FEM and Constitutive Modelling in Geomechanics

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Lecture: 25
Mapped Infinite Elements for Semi-Infinite Soil Medium

So, good morning students in the previous class we had seen the modelling of the interface elements and in today's class let us look at how we can model the semi-infinite nature of the soil. Because as you know the soil is a semi infinite in extent wherever you go there is soil then how do we accommodate that in our analysis because we are basically doing finite element analysis with some finite domain like finite length and finite height and so on.

But then how do we represent the semi-infinite nature and that is what we are going to see in today's class.

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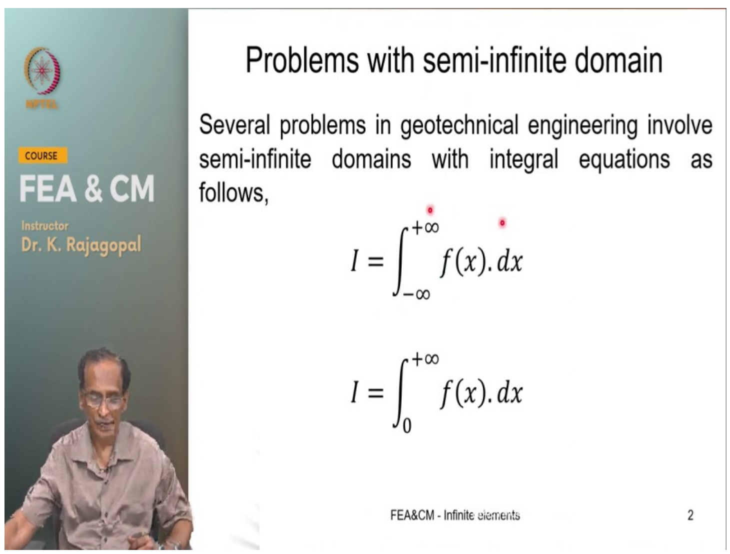

So, when we deal with many of the geotechnical problems we come across this type of integrals integral of function integrated from - infinity to + infinity 0 to infinity and so on. So, when we do this analytically we can easily handle but then the moment you mention the word infinity to any numerical program then they get jittery they will just simply collapse they will say that no we give up.

$$I = \int_{-\infty}^{+\infty} f(x).\, dx$$

$$I = \int_{0}^{+\infty} f(x).\, dx$$

And so, we need to find some other way and we cannot really substitute any Infinity in any of our parameters whether it is the Young's modulus or the limits and so on.

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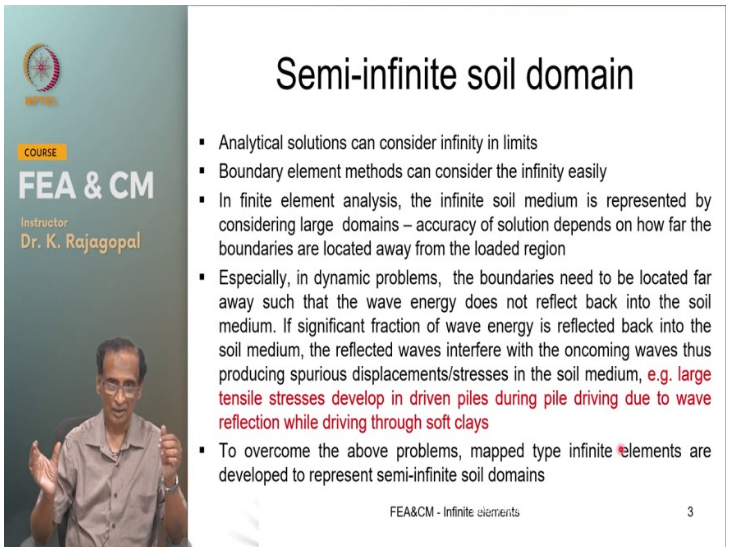

So, our say the analytical solutions that we have seen they can consider. So, you might have derived several Solutions in the elasticity and also even in within the geotechnical engineering we deal with. So, many things but cannot do them numerically and the boundary element method that is that is actually it is not similar to finite element but it is another numerical method where we consider the boundary.

And we can represent either the outside of the boundary or inside of the boundary and we can apply this boundary element methods because if you say outside of your boundary everything. Everything up to Infinity that is automatically considered in the analysis or some people they consider the finite element analysis but then prescribe some known displacements along the boundary.

These could be from your theory of theory of elasticity solutions and by prescribing these known displacements at the boundary we may be able to partially account for semi-infinite nature of the of the soil but then this may not be practicable in all the problems maybe in some cases we can get the solution but we cannot do that in all the cases. And some in investigators they tried combining finite elements and then the boundary elements.

Because the boundary elements are good in representing the boundary problems and the finite elements are good within the finite region. And so, when we deal with any dynamics problems the boundary plays a very important role. So, if you place the boundary too far then your computational effort could be very high because instead of having let us say 1000 degrees of freedom you may be having 10000 degrees of freedom because you have additional number of elements.

But if you place the boundary too close also you will have a problem you will have some numerical influence like your waves may be propagating once they reach the the resident they will reflect back and then your oncoming way and the reflected wave might interact. And they might they might produce some other extraneous stress condition within the soil and that is also not correct.

See one simple example that I can give you about the reflected wave is. So, when you have a driven pile you know that you are giving energy at the top and if the soil is strong enough it will absorb the energy and then it will allow the if the energy that you are imparting to the pile is much higher than the resistance then the pile will be driven into the soil. And if you are say when you are driving and you give a large energy and suddenly let us say you come into a soft clay.

And the soft clay obviously it will not be able to absorb the energy that is coming from the pile. So, it will get reflected back then that will reflect in the tensile stresses generated. So, if your waves are moving apart like let us say on coming wave and then the reflected wave if they move apart they will have they will produce tension . And so, if you see the is code for the driven piles they recommend a very large percentage of Steel almost 2 percent of the cross section whereas for a board pile it could be just a nominal reinforcement for the normal shrinkage purposes.

So, this additional steel in this the concrete of the driven piles is because of these extraneous stresses that that you could generate. And the same thing happens even in the case of finite element analysis of our geotechnical problems. And so, we need to take the boundary very very far away and to overcome that now we have this map type infinite elements that can represent infinite soil medium by using one simple element that we will see in this class today.

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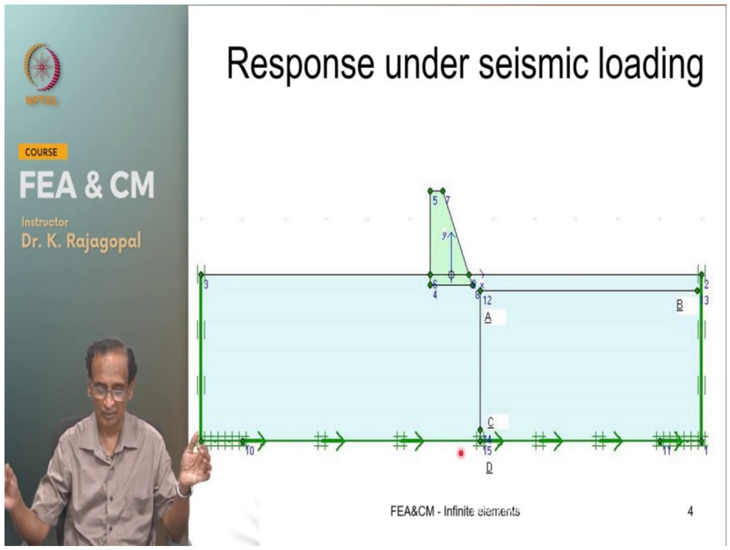

And just to illustrate I have a small problem like let us say we have a dam and that is resting on the soil and then we are giving some seismic excitation and there are some moving boundaries or viscous boundaries. And let us monitor the deformations at this point a very close to the dam.

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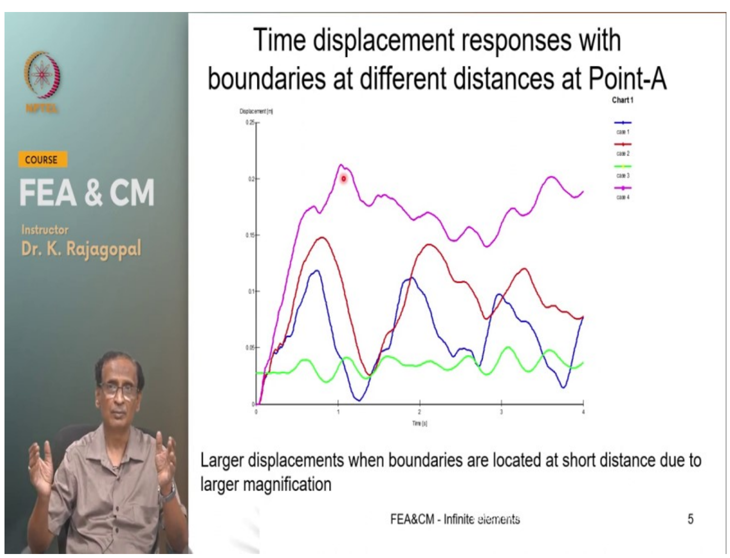

And these are all the results that we get by locating the boundary at different locations. The green line is when your boundaries at a far off location and then the blue red and then this pink line they are all with different distances and you see your numerical result does not; it not only depends on your properties but also on the boundaries. So, if you place the boundaries too close then your displacements are very high.

And as you move the boundaries far away then the result may be more appropriate corresponding to loading whereas with boundaries too close your results may be affected by oncoming wave and then the reflected wave. So, because of this you need to be very careful whenever we are doing this type of dynamics problems where we have some wave propagation we need to do something.

And one method is that by providing observing boundaries but absorbing boundaries also may not be effective in some cases. So, we can go in for these infinite elements that can represent the infinite soil medium.

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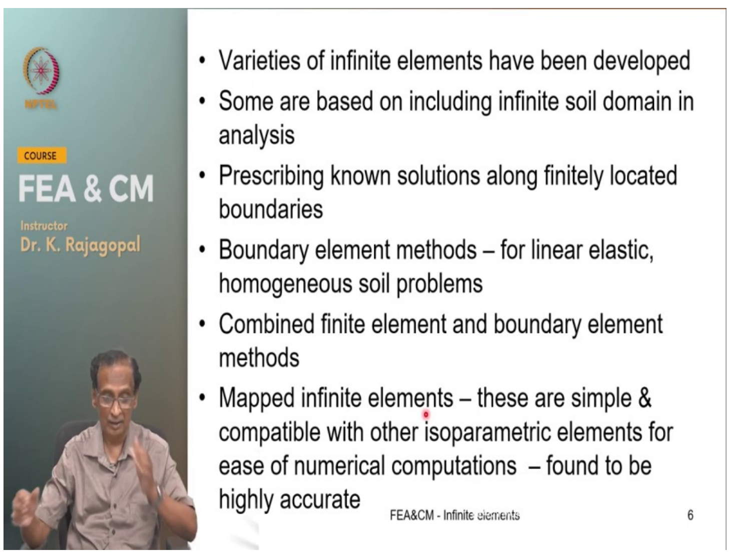

So, in the recent past almost about 40 years varieties of infinite elements have been developed. Some are based on including infinite soil domain in the analysis. We can define with very very large displacement or sorry the distance and we can prescribe some known solutions at some finite distance so, that we can represent the infinity or we can combine boundary element and finite element.

