FEM and Constitutive Modelling in Geomechanics
K. Rajagopal
Professor and PK Aravindan Institute Chair
Department of Civil Engineering
Indian Institute of Technology - Madras

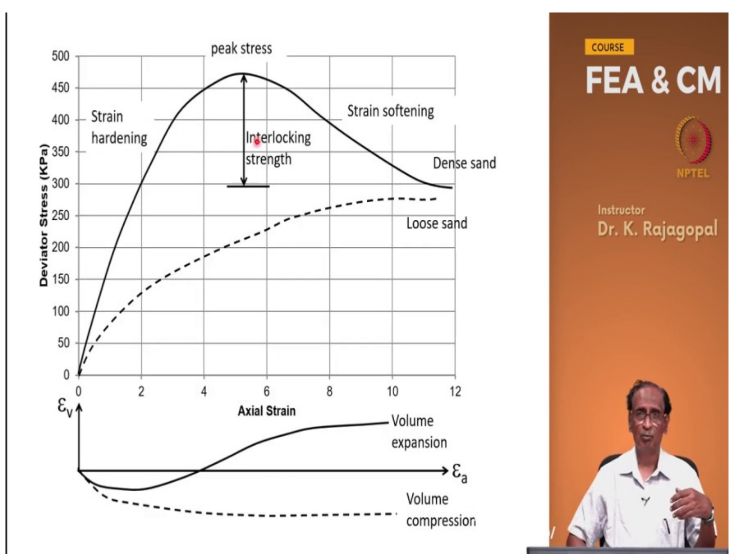
Lecture: 26
Some Observations of Soil Behaviour and Stress Invariants

Let us continue our lectures till the previous class we talked more about the finite element modelling. And we have also seen some particular aspects related to geotechnical engineering that is related to Sliding along any preferential direction like through the interface elements. Then modelling of semi-infinite soil media through our mapped infinite elements and then we have also seen the gradual construction and then removal of soil excavation.

And we have seen how to simulate the type of geotechnical aspects and from this class onwards we will focus more on the soil behaviour and it is constitute modelling. And then what are some aspects related to analysis with non-linear response or plastic response and so on. And before we go into the analysis aspects let us look at some observations on how the soil behaves so, that we can then we can think about the modelling.

Then of course towards the end I will also introduce the stress invariance that is one of the most important aspects of any constitute modelling.

(Refer Slide Time: 01:51)



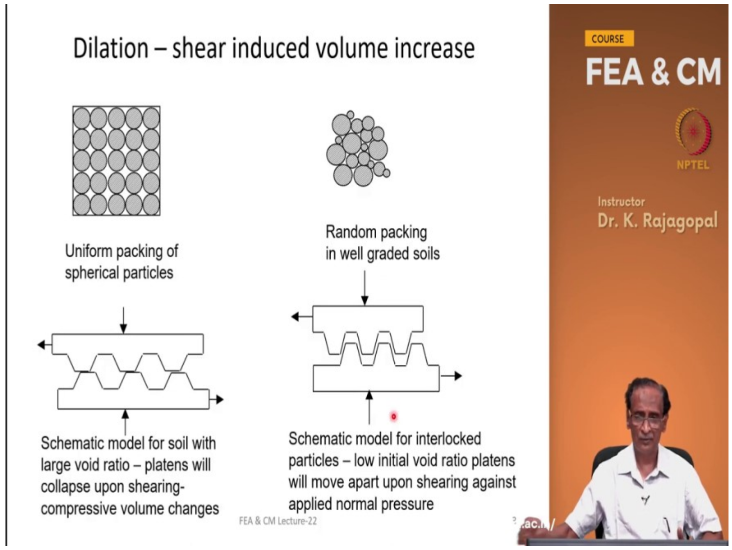
And this is what we see right from our undergraduate days on the soil behaviour we know that a dense sand or an over consolidated clay when it is subjected to Shear strain it will have a nice peak and then beyond that it will undergo some strain softening and then at a very large strain the stress will remain constant and that we call as the critical State. And in the case of lose loose sand or normally consolidated clays this stress will go on increasing with strain.

And then after very very long strain we might treat some asymptotic limit and we see that the ultimate limit of both the dense and the loose sands is the same. And we call this type of behaviour as a strain hardening behaviour. And in the case of dense sands this is our strain hardening behaviour and then we have the strain softening behaviour. And the additional strength that we have between the the critical State and then the peak stress we say its because of the interlocking effects.

Then if you look at the volume change response the loose sands will go on compressing then at some stage they will reach a constant volume state but let they will undergo volume compression predominantly. Then in the case of dense Sands they might undergo some volume compression depending on the initial relative density and then after some time the soil will expand in volume.

And this expansion in volume under the shear strains we call it as dilation that is a specific term that we have in the in the case of soil mechanics. Then at a very large Shear strain will reach a constant volume state and that is related to the critical state. And so, this is the basic difference between dense sand and the loose sand and then we have to see how to model them through our constituted models.

(Refer Slide Time: 04:24)



And the reason for this is very simple. So, if we look at the packing the soil packing and then the shape whether they are rounded particles or spherical particles and then angular particles and so on. Say if you have a uniform packing like this uniform packing of soil particles all spherical particles of uniform size and this will have an initial initially very high void ratio and then if you subject that to some Shear strain the particles tend to fill up the voids.

And with the result that you will the soil will undergo volume compression and the analogy for this we can look at by looking at these 2 serrated plates. Let us say that initially the 2 plates are arranged like this with a very large spacing between them and if you subject them to any Shear strain after some strain you will see that the upper plate will fill up the bottom plate and they will come together and that is similar to our volume compression of the loose Sands.

So, we can say that this is a analogy of the loose sand behaviour initially there is a large gap between the 2 plates and that is corresponding to our initially high void ratio. And then if you look at well graded soils or the sands with a very good angular particles there could be a lot of interlocking between the particles and then you can and when you subject them to any shear. Let me just it is actually in the case of the case of nicely arranged particles there are predefined Shear planes like this.

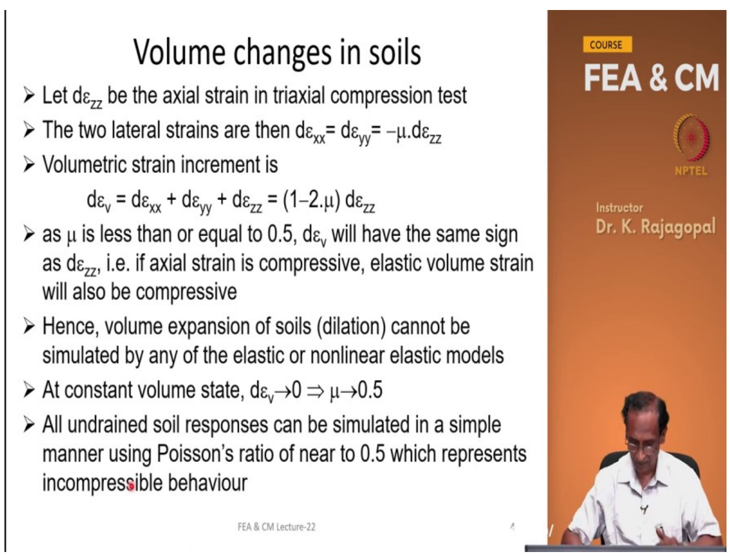
And then the soil will deform under Shear strain without much of volume change or there could be some volume compression. But then here there is no single plane on which all the particles are very nicely arranged. So, if you have to develop any initial then you have to deform or move the particle some particles away from the away from the shear plane so, that we can nicely form the firm the failure.

And that type of situation we can think of by these 2 interlocked plates let me just get back to my laser pointer. So, here we have a 2 interlocked plates if you subject them to some Shear strain the only way the shear strain will take place is when the 2 do when the 2 plates move away from each other that is similar to our dilation of well graded or densely packaged soils.

And this expansion in volume or the moving away of the plates is happening against the applied normal pressure. So, the shear strain has to do some additional work to or the shear stress should do some additional work to cause the shear strain apart from the shear strain it has to also act against the normal pressure. So, because of that the soil might have some additional strength.

And in fact that is what we see in the form of this additional strength that you have beyond your critical stress and this we call as the interlocking strength.

