in this case the reaction force is exactly equal to the applied loads at all the stages of analysis. So, there is no need for checking for the equilibrium because anyway it is understood that all the pressures that are applied are supported by the system.

So, that means that it is able to generate equal and opposite reaction. So, we do not normally check for equilibrium in such problem, but even if you do check for the equilibrium you will see that your outer balance force is very, very small and the entire load can be applied in the system in one single go and we can calculate our stresses and strains as the modulus is constant the slope of this line is constant.

See your slope is constant means your stress and strain are linearly related to each other or proportional to each other.

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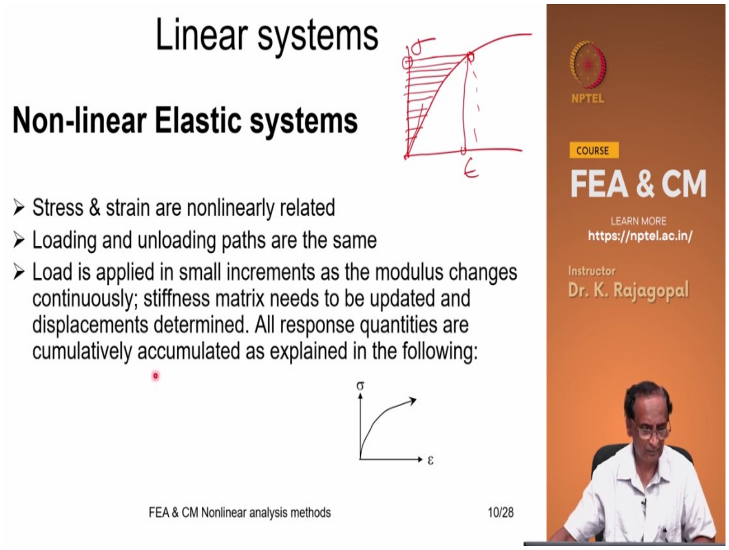

So, non-linear elastic systems are the stress and strain are non-linearly related to each other, but then the loading and unloading paths are the same and then the load is now we should be careful because we cannot apply the load as we wish. Let us say we have a stress-strain equation something like this and I want to know what is the strain at a stress of this, this is your this is your strain and that is your stress.

But then if you want to know what is this strain corresponding to this stress you cannot directly apply because we are working with only initial modulus, tangent modulus and so on and to get the accurate description of this strain we could apply this stress in very small increments and then calculate the strain increments and then accumulate to get our actual strain state and the analysis is a slightly more involved that we will see.

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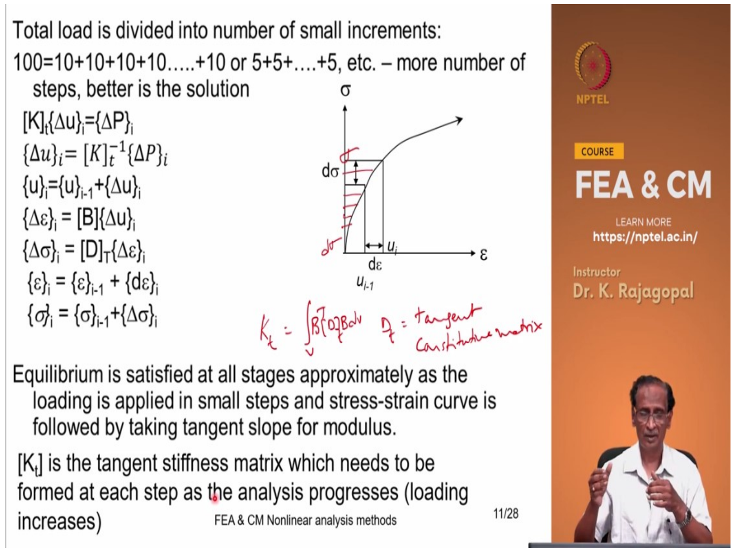
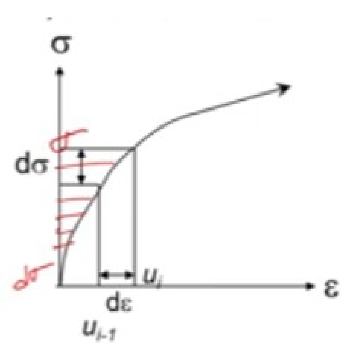


So, one of the simplest methods is apply the stress in small increments and suppose you want to apply say we apply the total stress in small increments. Let us say you want to apply a pressure of 100 you can split that into 10 kPa each and apply this 10 times and 5 kPa and applied 20 times and so on and the more number of steps that use the better is the solution. So, instead of doing the analysis one time we are going to repeat it several times.