And then the more recently map type infinite elements and these elements are very simple and also these are also isoparametric. So, they are compatible with the other finite elements that we have developed. So, they are easy to to implement and then we do not need to do much changes in the eXisting computer programs to implement this map type infinite elements.

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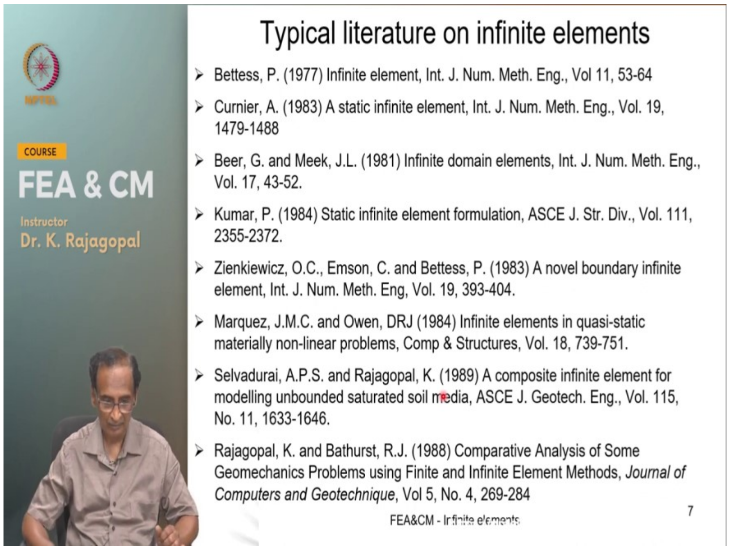

See these are all some of the papers can go through for some literature on the infinite elements and of course there are many more number of papers that I did not list here but I have just listed some more some very prominent papers.

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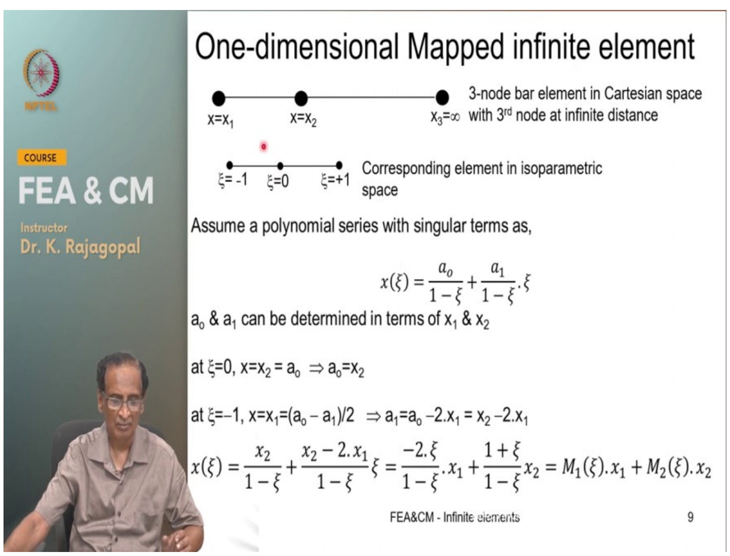
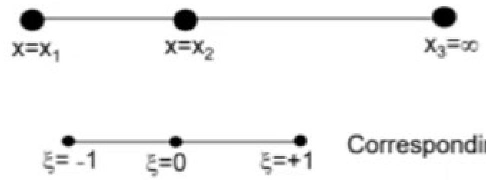


Let us look at one dimensional map infinite element to start with because that is the simplest one. And let us take a 3 node bar element an aXial element with the 2 nodes at some finite distances x 1 and x 2 and let us say the third node is at infinite distance.

3-node bar element in Cartesian space with 3rd node at infinite distance



Corresponding element in isoparametric space

x 1 and x 2 these are the finitely located nodes and then x 3 is at infinite distance this is the Cartesian space. But then the corresponding isoparametric element will be only in this space of - 1 to + 1.

So, we have 3 nodes one at Xi of - 1 are to 0 and Xi of + 1 and the node at Xi of + 1 is the one at infinite distance. And so, how do we simulate that in our numerical analysis. So, we can actually develop some mapping functions that are singular at Xi of + 1 and we can do the mapping in terms of these 2 nodes that are located at finite distances. So, when we develop our polynomial equations we can develop them with a singularity at Xi of + 1.

As the 3rd node is at infinite distance, the mapping functions are developed based on the two finitely located nodes .. singular polynomial terms are assumed with singularity at $\xi=+1$ to represent the 3rd node at infinite distance, denominator could be $(1-\xi)$, $(1-\xi)^2$, $\sqrt{(1-\xi)}, \sqrt{(1-\xi^2)}$, etc.

So, that we can represent the infinite distance at the third node and this Singularity we can achieve by having in the denominator 1 - Xi r 1 - Xi whole Square R square root of 1 - Xi r square root of 1 - Xi square and so on. like there could be anything like any combination such that when Xi is + 1 you should have Singularity that is you should have 0 the denominator so, that the value will blow up. It will go up to Infinity.

Assume a polynomial series with singular terms as,

$$x(\xi)=\frac{a_o}{1-\xi}+\frac{a_1}{1-\xi}.\xi$$

a_o & a_1 can be determined in terms of x_1 & x_2

at $\xi=0$, $x=x_2=a_o \Rightarrow a_o=x_2$

at $\xi=-1$, $x=x_1=(a_o-a_1)/2 \Rightarrow a_1=a_o-2.x_1=x_2-2.x_1$

$$x(\xi)=\frac{x_2}{1-\xi}+\frac{x_2-2.x_1}{1-\xi}\xi=\frac{-2.\xi}{1-\xi}.x_1+\frac{1+\xi}{1-\xi}x_2=M_1(\xi).x_1+M_2(\xi).x_2$$

And one of the simplest ones is by using a polynomial like this let us say you have 2 finitely located nodes at x 1 and x 2 and we can write the x as a function of a naught and a one these are the generalized coordinates a naught by 1 - Xi + a one by 1 - Xi times Xi. And purposely we had made them singular at Xi of + 1 because we want Infinity okay and we can determine a naught and a one in the same manner as how we determine them earlier by substituting Xi of - 1 our x is x 1 and at Xi of 0 our x is x 2.

So, by going through we get that a naught is x 2 OK that is when you substitute Xi of 0 our x is simply a naught that is equal to x 2 a naught is x 2 and then we see that a one is x 2 - 2 x 1 and our x of Xi we can write as x 2 that is a naught by 1 - Xi + x 2 - 2 x 1 by 1 - Xi times Xi. And by grouping these under x 1 and x 2 we can get this M 1 can be written as - 2 Xi by 1 - Xi and M 2 can be written as 1 + Xi by 1 - Xi and we can write our x as M 1 x 1 + M 2 x 2 where M1 and M2 are our mapping functions.

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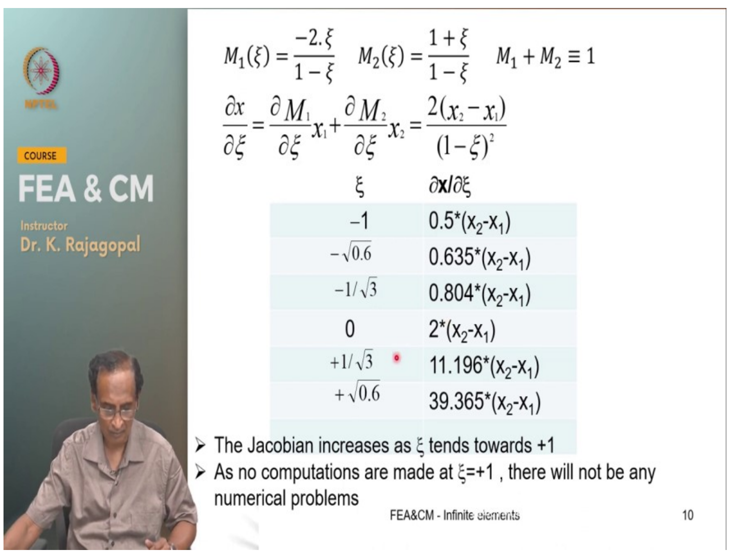

And our M 1 is - 2 Xi by 1 - Xi and M 2 is 1 + Xi by 1 - Xi and if we sum up these 2 mapping functions we get exactly one and M 1 + M 2 of 1 means there is a unique mapping unique mapping between the Cartesian space and then the isoparametric space. And so, and then at Xi of + 1 we have infinite distance and let us calculate the dou x by dou Xi that is our Jacobian determinant for the one dimensional problems.

$$M_1(\xi) = \frac{-2.\xi}{1-\xi} \quad M_2(\xi) = \frac{1+\xi}{1-\xi} \quad M_1 + M_2 \equiv 1$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial M_1}{\partial \xi} x_1 + \frac{\partial M_2}{\partial \xi} x_2 = \frac{2(x_2 - x_1)}{(1-\xi)^2}$$

ξ	$\partial x/\partial \xi$
-1	$0.5^*(x_2\text{-}x_1)$
$-\sqrt{0.6}$	$0.635^*(x_2\text{-}x_1)$
$-1/\sqrt{3}$	$0.804^*(x_2\text{-}x_1)$
0	$2^*(x_2\text{-}x_1)$
$+1/\sqrt{3}$	$11.196^*(x_2\text{-}x_1)$
$+\sqrt{0.6}$	$39.365^*(x_2\text{-}x_1)$

And that is equal to dou M 1 by dou Xi x 1 + dou M 2 by dou Xi x 2 and that is 2 times x 2 - x 1 by 1 - Xi whole Square and for finite element we have seen that the dou x by dou xI is equal to the half the length for bar element of some length l the dou x by dou Xi is just simply L by 2. So, at a different Xi values let us calculate this dou x by dou Xi at - 1 it is 0.5 times x 2 - x 1 and that minus of square root of 0.6 is 0.635 times x 2 - x 1.

See these are all the sampling points that we have with the 2 points and then the 3 points at 0 is 2 times x 2 - x 1 and that + 1 by root 3 it is this and at + of square root of 0.6 is about 39 times x 2 - x 1 and actually although we have the singularity at Xi of + 1 we will we will not have any numerical problems because we are not going to use any sampling point at Xi of + 1 and we are also not going to compute any stresses the strains at Xi of + 1.

So, we should not have any technical issues by using this type of singular mapping functions and as we see here as Xi is tending towards one our Jacobian value is increasing rapidly and in fact exponentially beyond Xi of 0. So, that means that our mapping functions with singularity in the in the in the mapping functions they are able to represent semi infinite domain.