(Refer Slide Time: 08:39)



And then what are the volume changes? So let us think of our triaxial compression test because that is the simplest stress state that we can imagine let us say that we have applied an axial strain increment of d Epsilon zz and the 2 lateral strains in the x and Y directions could be written as - mu times d Epsilon zz and the volumetric constrain increment is the sum total of all the 3 normal strain increments.

$$d\varepsilon_{xx} = d\varepsilon_{yy} = -\mu . d\varepsilon_{zz}$$

Volumetric strain increment is

$$d\varepsilon_v = d\varepsilon_{xx} + d\varepsilon_{yy} + d\varepsilon_{zz} = (1-2.\mu)\, d\varepsilon_{zz}$$

So, d Epsilon v is d Epsilon xx + d Epsilon yy + d Epsilon zz and that is equal to 1 - 2 mu times d Epsilon zz. And so, our Poisson’s ratio is less than 0.5 the theoretical range is from -1 to 0.5. But negative Poisson’s ratio is impossible because we see that if you elongate in one direction the other direction there is compression or if you compress in One direction there is elongation in the other direction.

And so, our Poisson’s ratio is always positive and it is less than 0.5. So, if it is if the Poisson’s ratio is less than 0.5 let us say 0.4; 1 - 2 mu is 0.2 times d Epsilon Z. So, that means that our volumetric strain increment will have the same sign as the axial strain. So, if your axial strain is compressive and volumetric strain is also compressive. So, we can say that all our elastic volume changes are only going to be compressive as long as you apply some compressive stresses.

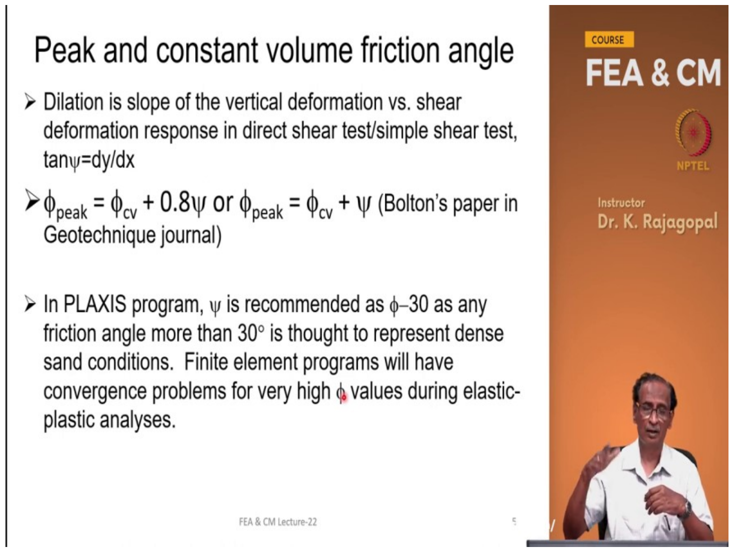
And on the soil particles we cannot anyway directly apply any normal tensile strain we can apply Shear strain but we cannot apply any tensile strain we can apply compressive strain or compressive stresses. oh So, our dilation induced volume expansion we cannot model through any of our elastic equations elastic or non-linear elastic models because our Poisson’s ratio will not allows like 1 - 2 mu D Epsilon z is our volume strain.

$$d\varepsilon_v \to 0 \Rightarrow \mu \to 0.5$$

And at a very large strain at constant volume state when d Epsilon v tends to 0 this should tend to 0 and we see that the MU should tend towards 0.5 to keep the volume constant that is the further increments of the volume strain should be 0. So, our mu is a 0.5 towards the critical State and in fact we do not write any equation like this we put we write it as Mu tangent because the Poisson’s ratio also keeps updating as we are applying strain.

The Poisson’s ratio might change along with the modulus values and if you have an undrained response of soil because that is associated with the 0 volume change we can simulate the undrained behaviour by setting the Poisson's ratio close to 0.5 maybe 0.49 or 0.499 and that is actually incompressible behaviour and this is one simple way of simulating the the undrained soil response.

(Refer Slide Time: 12:43)



So, we could have 2 friction angles one is corresponding to peak in the direct sorry in the in the shear test that we have performed on the on the dense Sands. We have a very nice Peak and then the strain softening and that peak stress corresponding to that peak stress we have the five Peak or the friction angle the peak friction frictional strength. And that the constant volume stage you will have your Tau CV and through that we can get Phi CV the friction angle at constant volume.

$\phi_{peak} = \phi_{cv} + 0.8\psi$ or $\phi_{peak} = \phi_{cv} + \psi$ (Bolton's paper in Geotechnique journal)

And in the direct shear test and simple shear test the tan psi is just simply the slope of that line dy by dx if x is the shear direction and Y is the volume or vertical direction tan psi just simply dy by dx and the phi peak is approximately equal to Phi CV + 0.8 psi or phi cv + psi is actually this is from one of the classical papers on soil dilatency that was published by Professor Bolton long time back.

And there he has described the soil dilation and its relation to the gradation and then the particle shape and so on. And in the PLAXIS program there is a thumb rule given that the dilation angle if you have not measured from the Laboratory test it could be taken as the Phi - 30 assuming that about 30 degrees we can think of as the starting point of any dense sand and any angle beyond the 30 degrees we can take it as a dilation angle.

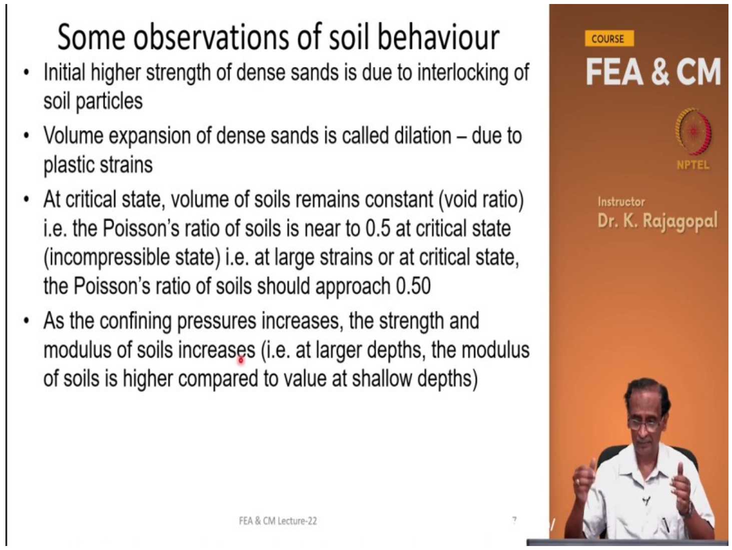
So, if your friction angle is 35 degrees your dilation angle could be 5 degrees. And one point that we have to remember is the finite element programs will have lot of difficulty converging to a solution not very high peak value peak shear strength values like if your Phi is beyond about 40 degrees if you try simulating any bearing capacity problem you will see that the program will just simply fight to converge. You will you will see lot of oscillations in the stress that you put the tube predict.

(Refer Slide Time: 15:33)

Say the dense Sands exhibit very sharp peak and its strength stress strain response will have a very nice strain hardening part and then a strain softening part while loose sands will have only a strain hardening part the stress will go on increasing with the strain then at a very large shear strain we might read some asymptotic limit. And the dense sands may initially compress in volume but undergo volume expansion under increasing Shear strains.

And then the loose sands they will exhibit only volume compression under Shear straining and at a very large strain both the dense and loose Sands will exhibit the same strength at the critical state.