So, that is the main difference between our linear elastic and this non-linear elastic or the inelastic systems and now we are not going to apply the total stress in one go, we are going to apply this in small steps d sigma or corresponding to that your incremental load of delta p. So, what we do is we write our equilibrium equations in terms of incremental firm K t times delta u i is equal to delta P i and in small steps we apply.



And then calculate your displacements delta u as K t inverse times delta P and then the total displacements u i is u i - 1 + delta u i where I - 1 refers to the previous load step and the i corresponding to the current load step in the current step you have applied a loading of delta P and the delta U is the displacement increment in the current step. So, delta u i is a K t inverse where K t is the tangent stiffness matrix that is formulated in terms of the tangent modulus.

Total load is divided into number of small increments:

100=10+10+10+10…..+10 or 5+5+….+5, etc. – more number of steps, better is the solution

$[K]_t\{\Delta u\}_i=\{\Delta P\}_i$

$\{\Delta u\}_i= [K]_t^{-1}\{\Delta P\}_i$

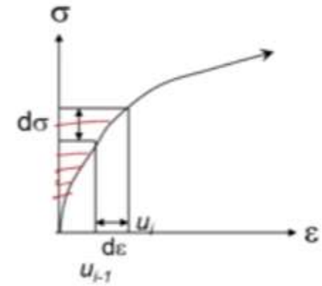
$\{u\}_i=\{u\}_{i-1}+\{\Delta u\}_i$

$\{\Delta\varepsilon\}_i = [B]\{\Delta u\}_i$

$\{\Delta\sigma\}_i = [D]_T\{\Delta\varepsilon\}_i$

$\{\varepsilon\}_i = \{\varepsilon\}_{i-1} + \{d\varepsilon\}_i$

$\{\sigma\}_i = \{\sigma\}_{i-1}+\{\Delta\sigma\}_i$



$K_t = \int_v [B]^T [D_t] [B] dv$ D_t = tangent constitutive matrix

So, our K t is the is the tangent stiffness matrix and that is written as integral B transpose D t B d v where D t is the tangent constitute matrix and that is updated every step based on the, current Young's modulus. We get our displacement increment and then the total displacement u i is u i - 1 + delta u i and delta epsilon is B times and delta u i and then delta sigma the change in the stresses is D t that is based on the current Young's modulus and then the current Poisson’s ratio multiplied by delta epsilon.

And then the total strain epsilon i is epsilon i - 1 + D epsilon i and sigma i is sigma i - 1 + delta sigma i and the equilibrium is also satisfied at all the stages approximately because our loading is applied anywhere in the small steps and so we do not really look for any equilibrium check in this case because anyway we are applying the load in very, very small steps and the material does not have any failure.

And it can take all the pressures that are applied on the system, but then we work with the tangent slope, the tangent modulus and then continue doing the analysis. So, our K t is the tangent stiffness matrix which needs to be formed at each step as the analysis progresses as the loading is increasing increased your K t we need to determine again. So, let us say you want to apply a pressure of 100 kPa in small steps of 1 kPa.

So, that means that you have to apply this hundred times so you have to perform this analysis hundred times. So, you have to form the stiffness matrices of all the elements hundred times assemble them and then you have to do the triangulation that is converting the stiffness matrix in an upper diagonal matrix and then you can do the back substitution. So, it is actually it is a very involved with a process.

And so it takes lot of time and effort for doing these analyses. So, I think I will stop with this and we will continue in the next class because I want to advance the same things to the next level of complexity and if you have any questions you please send an email to me.