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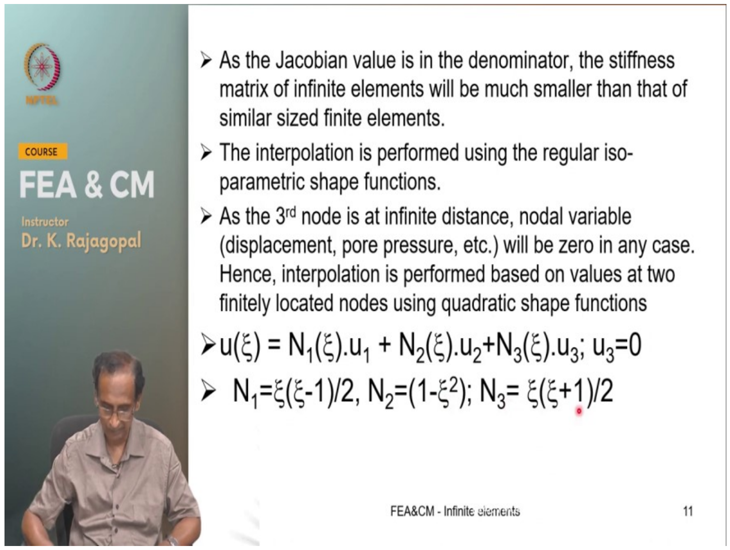

Does the Jacobian value is at the denominator whenever integral B transpose DB calculations are done our determinant of the Jacobian is in the denominator. So, in fact when you calculate the stiffness of any stiffness Matrix of an infinite element we will see that it is of very small value because you are determinant of the Jacobian matrix is in the denominator . And so, for doing the finite element calculations we can use the normal shape functions for interpolation say for interpolation of displacements or the pore pressures we can use the normal shape functions.

And then for forming the B Matrix we can use the singular mapping functions and then as the third node is at infinite distance the nodal variables like your displacement strain stress at the pore pressure everything will be 0. And so, we do not really need to do an interpolation based on that value. And our interpolation for the field values is N 1 u 1 + N 2 u 2 + N 3 u 3 where our N 1, N 2 and N 3 they are our well-known shape functions that we have derived for the for the isoparametric elements.

$$u(\xi) = N_1(\xi).u_1 + N_2(\xi).u_2 + N_3(\xi).u_3;\ u_3 = 0$$
$$N_1 = \xi(\xi-1)/2,\ N_2 = (1-\xi^2);\ N_3 = \xi(\xi+1)/2$$

And then in this case our U 3 is 0 so, in fact we may not be including this in our interpolation I think the reason is because your u 3 is anyway 0. and our N 1 is Xi times Xi - 1 by 2 N 2 is 1 - Xi Square N 3 is Xi times Xi + 1 by 2.

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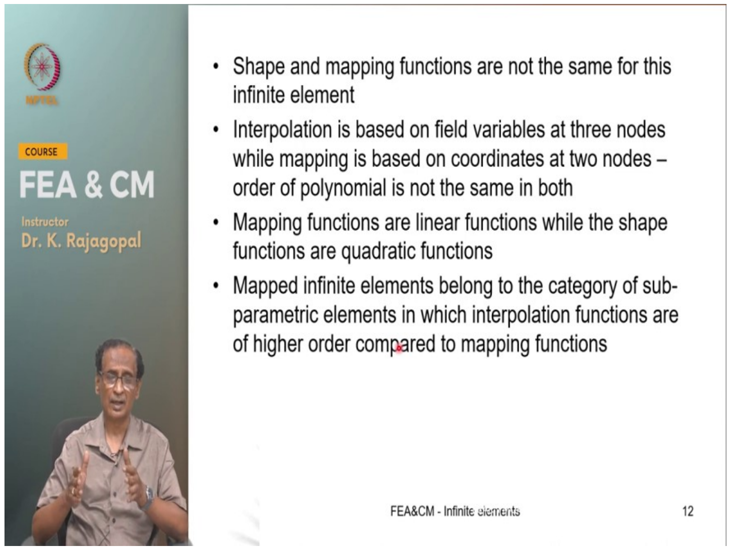

And in this case the shape of mapping functions are not the same we have 2 different sets of functions. See the mapping functions are - 2 Xi by 1 - Xi or 1 + Xi by 1 - Xi whereas our shape functions are Xi times Xi - 1 by 2 and 1 - Xi square and so on. Because we are basically 3 nodes for interpolation and the 2 only 2 nodes for the mapping and the 2 for the mapping we have singular mapping functions whereas the interpolation is done by using our conventional interpolation functions.

So, in this particular case our mapping functions are linear functions because we have used only 2 nodes for our mapping whereas we use the 3 nodes for interpolation. So, we have the quadratic functions a quadratic shape functions and linear mapping functions. So, these mapped infinite elements they belong to the category of sub parametric elements in which the interpolation functions are of higher order compared to the mapping functions.

See in the first class and the isoparametric elements we had seen 3 types of elements isoparametric in which both the mapping and shape functions are exactly the same and then the super parametric where the mapping functions are of higher order compared to the shape functions. And then sub parametric where our shape functions are of higher order comparative mapping functions. So our infinite elements they belong to this sub parametric category.

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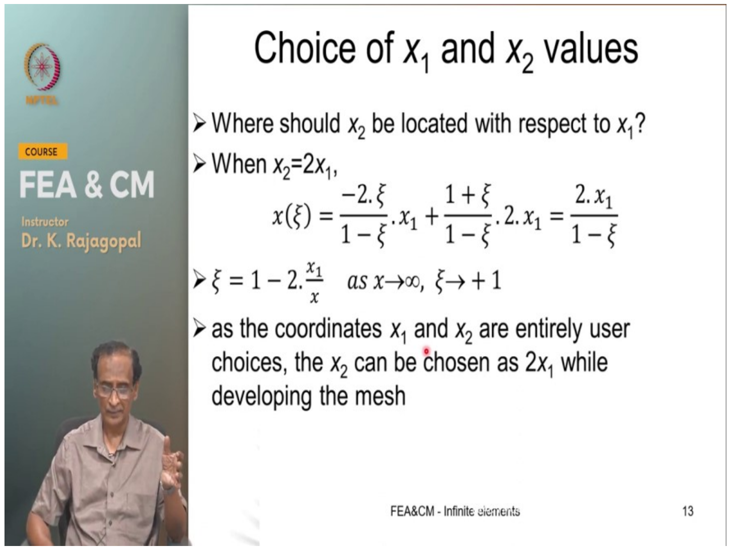

And then another thing that we need to notice is when we had discussed the convergence principles one of the requirements is that our mapping and shape functions should be well defined at all the nodes within the element and but in this case our mapping functions have Singularity at Xi of + 1. So, we are not really satisfying all the requirements of the convergence but then our Sigma of M is g 1.

So, that means that we can get unique mapping and in general once you have the unique mapping that means that you can get the convergence also because our Sigma of N i is one because we are using the conventional shape functions. So, here and we dealt with x 1 and x 2 and infinity and should there be any relation between the x 2 and x 1 let us see whether there needs to be anything see where should we locate x 2 with respect to x 1 this is our general equation for x in terms of Xi and x 1 and x 2.

- Where should x_2 be located with respect to x_1?
- When x_2=2x_1,

$$x(\xi) = \frac{-2.\xi}{1-\xi}.x_1 + \frac{1+\xi}{1-\xi}.2.x_1 = \frac{2.x_1}{1-\xi}$$

- $\xi = 1 - 2.\frac{x_1}{x}$ $as\ x \to \infty,\ \xi \to +1$
- as the coordinates x_1 and x_2 are entirely user choices, the x_2 can be chosen as 2x_1 while developing the mesh

Let us substitute x 2 as 2 x 1 and So our x of PSI is 2 x 1 by 1 – Xi and Xi in terms of the x can be written as 1 - 2 times x 1 by x. So, as x tends to infinity Xi tends to + 1 and then in general we can say that this has one over one by R times d k because our distance is increasing in the ratio of of this this Singularity is in the denominator 1 - Xi says Xi is tending towards one we are reaching the infinite distance in proportion to this denominator value.

So, our coordinates x 1 and x 2 are entirely user choices. So, x 2 is the normally we take as 2 times the x 1. So, that we get this relation that Xi is 1 - 2 times x 1 by x.

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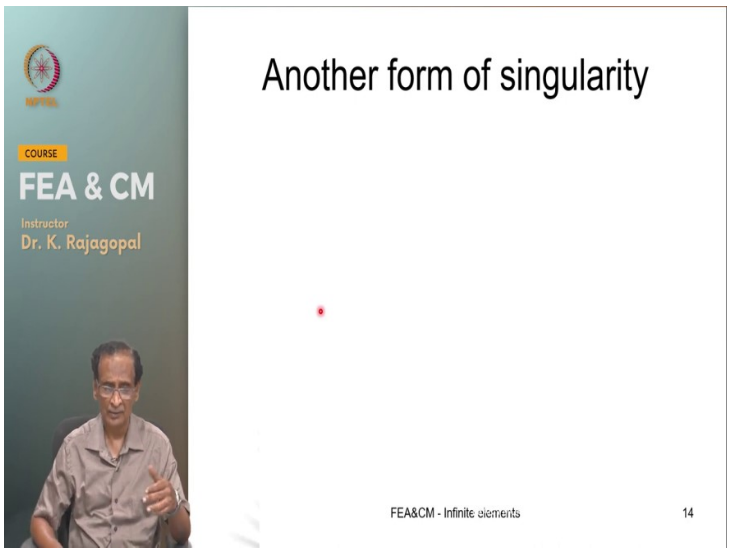

And we can actually there are other forms of singularity I have not explained here but you can refer to published papers there are several papers and I have referred to them in the list. And you will see the shape functions derived or sorry the mapping functions derived for other types of Singularity one by by Xi are one by square root of 1 - Xi are the denominator you may have 1 - Xi square and so on. You can you can see them from the published papers.

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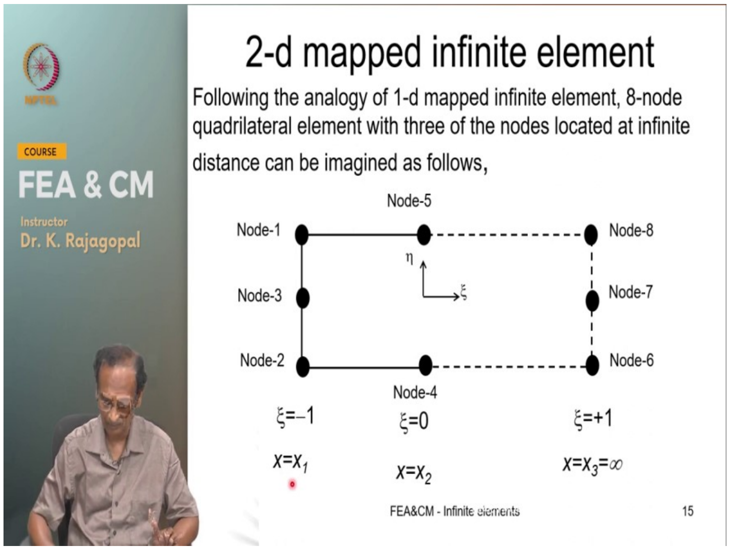
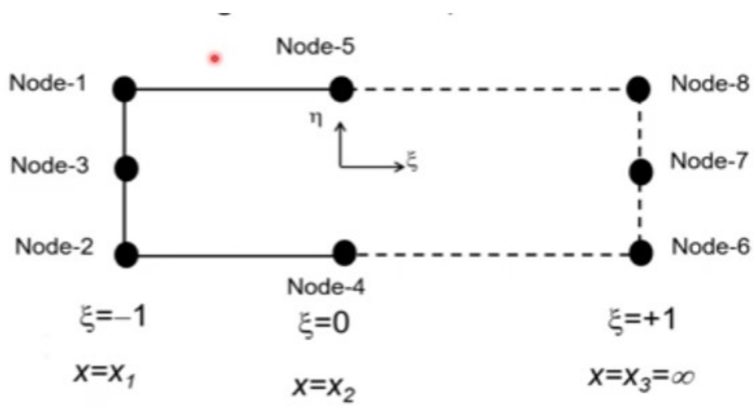


And whatever we derived for one dimensions we can also extend to the 2 dimensions very easily by using our Lagrange method. Let us imagine a night node quadrilateral with the 3 nodes 6 7 and 8 at infinite distance let us define an 8 node quadrilateral like this. Node 1 and node 2 are finitely located and these are the 2 coordinate nodes and nodes 3 4 and 5 these are the mid side nodes at different locations nodes.