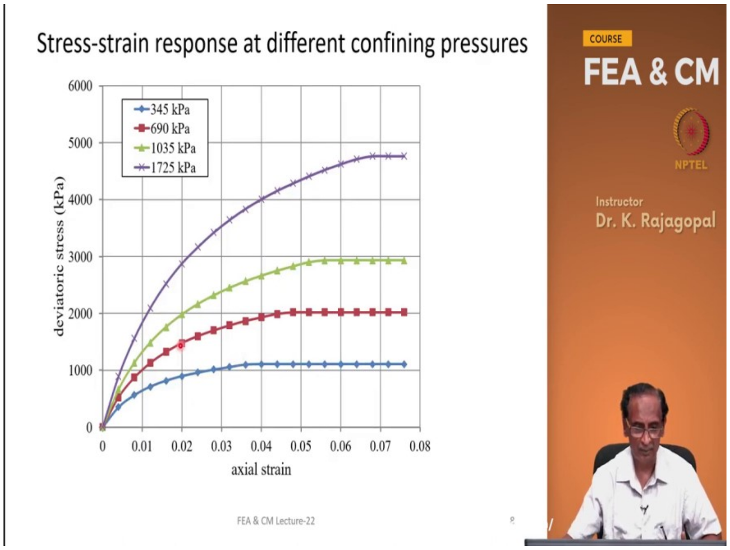
(Refer Slide Time: 16:22)



And when we develop any constituted model we should be able to to exactly replicate some or all of these features. We will see different types of constituted models that can simulate this type of behaviour. And the initial highest strength of dense Sands is due to interlocking of the soil particles either because of the gradation or because of the particle shape and the volume expansion is called as the dilation and its relate its due to the plastic straining.

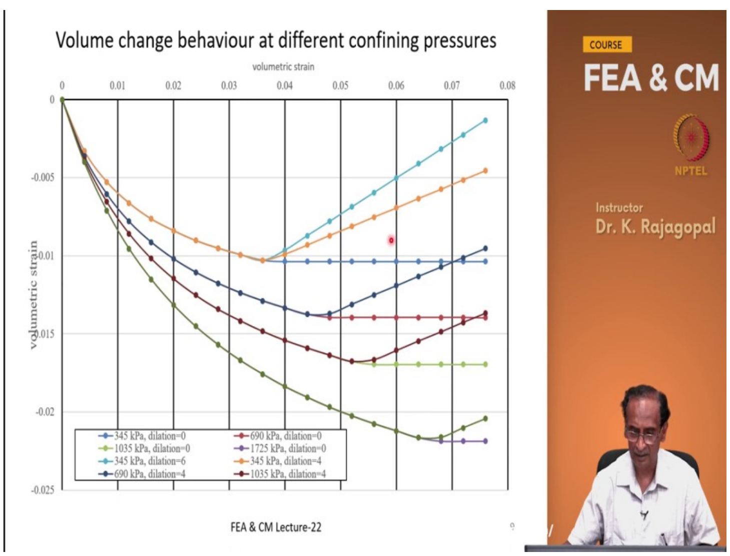
It is not during the elastic state or elastic straining but during the plastic state. At critical state or volume of the soil remains constant and the Poisson's ratio is near to 0.5 at the critical state to represent our incompressible state and as the confining pressure increases the strength and the modulus of the soil increases. That is if you have a soil deposit at a larger depth the same soil are the same relative density might exhibit a stiffer response and a higher strength because of more confined pressure.

(Refer Slide Time: 17:46)



And that we see from our triaxial compression test this is the same soil placed at the same relative density but tested a different confining pressures of 345, 690, 1035 and 1725 and you see that as the confining pressure is increasing the strength is going to the strength is increasing and it is not only the strength but also the slope of this initial slope of this stress strain curves they are increasing. So, that means that at higher confining pressure will have a higher modulus.

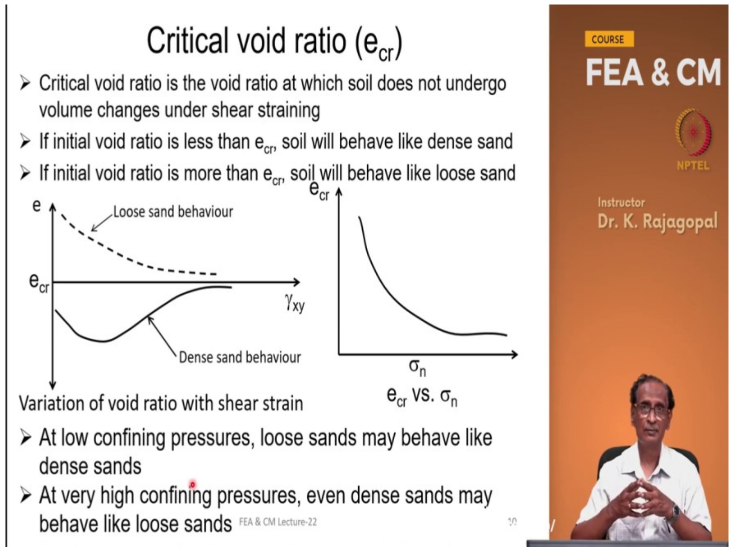
(Refer Slide Time: 18:26)



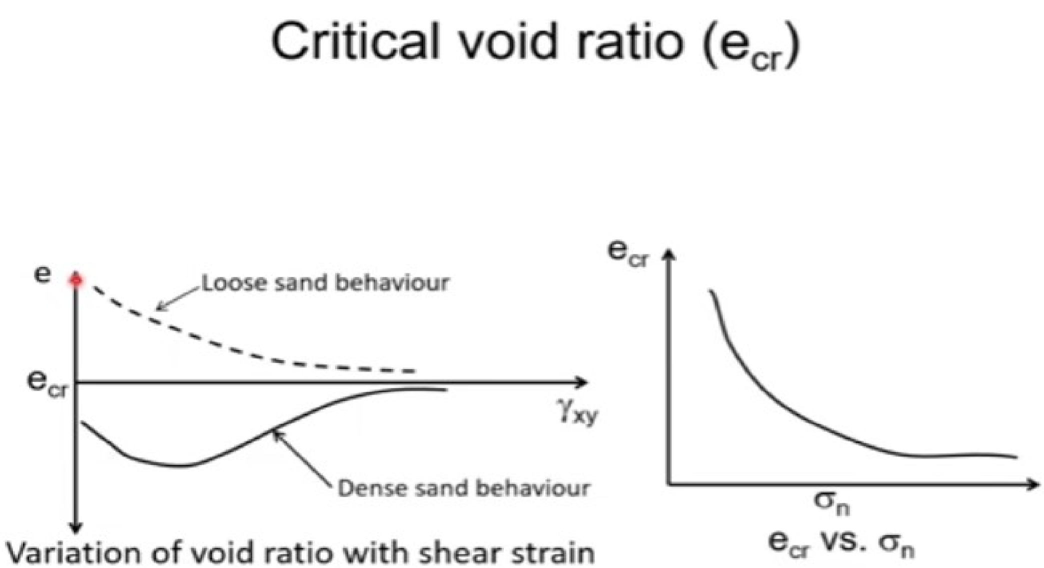
And this is the corresponfing volume change behaviour at larger confinement you will have more volumetric compression because the soil will have so, much strength that you have to mobilize so, much of shear stress before the soil starts failing or undergoing plastic deformations. So, you will see that at a larger confining pressure the soil will compress for longer duration or longer up to larger Shear strains and then only they will start expanding.

And this is at 345 and then this is at 690 and then 1035 and then 1725 and these are all the responses that we get with the different dilation angles. So, after plastic failure if this curve is flat that means that it is a dilation angle of 0 and with the higher dilation angle we will see higher slope.

(Refer Slide Time: 19:38)



Then what distinguishes between what is the factor that distinguishes between the dense and behaviour and the loose sand behaviour that is the critical void ratio. So, initially if we place the soil at a very large void ratio and then start shearing it will it will go on compressing up to some critical state when you are void ratio reaches a critical value ECR then if you perform the same test by placing the soil at a very low very low void ratio.



Initially there might be some compression because of rearrangement of particles then because of dilation soil will expand and then at a large shear strain it might also reach the same constant void constant wide ratio that we call as the critical void ratio. So, the critical void ratio is the void ratio at which the soil does not undergo any volume changes under Shear straining. So, if it is a loose sand it will compress and if it is then sand it will expand in volume.