1, 2, 3 or at Xi f - 1 and nodes 4 and 5 are at Xi of 0 and the nodes 6 7 8 are at Xi of + 1 and the Xi of - 1 refers to x 1 Xi of 0 refers to x to and usually x 2 is taken as 2 times x 1.

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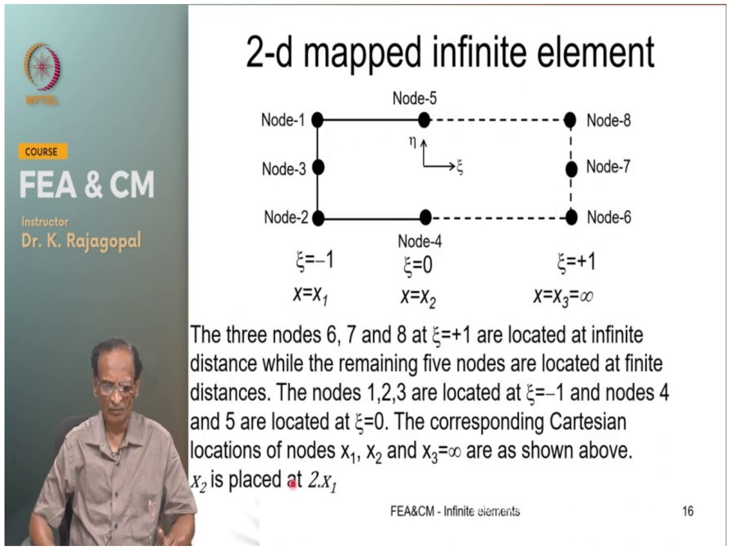

And we can imagine this map infinite element in 2 dimensions and then write our shape functions and mapping functions. So, let us let us look at see our mapping functions in the desired Direction they are developed in terms of the singularity at Xi of + 1. So, our M 1 is - 2 Xi by 1 - Xi and M 2 is 1 + Xi by 1 – Xi. Then in the eta direction our along this line there are 3 nodes.

So, we have quadratic function in the eta whereas along this line we have only linear function 1 - eta by 2 1 + eta by 2 and by using our Lagrange method we can get our mapping functions in terms of Xi and data as the product of these 2 independent functions.

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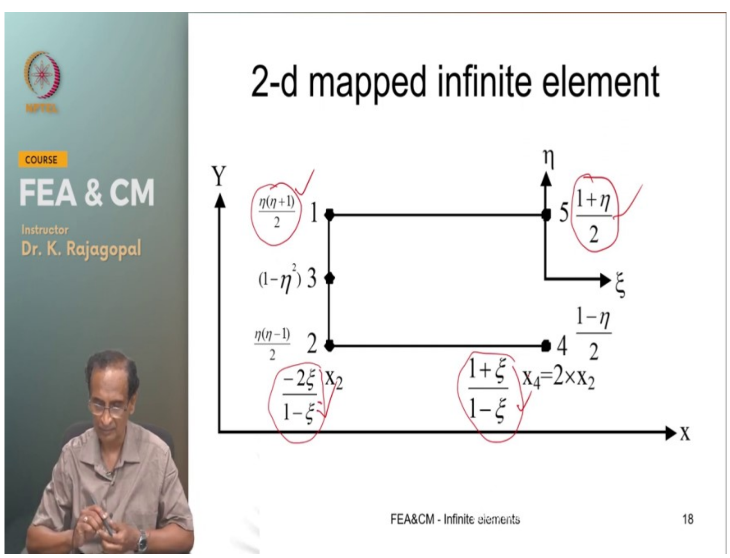
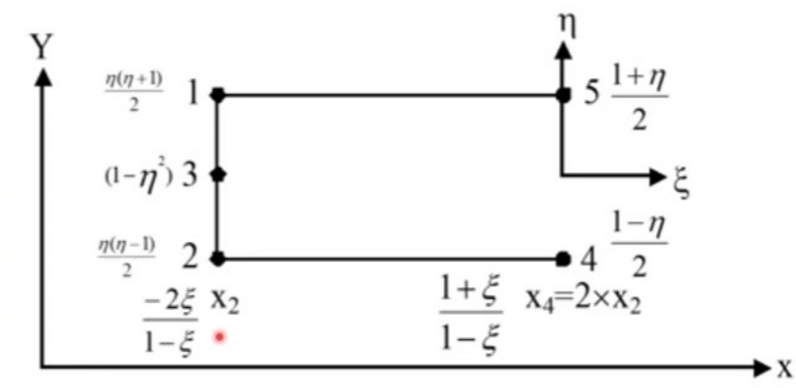


So, our M 1 in terms of Xi eta can be written as - 2 Xi by 1 - Xi multiplied by eta times eta + 1 by 2 that is by looking at this Xi direction these 2 are our mapping functions in the Xi direction these 2 are our mapping functions then the eta direction this is the one. So, if you want for node one we can directly take a product of this and this ok to get our mapping function for node one.



And similarly for getting the mapping function at node 5 we ill use this singular mapping Function One + Xi by 1 - Xi multiply with this to get our mapping function at node 5 you turn it off.

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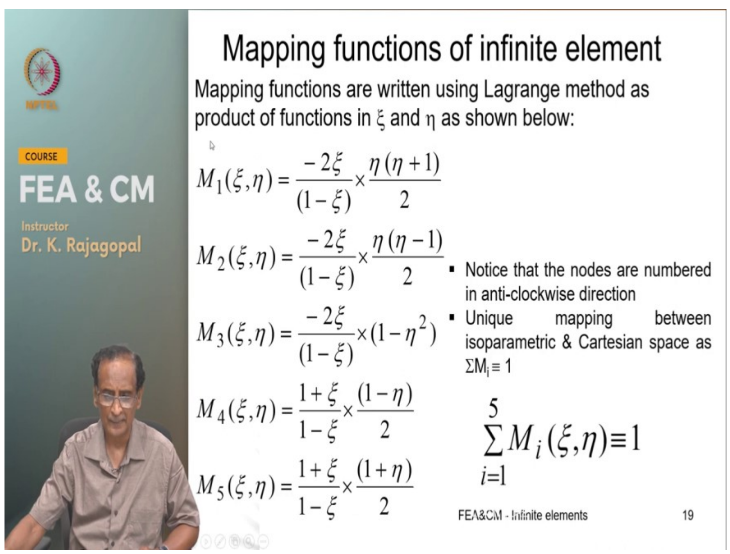
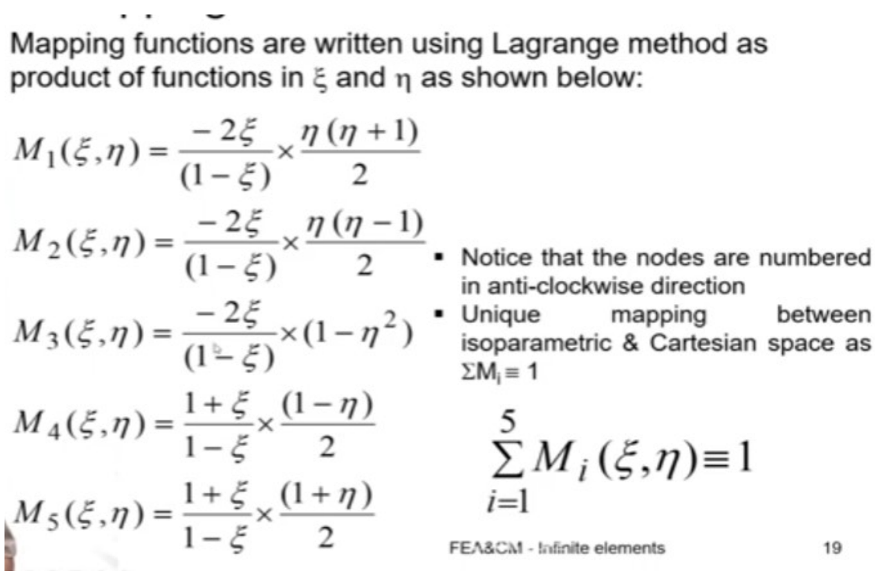


So, by using the Lagrange principle we can get our 5 mapping functions like this and you I want you to notice that we have numbered these nodes in the 90 clockwise direction. Unless you number the nodes in the anti-clockwise direction you will not get your your positive Jacobian matrix or the Jacobian determinant. So, 1 2 3 4 5 and as the convention we numbered the corner nodes first and then the mid side notes later.

Say 1 and 2 and 3 along the direction of 1 and 2 and then the 4 and 5 and although we got these mapping functions directly by using the LaGrange method we see that our Sigma of M i is exactly equal to one and actually that is very simple to notice because the M i's are one are their own nodes. Say this the shape function M 1 is one at this point Xi of - 1 and the eta of + 1. And at this point your M 3 is one and if you evaluate M3 at any other points its going to be 0 and so on.



So, our Sigma of M i is exactly equal to 1 that means that we can expect unique mapping between the Cartesian space and then the isoparametric space.