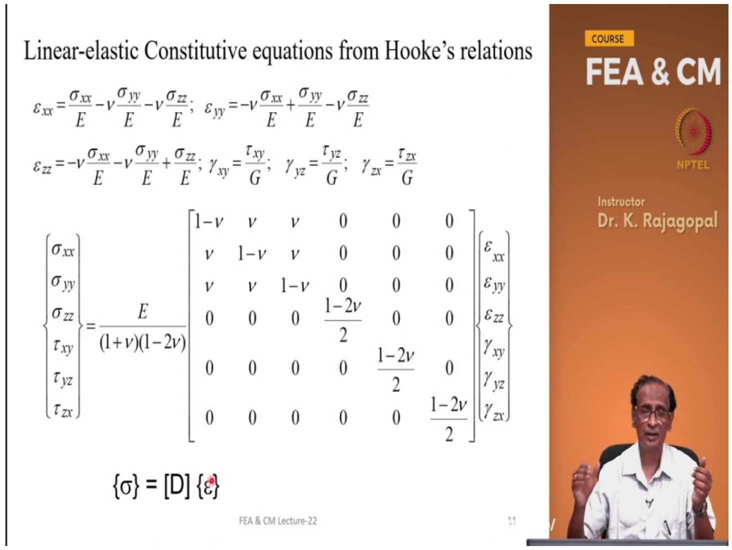
And at a very large Shear strain the soil will reach a critical state when it will deform or it will undergo Shear strains without any corresponding volume changes and that void ratio is called as ECR or critical void ratio. And this critical void ratio is not a constant is going to change with normal pressure. So, at a lower normal pressure Sigma n is at a very low Sigma n ECR is very very large.

Whereas at very high confining pressure you are ECR could be could be small. And if your initial void ratio is less than ECR the soil will behave like a like a dense sand that is expanding in volume and if your initial void ratio is more than that of ECR the soil will behave like a loose sand. And so, at very low normal pressure even a soil with a very high initial wide ratio might behave like a dense sand because this wide ratio is less than the less than the ECR.

So, that explains why some the so-called loosens the expanded volume at some confining pressures. So, at very high confining pressures even dense sands may behave like loose ends that is undergoing constant increase in the in the shear stress with the shear strain and then volumetric compression and is actually this we need to know their behaviour. So, that we can choose an appropriate constitute model.

So, if it is loose sand it is enough if you have some elastic model because anyway it is going to undergo volumetric compression it is not going to undergo any expansion but if you want to simulate the dense sand behaviour you need some Elastic Plastic model. So, that we can represent the dilation the shear induced the volume expansion.

(Refer Slide Time: 23:20)



And see these linear elastic constitute equations we have seen earlier these are the generalized hooks relations relating the the different strain components to the stress components. The 3 normal strains Epsilon xx Epsilon yy and Epsilon z z are related to to the normal stresses Sigma x Sigma y and sigma z and the Poisson's ratio mu. And the shear strains gamma xy is Tau by G gamma yz is Tau y by Tau y z by G and so on.

Linear-elastic Constitutive equations from Hooke's relations

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E}; \quad \varepsilon_{yy} = -\nu\frac{\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E}$$

$$\varepsilon_{zz} = -\nu\frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} + \frac{\sigma_{zz}}{E}; \quad \gamma_{xy} = \frac{\tau_{xy}}{G}; \quad \gamma_{yz} = \frac{\tau_{yz}}{G}; \quad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

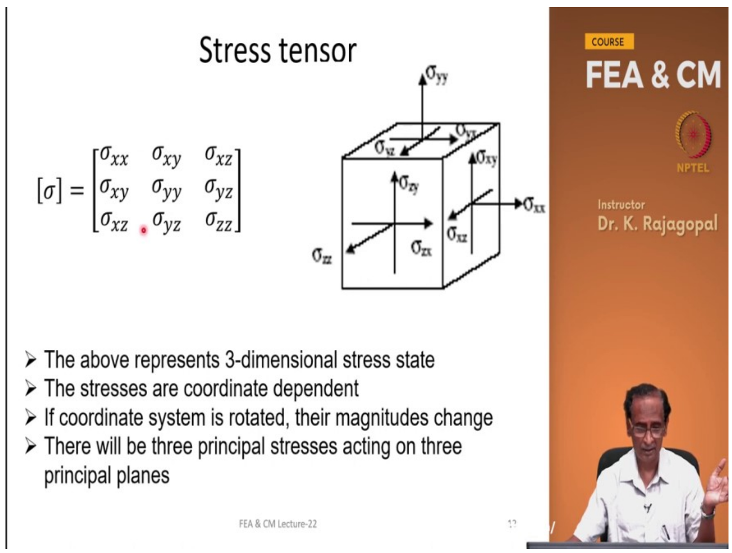
And the reverse the inverse relation that expresses the stresses in terms of strains is through this. This is our generalized 3 dimensional stress state where we have 3 normal stresses Sigma xx Sigma yy and sigma zz and then 3 Shear stresses Tau xy Tau yz and Tau zx and in general we write stress as some constitute Matrix D multiplied by Epsilon. And since our finite element analysis is based on displacement.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

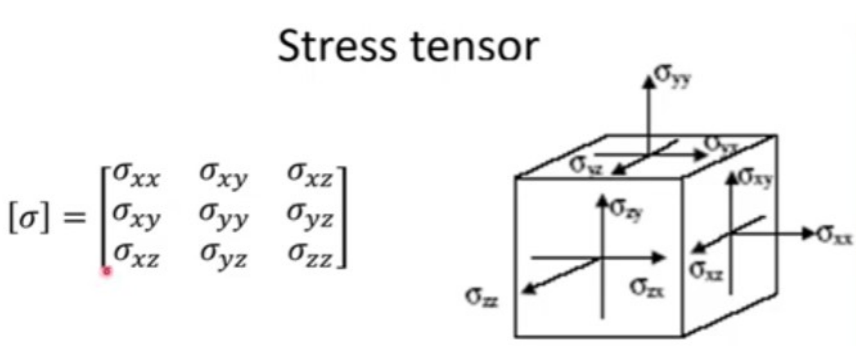
We are using a displacement approach where our displacements are the basic unknowns from the displacements we get the strains and from this the strains we can get the stresses through the constituent equations. And the relation between these strains and displacement is through our compatibility equations that is Epsilon xx is dou u by dou x and so on.

$$\{\sigma\} = [D]\,\{\varepsilon\}$$

(Refer Slide Time: 25:16)



And let us look at the stress tensor once again a 3 dimensional stress tensor is written like this Sigma xx Sigma yy Sigma zz. These 3 are the normal stresses Sigma xy Sigma xz Sigma yz are the shear stresses. And we have seen this diagram earlier and the convention is the first index Sigma x is the is the stress that is acting and normal to a plane perpendicular to that axis.

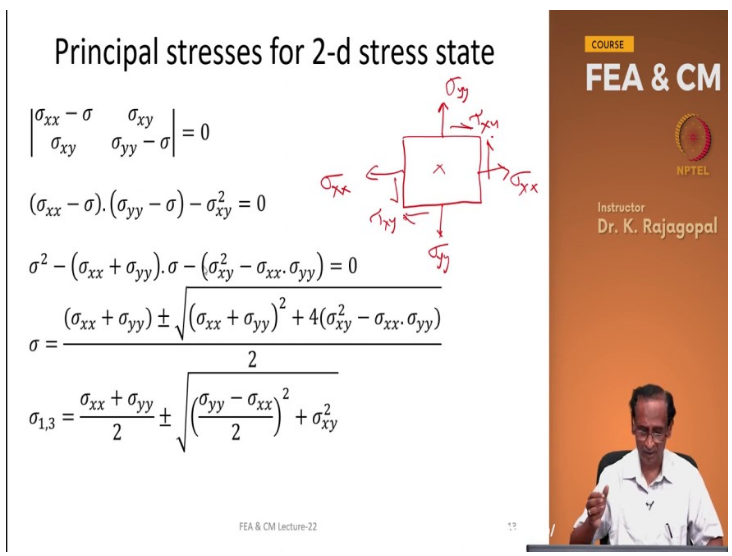


And then Sigma xx means it is acting an f and a plane normal to the on a plane normal to the x axis and in the direction of x and sigma xy means the distress that is acting on a plane normal to x but in the direction of y and sigma xz. And because of the equilibrium considerations Sigma yx and sigma xy should be the same. So, we write the equations like this and this is the generalized 3 dimensional stress state.

And these stresses are Coordinate dependent. So, if I change the coordinate system the stress magnitudes might change. So, that is why we call it as a stress tensor it is not a stress matrix but it is a stress tensor because a tensor quantity depends on the coordinates. If you rotate the coordinate system your stresses also will change. And corresponding to to the 3 directions will have 3 principal stresses acting on 3 principal planes.