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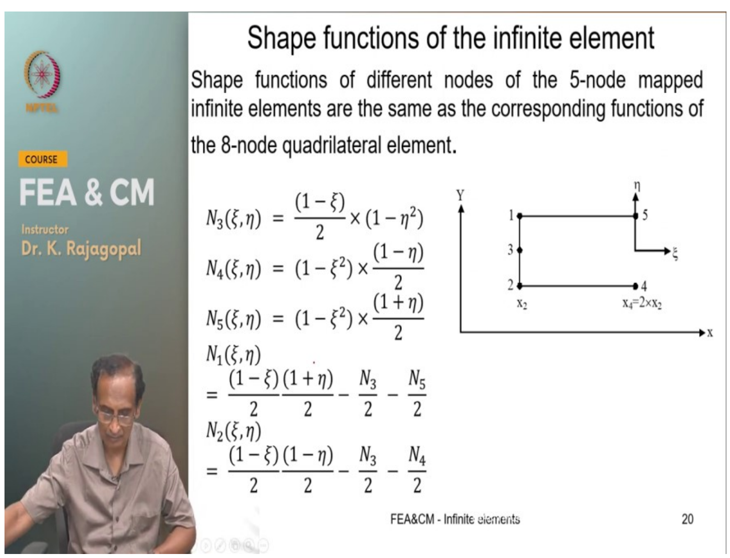
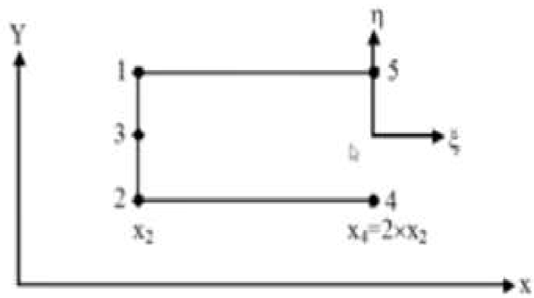


So, now let us look at the shape functions for the infinite elements and the infinite element of considered only 5 nodes out of the 8 nodes and the other 3 nodes are at infinite distance.



So, our the map the shape functions we can get by the by our the progressive correction methods that we had seen earlier. So our N 3 is a 1 - Xi by 2 times 1 - eta Square in Theta direction we have 3 nodes and along the Xi direction there are only 2 nodes.

$$N_3(\xi,\eta) = \frac{(1-\xi)}{2} \times (1-\eta^2)$$

$$N_4(\xi,\eta) = (1-\xi^2) \times \frac{(1-\eta)}{2}$$

$$N_5(\xi,\eta) = (1-\xi^2) \times \frac{(1+\eta)}{2}$$

$$N_1(\xi,\eta) = \frac{(1-\xi)}{2}\frac{(1+\eta)}{2} - \frac{N_3}{2} - \frac{N_5}{2}$$

N 4 is 1 - Xi Square because there are 3 nodes in the Xi direction then in Theta direction there are only 2 nodes, so, 1 - Theta by 2. N 5 is 1 - Xi square 1 + eta by 2 and n one for the 4 node element see for the 4 node sorry see the shape function for the 4 node element is this and the correction because of this mid side node along this line one 2 and along that line the horizontal line at one at 5.

$$N_2(\xi,\eta) = \frac{(1-\xi)}{2}\frac{(1-\eta)}{2} - \frac{N_3}{2} - \frac{N_4}{2}$$

So, this is the the N 1 4 I can write it as N 1 for the 4 node element - N 3 by 2 because we have defined one mid side node in this line then N 5 by 2 because we define another mid side node along the horizontal line. And then similarly N 2 is N 2 for the 4 node element - N 3 by 2 - N 4 by 2 and then we have the 3 other shape functions for node 6, 7, 8. But we do not bother about them because anyway we are not going to use them for interpolation.

Because the displacements and then the pore pressures are 0 at these at that far off location. So, let me go back to the laser pointer.

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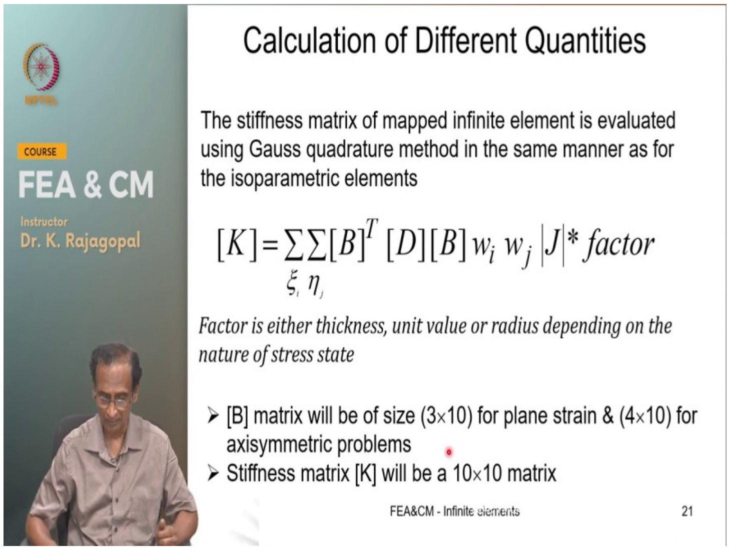

So, how do we do the computations is very simple our stiffness Matrix is written as integral B transpose DB integrated of the volume. And as we had seen with the other Continuum elements we can write the K as a double summation over Xi and Theta B transpose DB multiplied by W i times W J these are the weight factors in the Xi and ea directions multiplied by this determinant of the Jacobian matrix and the factor.

$$[K]=\sum_{\xi_i}\sum_{\eta_j}[B]^T [D][B] w_i\, w_j\, |J|*factor$$

The factor is one for the plane strain and the thickness for the plane stress and then the radius for the axis symmetric problems. For example for a plane strain element this B Matrix will be of 3 by ten for the 5 node semi infinite element or the map at infinite element 4 by ten for the axis symmetric problems. And the stiffness Matrix will be a 10 by 10 matrix for the 2D problems for the 5 node infinite element.

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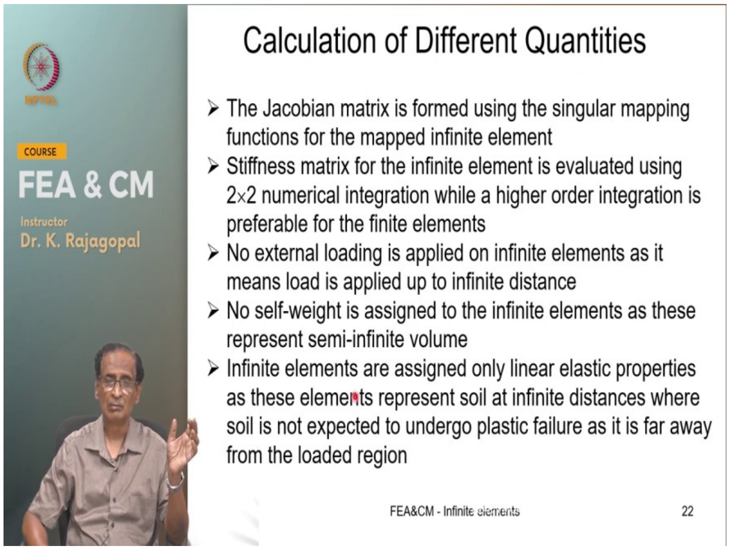

Our Jacobian matrix is formulated using our singular mapping functions. So, when you calculate dou n by dou x dou n by dou Y and so on. We use the mapping functions because we need this dou x by dou Xi dou y by dou Xi and so on. . And so, by using the mapping functions of the singular mapping functions we can get our infinity into the calculations then our stiffness Matrix is formulated using our regular numerical Integrations.

And for the infinite elements we use a 2 by 2 integration lower order integration whereas we use 3 by 3 for for the finite elements. And we should not apply any external loading and the infinite elements because if you apply any external loading either a uniform pressure or a concentrated load that means that you are loading is applied up to infinite distance because the our infinite elements are basically representing the semi infinite length.

So, we should not be applying an external loading and similarly self weight should not be assigned to the infinite elements because basically these infinite elements the representing infinite soil volume and if you define any unit weight you will see some peculiar results because you are gravity forces will be. So, high that your displacements will be very very large and other precaution that we need to take is we should not allow any failure to take place within the infinite element.

So, if any plastic failure takes place that means that up to the soil up to some infinite distance has failed that is not the reality. Because our applied load on the surface may be spread over sometime some distance like 3 to 4 times or 3 to 5 times the footing width and beyond that the pressure is so, small that you cannot have any failure. So, we make sure that all the infinite elements they are assigned only linear elastic properties. So, that you do not have any failure.

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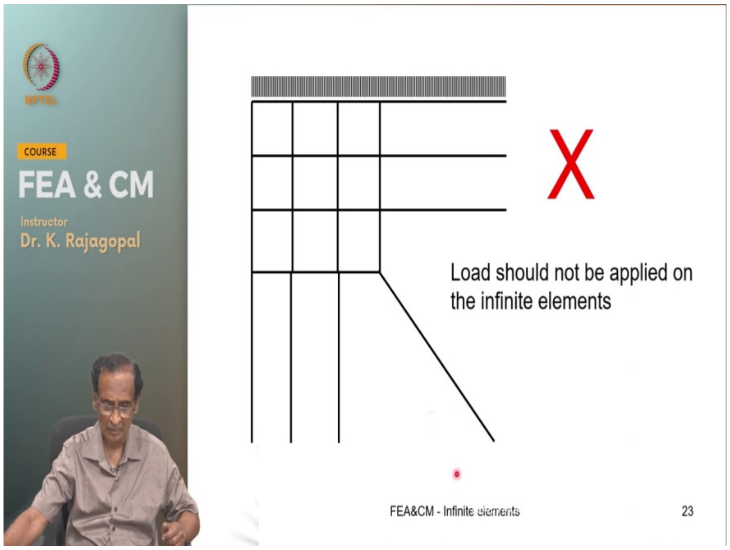

So, we should not do this like you cannot apply any uniform pressure all through because this is not a finite distance this is actually infinite distance because this is part of an infinite element and you see here actually this is our finite domain or the elements represented by finite elements and then these are the infinite elements. So, 5 node infinite elements then along this direction we have Infinity in one direction whereas at the corner we should technically have an infinite element that has the infinity in both Xi and eta directions.

But by experience it is seen that by just simply moving the this line at 45 degrees we can by using our one dimensional mapped infinite elements we should be able to represent 2 dimensional Infinity. See here if the local direction will be something like this and for this the local direction is something like this and that shows that you have Infinity in both Xi and eta directions.

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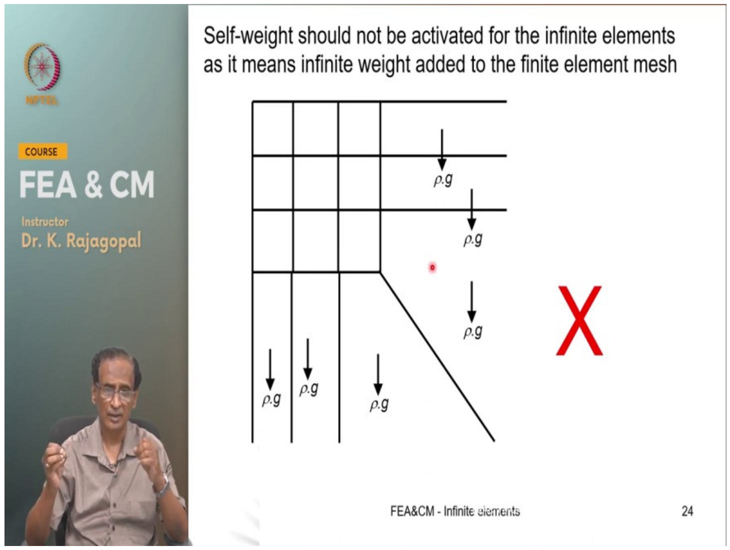

And the other precaution is we should not activate any self weight on the infinite elements. It is also very important because these infinite elements they represent infinite volume and your loading on the infinite elements means you will have tremendous loads in the load vector.