So, actually that we have to do some analysis to get the principal stresses and their and the planes on which they are acting. And the shear stresses are absent on the on the principal planes.

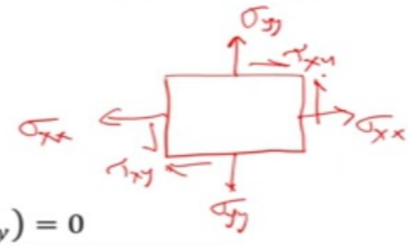
(Refer Slide Time: 27:38)



Let us look at our 2 dimensional stress state for simplicity. Let us consider our stresses like this and our convention for the stresses is tensile normal stresses are taken as positive that is the elasticity convention. So, Sigma yy Sigma xx are both tensile. So, we take them as positive and then any shear stress that is causing clockwise moment about a point in the interior it is called as a as a as a positive shear stress.

Principal stresses for 2-d stress state

$$\begin{vmatrix} \sigma_{xx} - \sigma & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} - \sigma \end{vmatrix} = 0$$



$$(\sigma_{xx} - \sigma).(\sigma_{yy} - \sigma) - \sigma_{xy}^2 = 0$$

$$\sigma^2 - (\sigma_{xx} + \sigma_{yy}).\sigma - (\sigma_{xy}^2 - \sigma_{xx}.\sigma_{yy}) = 0$$

$$\sigma = \frac{(\sigma_{xx} + \sigma_{yy}) \pm \sqrt{(\sigma_{xx} + \sigma_{yy})^2 + 4(\sigma_{xy}^2 - \sigma_{xx}.\sigma_{yy})}}{2}$$

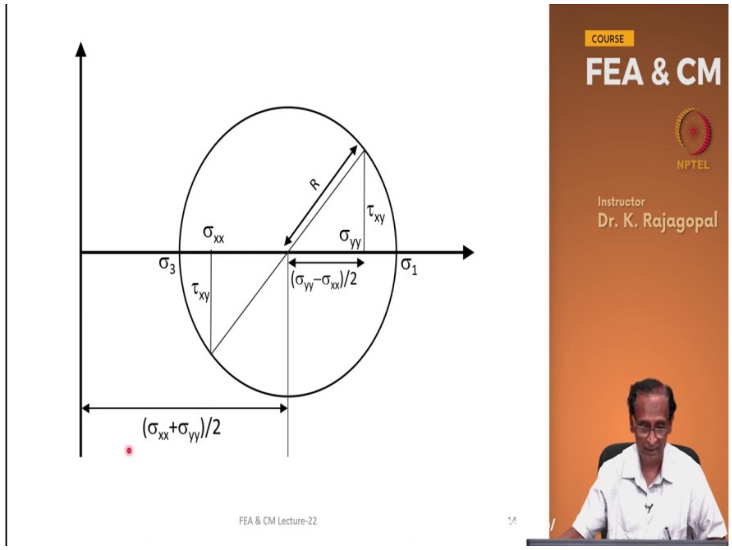
$$\sigma_{1,3} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{yy} - \sigma_{xx}}{2}\right)^2 + \sigma_{xy}^2}$$

So, the Tau xy acting on the horizontal plane is taken as positive whereas the Tau xy acting on the vertical plane is taken as negative because it is producing anticlockwise moment. So, how we can determine the principle stresses is very simple like we can draw the more Circle and we can also determine them as the roots of the of this the determinant equation like this say Sigma are the roots.

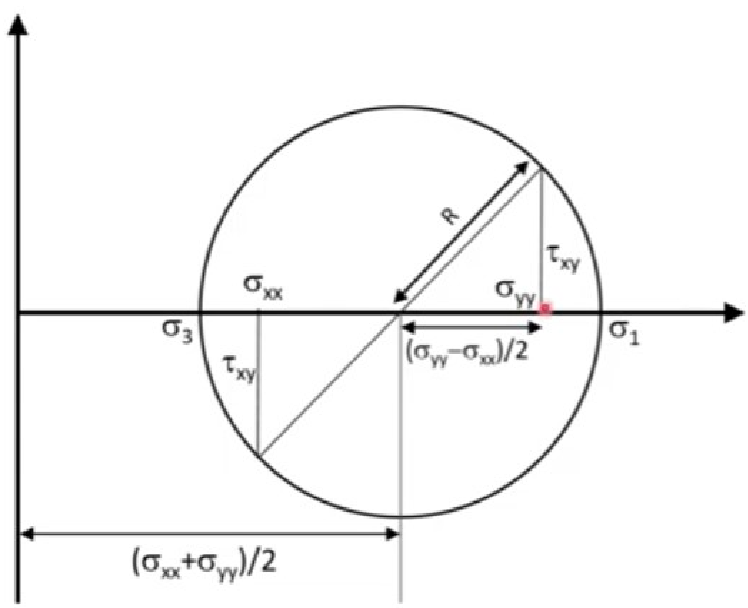
We can set up the equation for determining the roots like this Sigma xx - Sigma Sigma xy Sigma xy Sigma yy - Sigma the determinant should be 0 let me just get back to my laser . So, this is our governing equation for getting our principle stresses. So, if you expand this is Sigma xx - Sigma multiplied by Sigma yy - Sigma - sigma xy square is 0 and so, after simplifying your sigma 1 and sigma 3.

The sigma 1 is the major principle stress and sigma 3 is the minor principle stress that is equal to Sigma sorry I think it should be + or minus. So, here our major and minor principle stresses can be determined and as the roots of this equation. And this is our familiar equation it is actually it is a quadratic equation and we know how to get the roots and we can draw the Mohr Circle and get our principal stresses and then the directions of the of these principal planes.

(Refer Slide Time: 31:00)

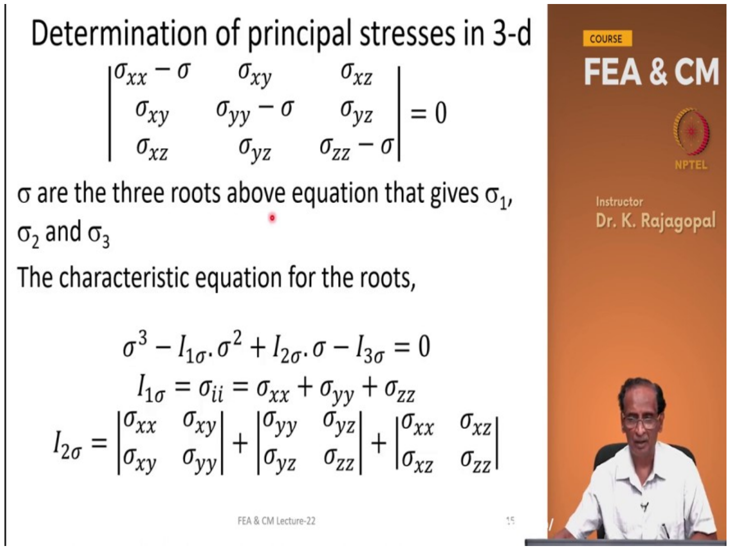


So, our Sigma xy sorry Sigma yy and sigma xx let us say our Sigma yy is more than the sigma xx and then the Tau xy acting on the on the horizontal plane where Sigma yy is is the normal stress is positive. So, this is positive and the Tau xy that is acting on the on the vertical plane is taken as negative. So, we get the we get 2 points and we can connect both of them with a straight line and this is your center point are the mean normal stress Sigma xx + Sigma yy by 2.



And if you draw a circle we know that we get your major principle stress here and sigma 3 is the minor principle stress and sigma 1 1 Sigma 1 is basically Sigma xx + Sigma yy by 2 + the radius and sigma 3 is the mean stress minus the radius and we get the at the same equation as shown here. So, this is actually it is a very convenient method like we can draw the Mohr Circle and then determine our major and minor principal stresses.

(Refer Slide Time: 32:38)



So, now let us see the the 3 dimensional stress state and how to determine the 3 principle stresses. So, we can set up the same equation the determinant of this Matrix is 0 Sigma xx - Sigma Sigma yy - Sigma Sigma zz - Sigma the determinant is 0. And this procedure is the same as how we find the eigen values in the dynamics. So, the sigma are the 3 roots of the above equation that give Sigma 1, Sigma 2 and sigma 3.

Determination of principal stresses in 3-d

$$\begin{vmatrix} \sigma_{xx}-\sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma \end{vmatrix} = 0$$

σ are the three roots above equation that gives σ_1, σ_2 and σ_3

The characteristic equation for the roots,

$$\sigma^3 - I_{1\sigma}.\sigma^2 + I_{2\sigma}.\sigma - I_{3\sigma} = 0$$

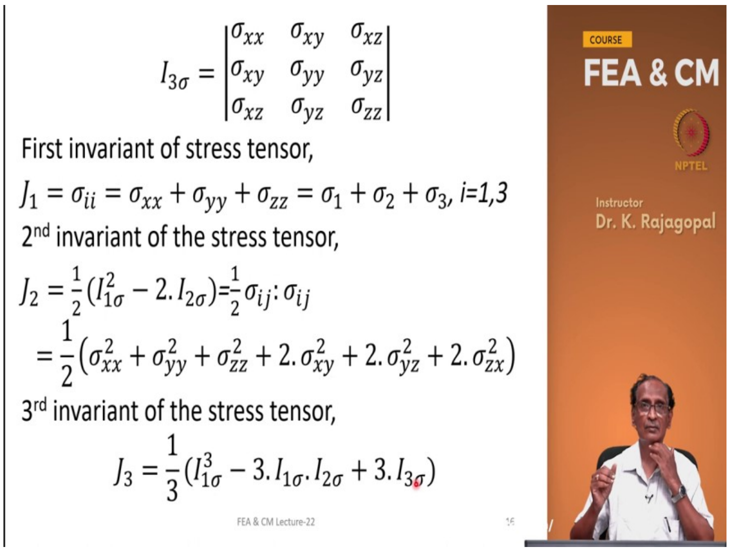
$$I_{1\sigma} = \sigma_{ii} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_{2\sigma} = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix}$$

And we can expand and write a characteristic equation like this it is a cubic equation Sigma cube - I 1 Sigma Sigma square + I 2 Sigma times Sigma - I 3 Sigma is 0. And here I 1 Sigma is the sum total of the 3 normal stresses Sigma II is actually in the if you have an index notation with a repeated index means it is a summation and I varying from one to 3. And in 3 dimensions I varies from one to 3 and in 2 dimensions it varies from one to 2.

Sigma II basically means Sigma xx + Sigma yy + Sigma zz and I 2 Sigma is actually its determinant of the sub matrices the sum total because we have 3 coordinates. So, we have 3 determinants like this.

(Refer Slide Time: 34:22)



Then I 3 sigma is the determinant of our stress tensor and if we get these 3 quantities I one Sigma I 2 Sigma and I 3 sigma. We can now write our invariance of the stress tensor J 1 is a sigma I I that is Sigma xx + Sigma yy + Sigma zz. And that is also equal to the sum total of the 3 the 3 principle stresses Sigma 1 + Sigma 2 + Sigma 3 and the second invariant of the stress tensor J 2 is one half of I one Sigma Square - 2 times I 2 Sigma where I 2 Sigma is the is the sum total of the 3 sub matrices determinants of the 3 sub matrices.

$$I_{3\sigma} = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}$$

First invariant of stress tensor,

$$J_1 = \sigma_{ii} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3, \; i=1,3$$

2nd invariant of the stress tensor,

$$J_2 = \frac{1}{2}(I_{1\sigma}^2 - 2.I_{2\sigma}) = \frac{1}{2}\sigma_{ij}:\sigma_{ij}$$

$$= \frac{1}{2}\left(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 + 2.\sigma_{xy}^2 + 2.\sigma_{yz}^2 + 2.\sigma_{zx}^2\right)$$

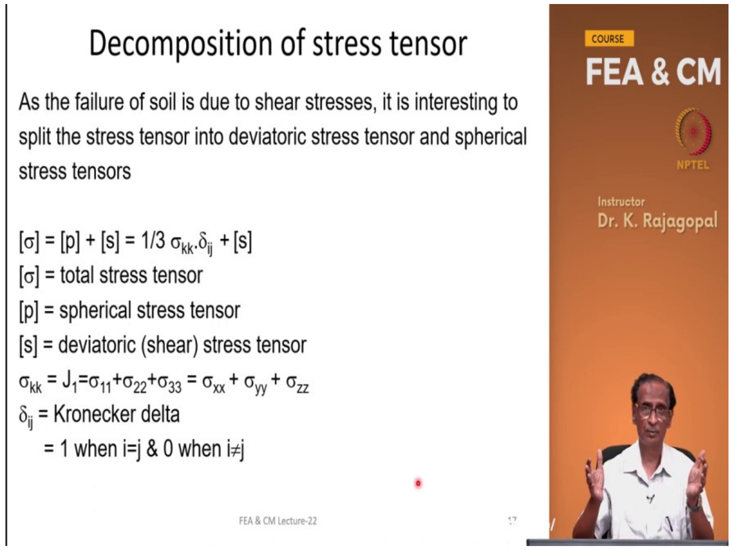
3rd invariant of the stress tensor,

$$J_3 = \frac{1}{3}(I_{1\sigma}^3 - 3.I_{1\sigma}.I_{2\sigma} + 3.I_{3\sigma})$$

And that we can also write as one half the product of Sigma ij and sigma ij the term by term product like this one half of Sigma xx square + Sigma yy square + Sigma zz square + 2 times Sigma xy Square because we have 2 Sigma xy's similarly to Sigma yz and 2 Sigma xz xz dot zx both are the same and the third invariant of the stress tensor J 3 is one third of I 1 Sigma Cube - 3 times I one Sigma I 2 Sigma + 3 times I 3 sigma this is these are the invariants of this stress tensor.

And this stress tensor if you apply this on a soil would it cause any failure or not see that depends on the predominance of the shear because our the soil will fail only at the shear stress and if you apply compressive stresses by pure compression the soil will have Infinite Strength In fact it is. So, high that we do not really bother defining the compressive strength of the soil.

(Refer Slide Time: 36:51)



And similarly we do not Define any tensile strength because we know that it is a negligible but it has some finite Shear strength. So, we talk in terms of the shear strength of the soil. Say the failure of the soil is due to Shear stresses it is interesting to split our stress tensor into 2 parts one is a deviatoric part and the other is the spherical stress tensor, spherical stress tensor that is also called as the hydrostatic pressure state.

$[\sigma] = [p] + [s] = 1/3\ \sigma_{kk}.\delta_{ij} + [s]$

$[\sigma]$ = total stress tensor

$[p]$ = spherical stress tensor

$[s]$ = deviatoric (shear) stress tensor

$\sigma_{kk} = J_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$

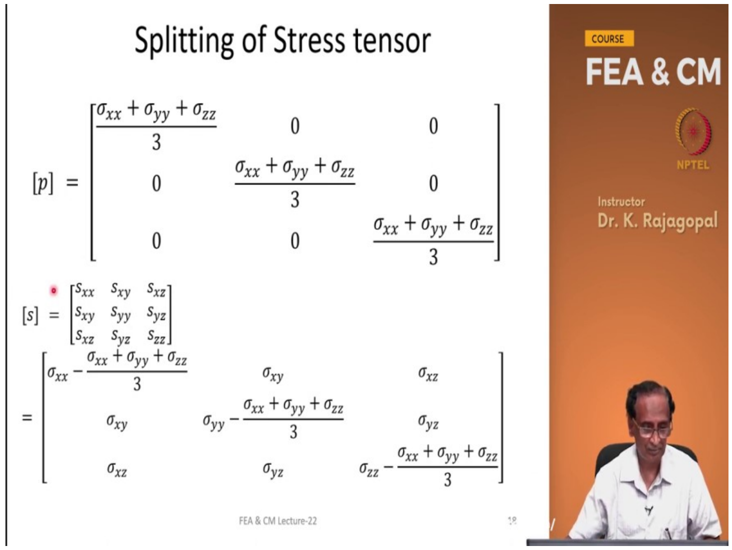
δ_{ij} = Kronecker delta

= 1 when i=j & 0 when i≠j

And the hydrostatic pressure state is when all the 3 normal pressures are the same. So, Sigma we can write as p + s, p is the hydrostatic pressure s is the is the deviatoric stress tensor and the p is one third Sigma kk Delta i j Sigma k is basically it is a its repeated index. So, it is Sigma xx + Sigma yy + Sigma zz or I can call it as a mean normal stress. Delta ij is a is the kronecker Delta and it is one when i is equal to j and 0 when i is not equal to j if in fact Delta ij is the is our diagonal matrix with a unit value.