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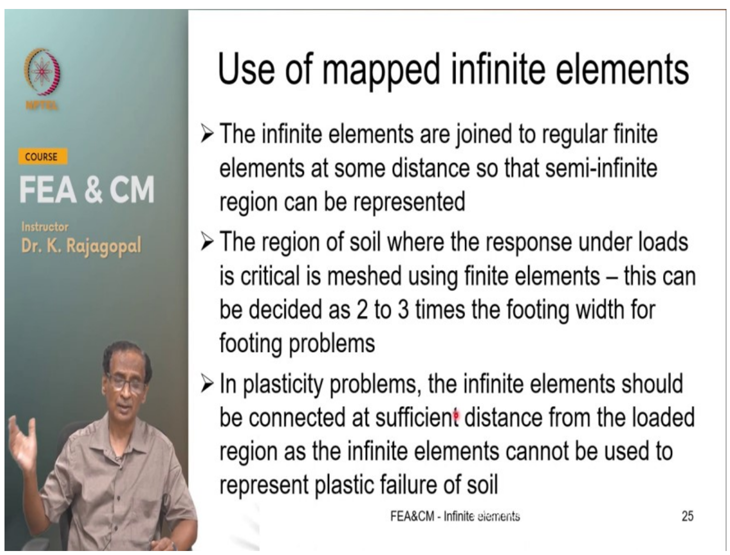

See the use of the mapped infinite elements is very simple the infinite elements are joined to regular finite elements at some distance so, that our semi infinite nature of the soil can be represented. And the reason of the soil where the response under the loads is critical is missed using the finite elements. Normally or the predominant influence of any applied load on the surface is felt up to some 2 to 3 times the footing diameter or say even 5 times the footing size.

And within the domain you can have normal finite elements and beyond that you can place your infinite elements. And in the plasticity problems infinite elements are connected at sufficient far off distance that we can directly assign a linear elastic properties for the infinite element. So, that there is no failure.

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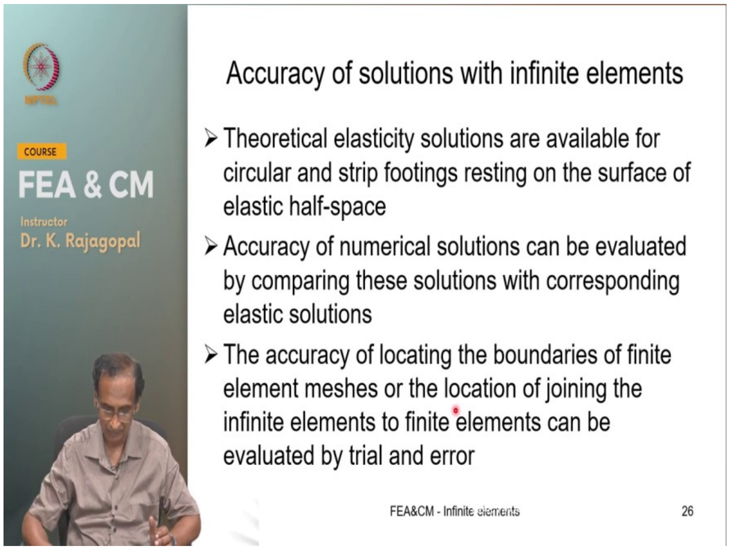

So, the accuracy we can easily check because we have number of theory of elasticity solutions for the loading on the surface of an infinite soil domain the circular strip fitting sorry the circular loading and a circular footing or a continuous loading slip loading on a wall footing and so on. We can compare our numerical results with these theoretical solutions for verifying our accuracy.

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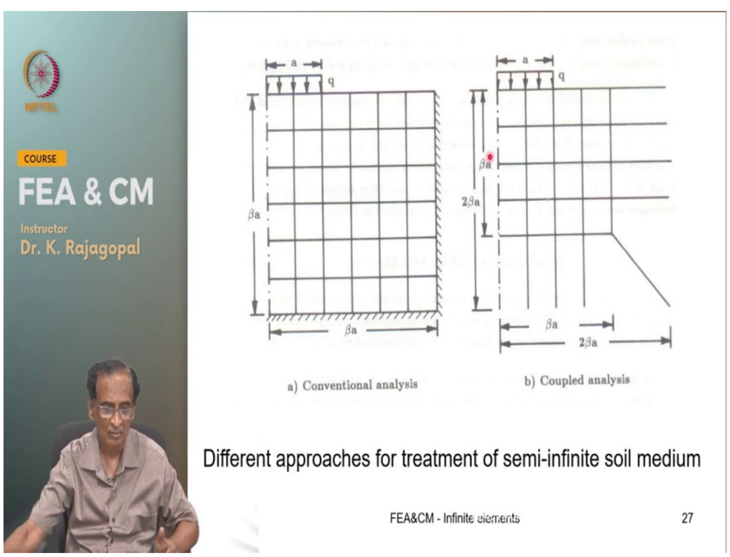

So, here we have 2 meshes one is a purely a conventional mesh with finite elements whereas the other one is called as a coupled analysis where you have some finite elements or where you have a region with finite elements and then another reason with infinite elements. And let us say that we connect the infinite elements at a distance of beta times or the radius, radius of the footing or half width of the footing.

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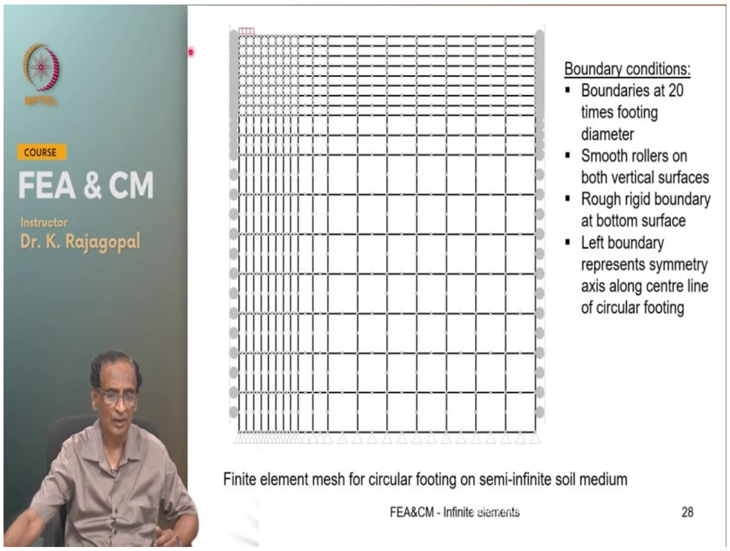

And this is one typical finite element where your footing is here and then but your domain is very very large.

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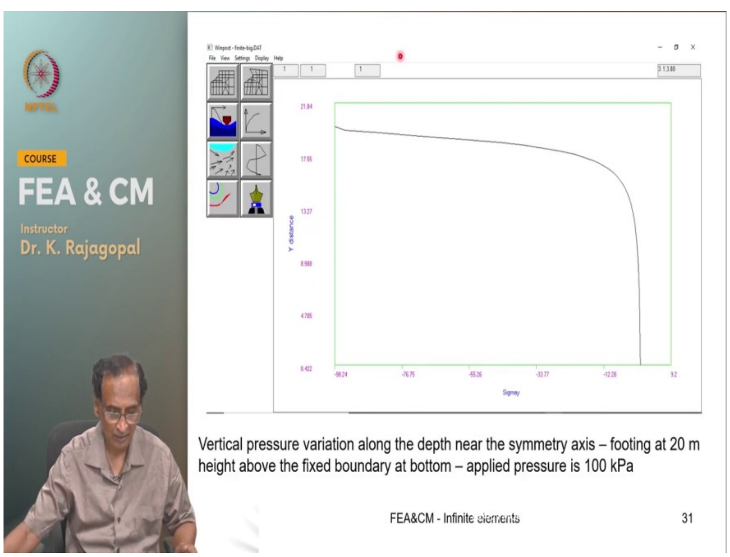

And this is our pressure distribution along the entral axis along the sorry this is sorry this is the surface settlement bowl and this is the the vertical pressure variation along the Symmetry plane symmetry axis.

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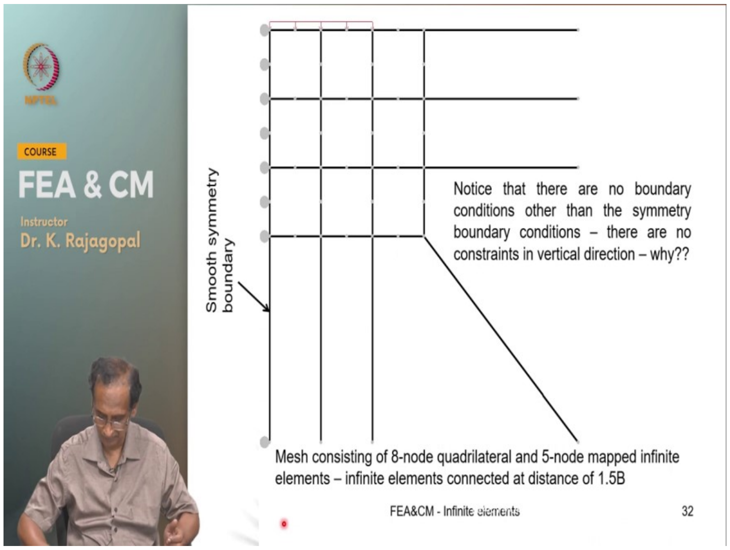

And we can do the same thing by connecting infinite elements at different distances. So, in this case the infinite elements are connected at 1.5 times the fitting width and when we use finite elements we should be careful about the boundary conditions. So, on both the sides we have supported the vertical surface on smooth rollers whereas at the bottom surface both x and y direction displacements are constrained by connecting hinges to all these nodes.

Basically that is to represent a rough rigid surface and the 2 vertical surfaces are treated as a smooth rigid smooth rigid boundaries whereas with infinite elements we enforce only the the boundary condition corresponding to the Symmetry and no nothing else. And in fact if you do the same thing with finite elements and try to solve you will have a singularity problem. You will the program will complain that your stiffness Matrix 1 of the diagonal terms is 0.

Because there are not enough number of constraints in the case of finite element analysis but then in the case of infinite elements there is another node at infinite distance where we are saying that your displacement is 0. And so, technically speaking this is like this mesh with these boundary conditions is representing a very very large mesh within with the 0 displacement at infinite distance. So, because of that you will be able to solve.