And our p is basically a stress tensor with the mean normal stress along the along the diagonal and then our s is the deviatoric stress tensor.

(Refer Slide Time: 38:48)



Splitting of Stress tensor

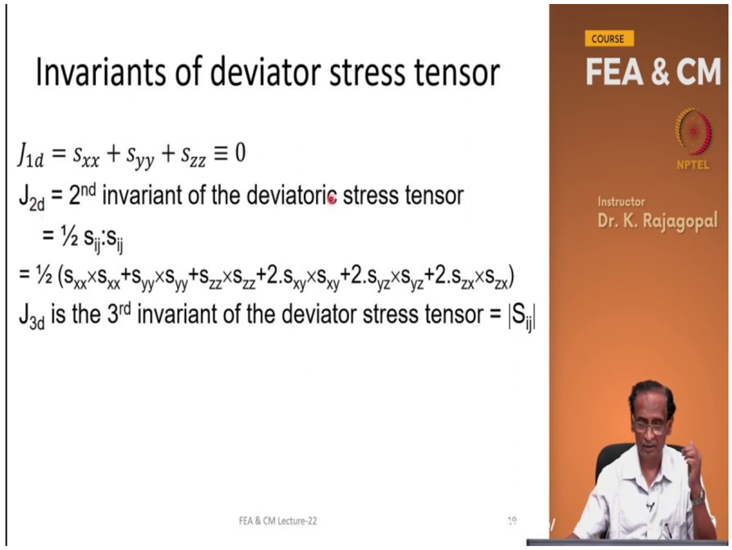
$$[p] = \begin{bmatrix} \dfrac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} & 0 & 0 \\ 0 & \dfrac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} & 0 \\ 0 & 0 & \dfrac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} \end{bmatrix}$$

$$[s] = \begin{bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{xy} & s_{yy} & s_{yz} \\ s_{xz} & s_{yz} & s_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{xx} - \dfrac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \dfrac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \dfrac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} \end{bmatrix}$$

And our p are the spherical stress tensor is the mean normal stress and we have along the diagonal we have the same the quantities Sigma xx + Sigma yy + Sigma zz by 3 0 0 0 everywhere all the off diagonal terms are 0 and the deviatoric stress tensor is is s xx s xy s zz and so on. And that is equal to to total stress tensor - the p. And so, this is Sigma xx - J 1 by 3 Sigma xy Sigma xz and then Sigma yy - J 1 by 3 where J 1 is the first invariant of the stress tensor that is Sigma xx + Sigma yy + Sigma zz same thing with the third quantity Sigma zz - J 1 by 3.

(Refer Slide Time: 39:48)



And we can similar to the invariance for the total stress tensor we can write the invariance for the deviatoric stress tensor J 1 D is s xx + s yy + s z z the sum total of the 3 normal deviatoric stresses. And if you look at this these are the 3 normal deviatoric stresses. So, Sigma xx - J 1 by 3 sigma yy - J 1 by 3 Sigma zz - J 1 by 3. So, if you add them up you get Sigma xx + Sigma yy + Sigma zz - 3 times J 1 by 3 that is J 1.

$$J_{1d} = s_{xx} + s_{yy} + s_{zz} \equiv 0$$

J_{2d} = 2nd invariant of the deviatoric stress tensor

$$= \frac{1}{2} s_{ij}:s_{ij}$$

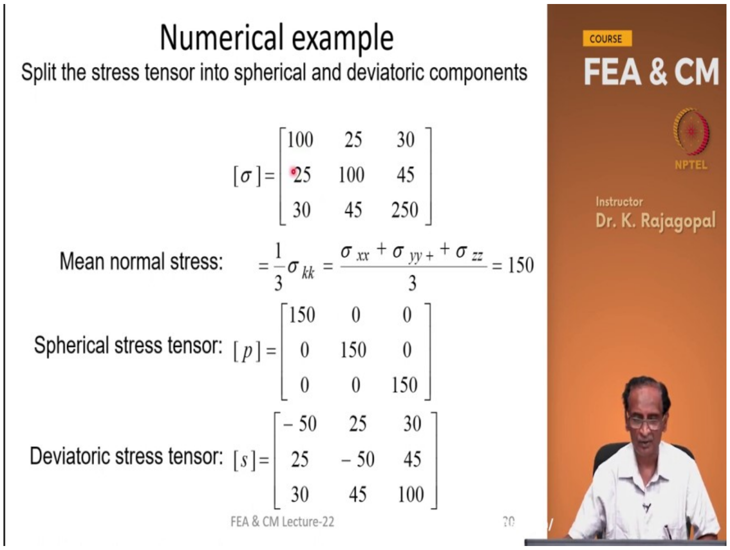
$$= \frac{1}{2} (s_{xx} \times s_{xx} + s_{yy} \times s_{yy} + s_{zz} \times s_{zz} + 2.s_{xy} \times s_{xy} + 2.s_{yz} \times s_{yz} + 2.s_{zx} \times s_{zx})$$

J_{3d} is the 3rd invariant of the deviator stress tensor = $|S_{ij}|$

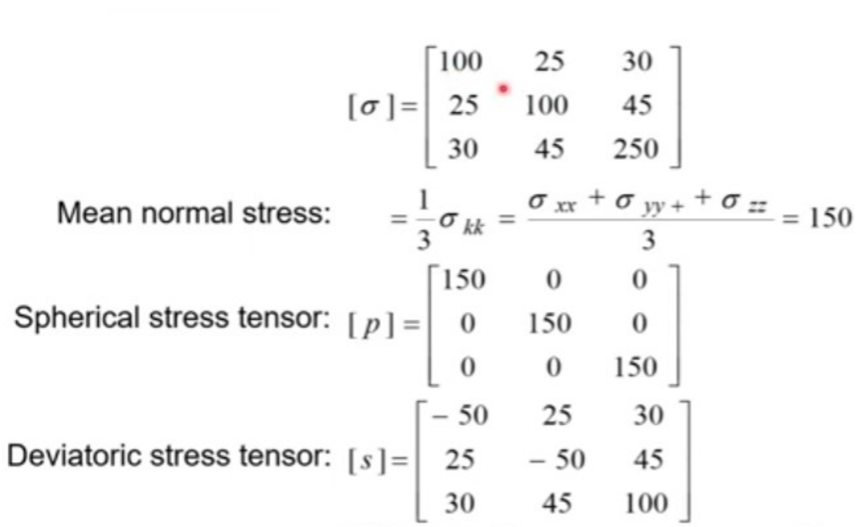
And the J 1 is nothing but Sigma xx + Sigma yy + Sigma zz. So, at the J 1 D is 0 and this subscript D indicates corresponding to the deviator stress tensor J 1 is the invariant for this stress tensor the total stress tensor whereas J 1 D is the invariant for the for the deviatoric stress tensor and J 2 D is the second invariant of the deviatoric stress tensor that is one half of s ij multiplied by s ij and then the J 3D is the third invariant of this stress tensor that is equal to the determinant of the of the deviatric stress tensor.

In fact if you look at these equations the the J 2 d is basically I 1 Sigma is 0 for the deviatoric stress tensor and you are left with only I 2 Sigma because the 2 gets canceled out and the I 2 Sigma is is our is this this quantity ok and and then the j3 i1 Sigma is 0 for the deviated stress tensor. So, you are left with only I 3 sigma I 3 sigma is the determinant and for the deviatoric stress tensor this is just the determinant of the of the the deviatoric stress tensor.

(Refer Slide Time: 42:28)



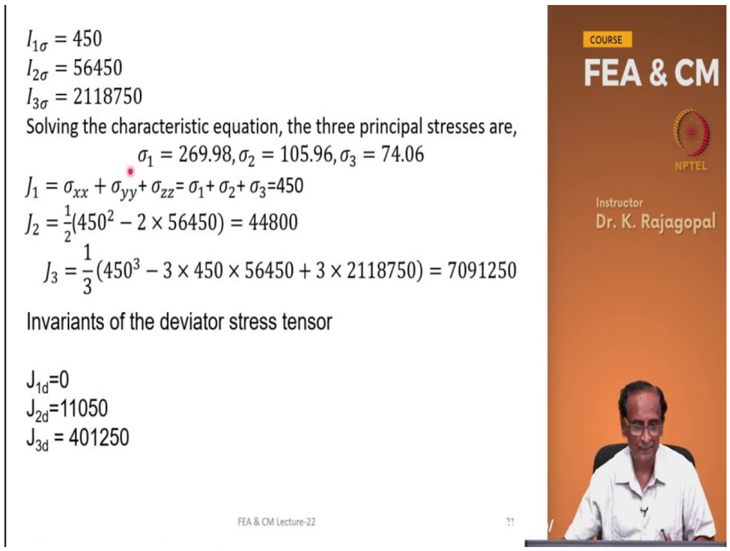
Let us look at a small numerical example let us take a stress state where our Sigma x and sigma y are the same normal stresses are the same in both x and y directions and then Sigma zz is 250 and then we have some normal stresses. And the mean normal stress that is one third of Sigma kk that is 100 + 100 + 250 there is 450 by 3 that is 150 and actually even before you calculate you can also say that the sum total of the 3 principles stress is Sigma 1 + Sigma 2 + Sigma 3 is equal to 450.



Because that is an invariant and that is not going to change. So, once you get your mean normal stress as 150 the spherical stress tensor p is 150 150 150 and the deviatoric stress tensor is Sigma - p that is -50 25 30 25 -50 45 30 45 and 100. So, this is your deviatoric stress tensor and this is your spherical stress tensor. And the spherical stress tensor is representing hydrostatic compression.

And if you apply compressive stress the soil will never fail and your failure will not happen because of your spherical stress tensor the failure will happen only because of the deviatoric stresses.

(Refer Slide Time: 44:19)



So, our I 1 Sigma is 450 that the sum total of the 3 normal stresses I 2 Sigma is 56450 and then I 3 Sigma is 218715 and we can solve our cubic equation to get our 3 principal stresses. So, most of the calculators have the facility to solve a cubic equation and you can easily set up this matrix and then find the roots and then the determinant and all the all the quantities that you require for doing these computations.

$I_{1\sigma} = 450$
$I_{2\sigma} = 56450$
$I_{3\sigma} = 2118750$
Solving the characteristic equation, the three principal stresses are,

$$\sigma_1 = 269.98, \sigma_2 = 105.96, \sigma_3 = 74.06$$

$J_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3 = 450$

$J_2 = \frac{1}{2}(450^2 - 2 \times 56450) = 44800$

$$J_3 = \frac{1}{3}(450^3 - 3 \times 450 \times 56450 + 3 \times 2118750) = 7091250$$

Invariants of the deviator stress tensor

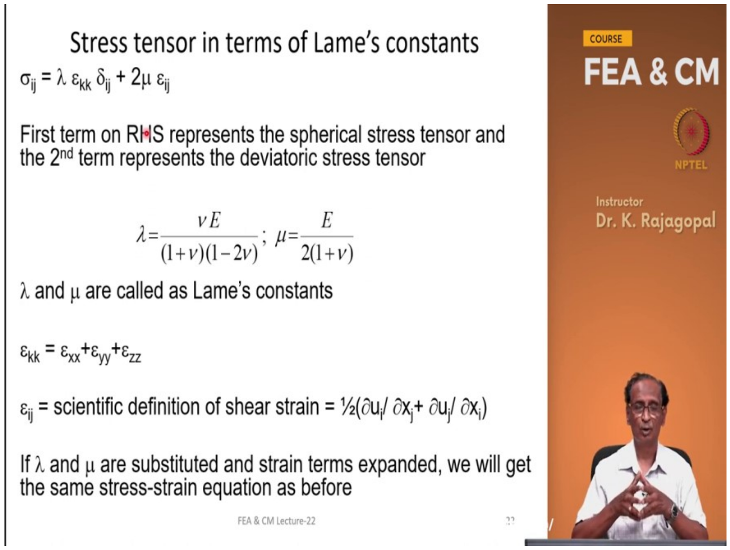
$J_{1d} = 0$
$J_{2d} = 11050$
$J_{3d} = 401250$

So, if you do that Sigma 1 is a 269.98 sigma 2 is this and sigma 3 is this Sigma 1 is our major principles list Sigma 2 is the minor principle stress Sigma 3 sorry a sigma 2 is the intermediate principle stress and sigma 3 is our minor principle stress. And J 1 is the sum total of the 3 diagonal terms or Sigma 1 Sigma 2 and sigma 3 that is 1 450 and J 2 is one half of I 1 Sigma Square - 2 times I 2 Sigma that is 44800 and J 3 is 1 1 3 of I 1 Sigma Square - 3 times I 1 Sigma times I 2 Sigma + 3 times I 3 Sigma.

I 3 sigma is the determinant of that stress Matrix or this stress tensor sorry and that is this much. And similarly we can take the deviatoric stress tensor and determine the invariance J 1d that is the s xx + s yy + s zz that is equal to 0, J 2 D is is 11050 and J 3DS 401250 is actually J 2D is just simply one half of the term by term product of the deviatoric stress tensor terms. And the answers are given so, you can work out on your own.

Because all the equations are given and then the stress tensors are given this is this and J 2 D is one half of 50 square + 50 square + 100 square + 2 times 25 Square 2 times 30 square + 2 times 45 Square. And if you do that correctly you will get this and then the J 3D is the determinant of this Matrix.

(Refer Slide Time: 47:33)



So, around the time of when Hook proposed his generalized stress strain equations there was another mathematician by name Lame he proposed a similar equation like similar to what we have seen just now. The stress tensor being equal to the to the some of the spherical stress tensor and then the deviatoric stress tensor. He has given an equation in terms of Lambda and mu which are called as Lama's parameters.

$$\sigma_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + 2\mu\, \varepsilon_{ij}$$

And the first term here represents our spherical stress tensor. Lambda is is some parameter that relates to the bulk modulus and Epsilon KK is our volumetric strain Epsilon xx + Epsilon yy + Epsilon zz and our Epsilon ij is we will see that that is that is we call as the engineering Shear strain Lambda and mu are called as Lames constants and our Epsilon x sorry Epsilon kk is Epsilon xx + Epsilon yy + Epsilon zz.

And Epsilon ij is the scientific definition of Shear strain that is one half of dou u by dou x + now dou u j by dou x i are basically it is dou u by dou x and dou sorry a dou you actually you see this index I here and J there dou u i by dou x j + dou u j by dou x i that is the the cross differentiation like dou u by dou y + dou v by dou x and this is half the average of the of the change in the right angle is called as the as the shear strain in the scientific definition.

λ and μ are called as Lame's constants

$$\varepsilon_{kk} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

ε_{ij} = scientific definition of shear strain = $\frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$

And the engineering definition is the change in the right angle because of some Distortion ok and our Lambda and mu are given like this Lambda is a new e by one + Nu times 1 - 2 Nu and mu is our Shear modulus e by 2 times 1 + Nu. And this Lambda is nothing but our constrained modulus that we; derived long back in one of the classes the constraint modulus are the odometer modulus.

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}; \mu = \frac{E}{2(1+\nu)}$$

And if we substitute this Lambda and mu in this equation we can get back our relation between the stresses and strains in terms of that six by six Matrix we will get back the same stress strain equation. So, this is actually this is what we have done when we decompose the total stress tensor into a deviatoric part and then the spherical part. And do not confuse the total stress tensor here with the total stresses that we have in the in the soil mechanics.

$\varepsilon_{kk} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$

ε_{ij} = scientific definition of shear strain = $\frac{1}{2}(\partial u_i / \partial x_j + \partial u_j / \partial x_i)$

If λ and μ are substituted and strain terms expanded, we will get the same stress-strain equation as before

Because here the total stress tensor is the sum total of the spherical stress tensor plus the deviatoric stress tensor whereas in soil mechanics the total stress is effective stress plus the pore pressure and your pore pressure is a similar to your spherical stress sensor and then the effective stresses could be similar to your deviator stress sensor. So, I think this may be my last slide.

So, thank you very much and if you have any questions please write to me at this address and then I will respond back to you. So, thank you very much we will meet in the next class.