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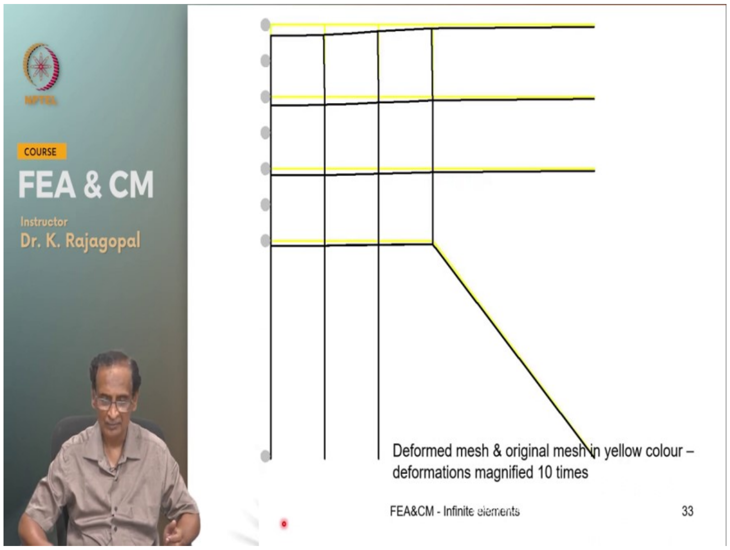

And this is our deformed mesh when we connect the infinite elements at 1.5 times the footing width.

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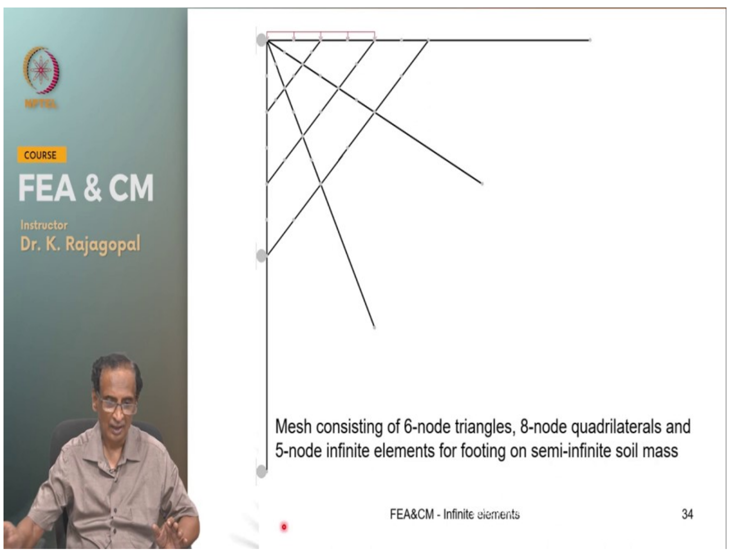

And we can also use combination of quadrilaterals and triangles. So, here drawn some radial lines for representing your infinite distance.

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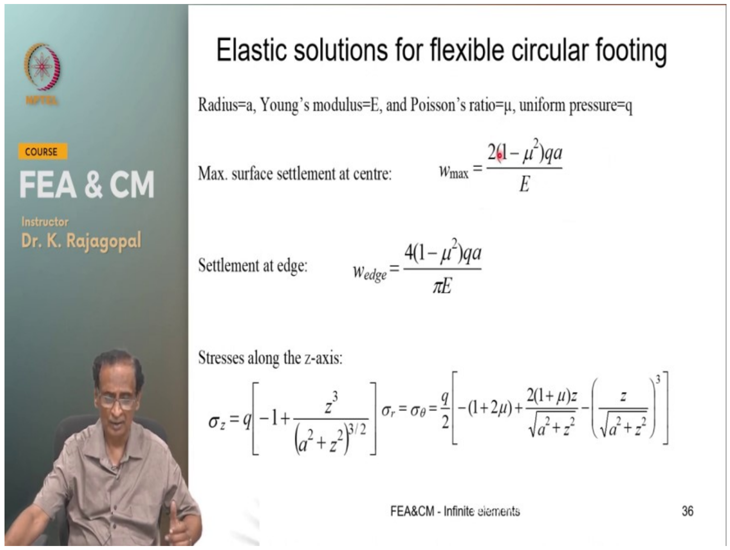

And we can compare our finite element results with the corresponding theory theory of elasticity Solutions. This is the maximum settlement at the center of a bowl center of a flexible circular footing and this is the settlement or the edge of the fitting and these are the stresses along the Z axis Sigma Z then Sigma R Sigma Theta and so on.

Radius=a, Young's modulus=E, and Poisson's ratio=μ, uniform pressure=q

Max. surface settlement at centre: $w_{\max} = \dfrac{2(1-\mu^2)qa}{E}$

Settlement at edge: $w_{edge} = \dfrac{4(1-\mu^2)qa}{\pi E}$

Stresses along the z-axis:

$$\sigma_z = q\left[-1+\frac{z^3}{\left(a^2+z^2\right)^{3/2}}\right] \quad \sigma_r = \sigma_\theta = \frac{q}{2}\left[-(1+2\mu)+\frac{2(1+\mu)z}{\sqrt{a^2+z^2}}-\left(\frac{z}{\sqrt{a^2+z^2}}\right)^3\right]$$

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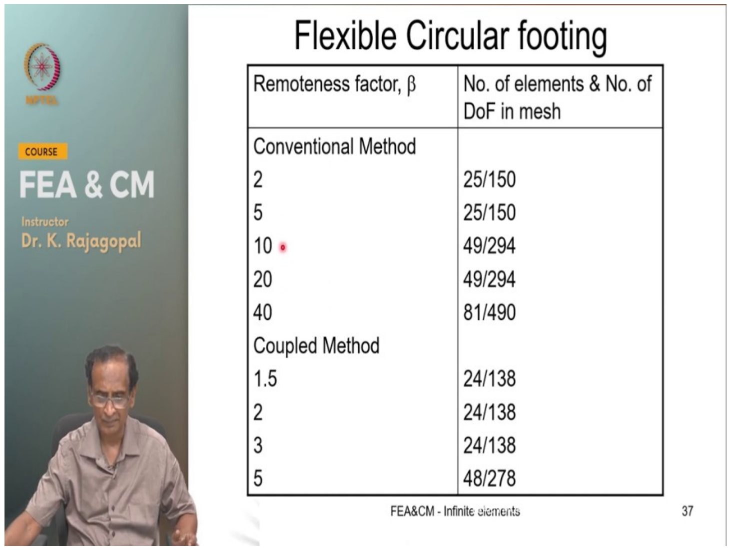

And the conventional method is with with regular finite elements where the boundary is placed at the 2 times the footing width 5 times 40 times and so on. Then in the coupled methods 1.523 and so on and in all the cases the couple methods with finite and infinite elements they have much lower number of elements and degrees of freedom.

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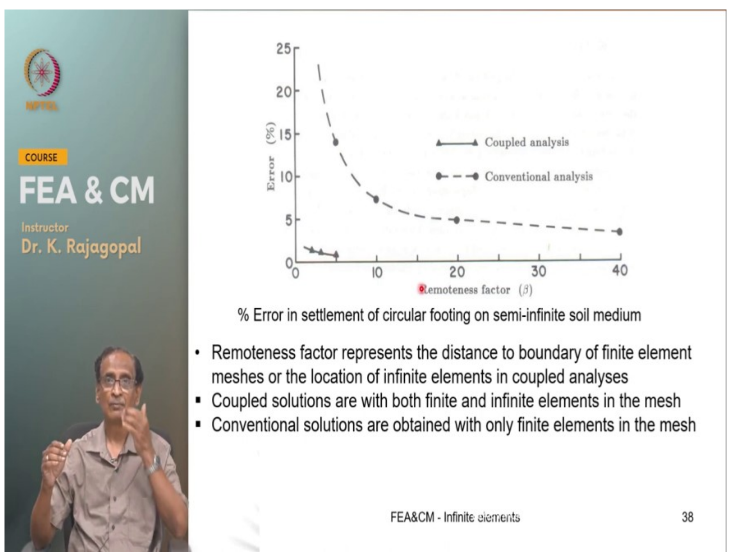

And here is a plot between the remoteness factor that is the distance up to the boundary in the case of finite elements the distance where the infinite elements are coupled with the regular finite elements and on the y axis we have the error. And this line is corresponding to coupled analysis where you have used both finite and the infinite elements and the percentage error is very very small of the order of one percent.

Whereas with conventional finite element analysis even when you join the when you put a boundary restraint at 20 times the footing width there is still about 5 percent error in the displacement that you predict or if you locate them at 5 times then your displacements are have an error of 15 percent. And if you have 15 percent error in displacements your stresses will be will not be or will have more error like might have about a 25 or 30 percent error.

So, will not be able to; even maintain the equilibrium between the applied loads and then the reaction forces.

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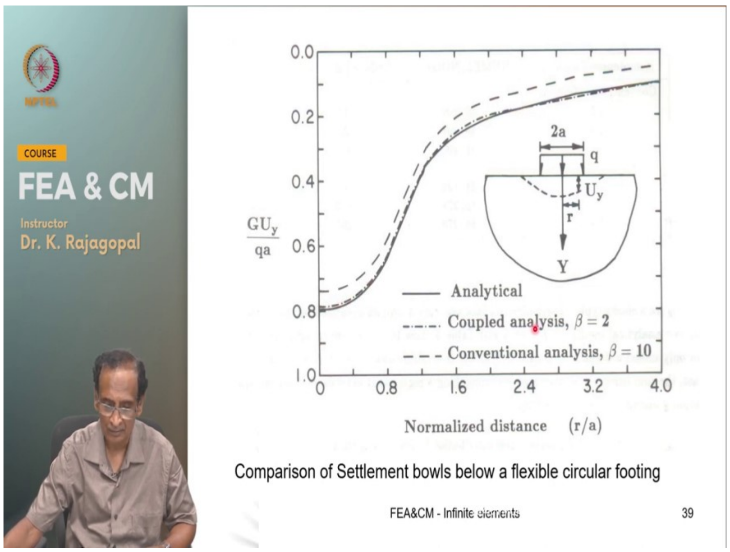

This is the settlement bowl this dark line is the analytical solution and this is our coupled analysis and this is with sorry yeah this is with beta of 2 and this is the conventional analysis with the beta of 10. There is still so, much variation between the theoretical result and then the finite element result when we used our conventional analysis where we fixed the boundaries at 10 times the footing width.

Whereas with infinite elements even when you couple at a distance of 2 times the footing width the result is very close to the to the exact analytical solution.

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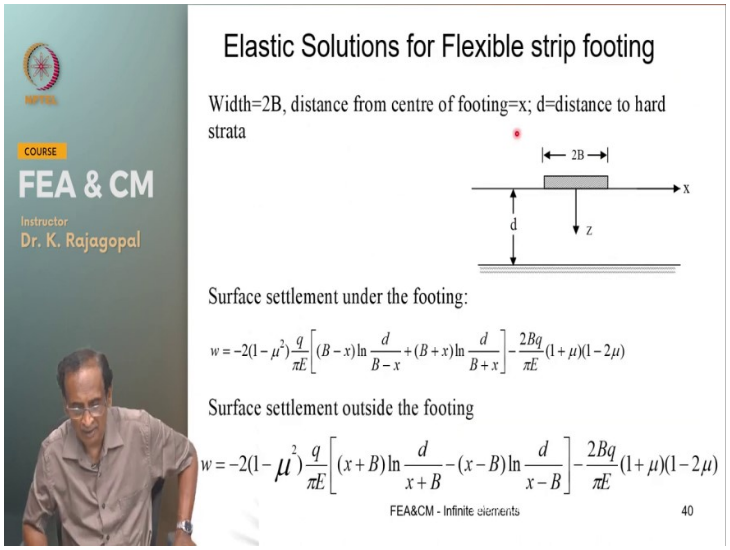
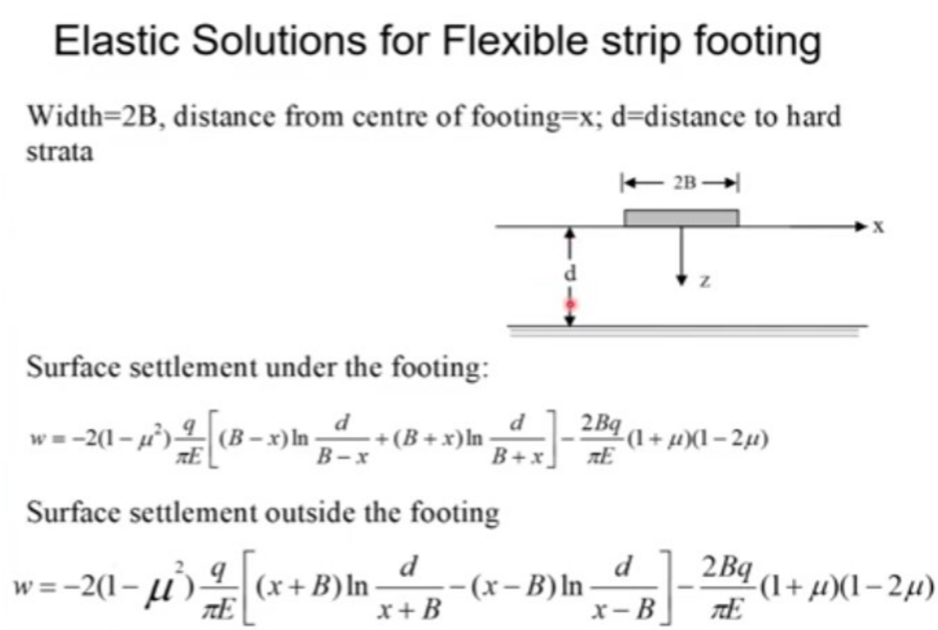


And we can verify our results even for strip footing and the solution for a strip footing are given in terms of the depth to the hard strata. So, let us say that the D is the depth to the hard stratum and this is our equation for settlements W within the for a point within the footing and the settlement equation for a point surface point outside the footing.



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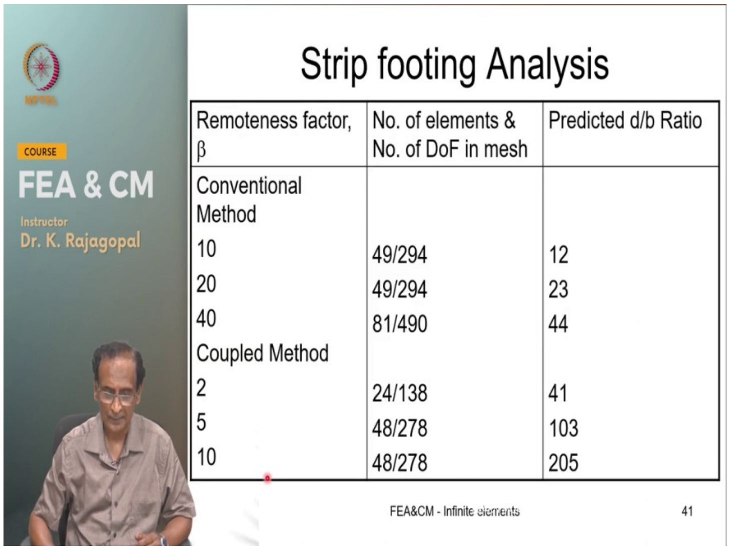

And once again a very small number of elements and degrees of freedom were required in the case of infinite elements and say this is these are the results with conventional finite elements when you place the boundary at 10 times if you back predict it is coming to about 12 and with the 20 it is coming to 23 and with 40 is coming to about 44 that is we can back predict to get back our analytical result.

And so, with finite element analysis there is no benefit because if you place the boundary at 10 times the footing width then you get current results only corresponding to that. But then if you use a couple method if you place the boundary at 2 times the footing width is equal to placing the boundary at about 40 times the footing width. And if you place it at 5 times if you place the infinite elements at 5 times the footing width is equivalent to fixing the boundary at 103 times the footing width.

So, it is actually it is a very huge difference between the finite element and infinite elements and the advantage with infinite elements is we use a lesser number of elements and lesser number of nodes or degrees of freedom. So, our solution is going to be faster especially if you are solving any non-linear problem are the Dynamics problems the solution times are the order of say in terms of this 24 hours or one day or one week and so on.

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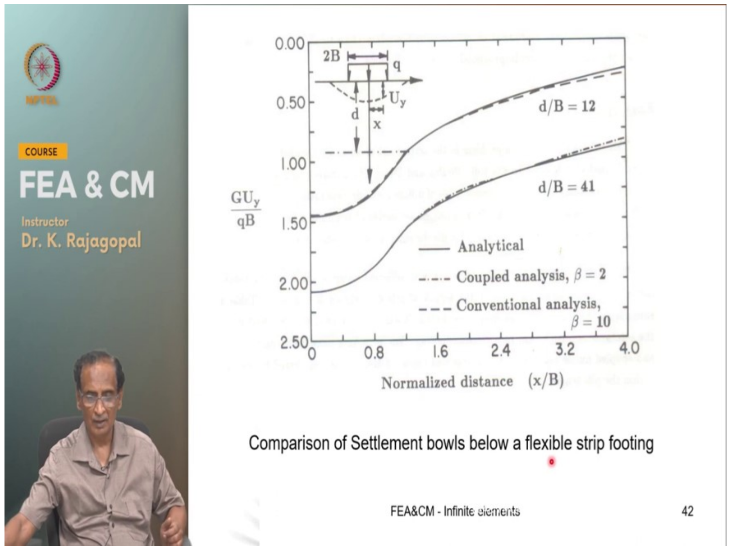

So, here you see the comparison between the analytical and then this finite element Solutions. So, this is for a beta of 2 the D by B comes out about 41 whereas for finite element analysis the D by B is about 12. Although the boundaries fixed at 10 times there is a slight difference it is not coming out exactly as 10 because of the numerical my address.

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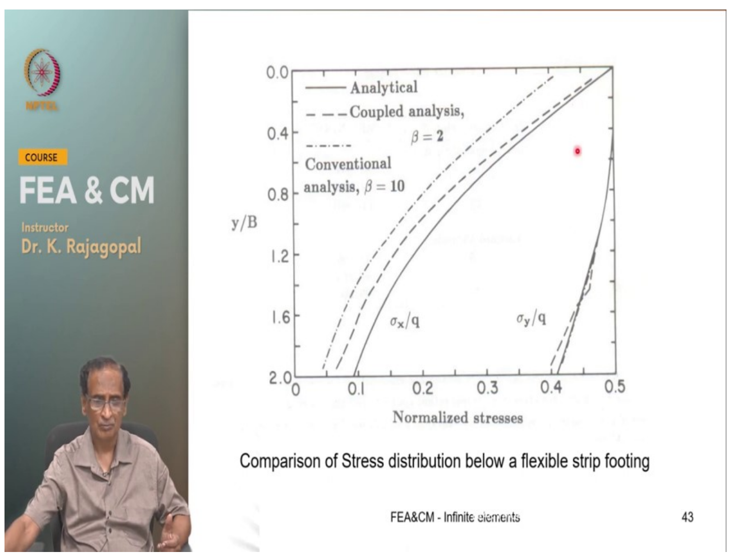

And these are the predicted stresses. The black lines are the analytical stresses and then the dashed lines are from different finite element analysis.

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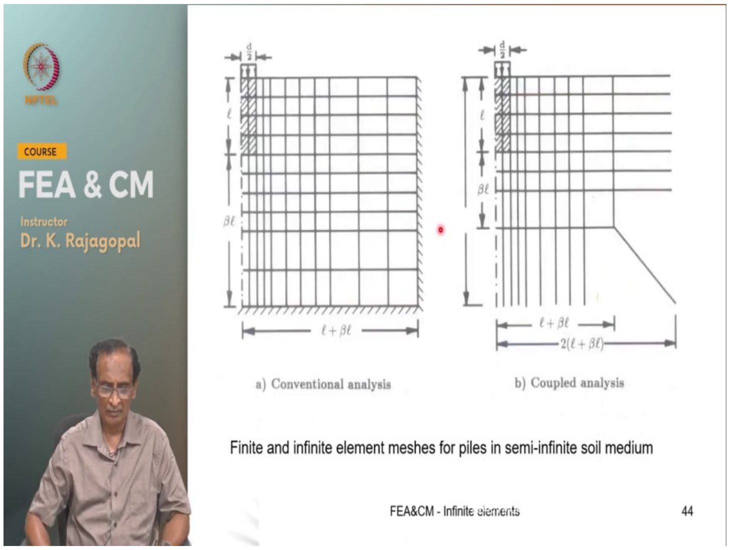

And you can do the same thing even with pile foundations and especially when you have very long pile foundation is hard to decide where you need to keep your boundary because the file length itself is it could be 30 to 40 meters. And beyond that how much soil we need to place either one times 2 times half a time and so on. And so, so in the analysis finite element analysis of piles usually involves in very very huge meshes.

So, if we can save on the and the size of the problem by using intelligently using these infinite elements then we can have much faster solutions.

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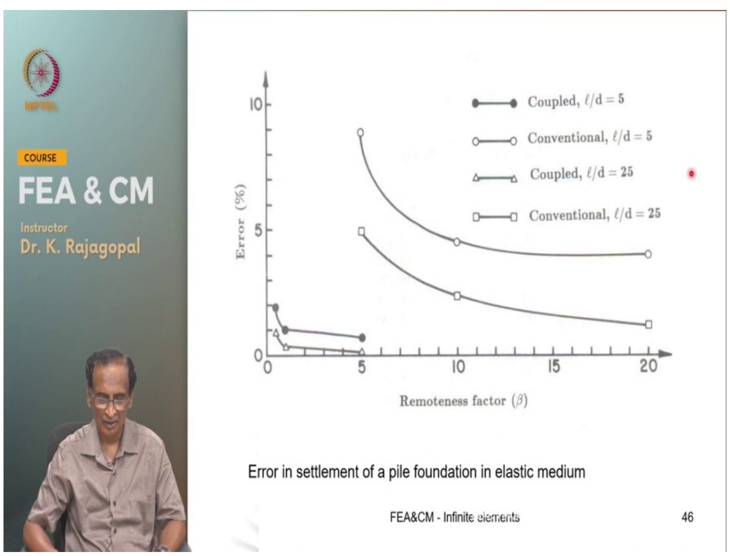

And similar to what we found with the surface footings the percentage error with finite elements or infinite elements is very very small of the order of about one to 2 percent whereas with finite elements is a huge order. Even with say 10 times the footing width your percentage error is about close to 4 percent. But if you use a length of mesh 5 times the file length you will get you will still get very large error.

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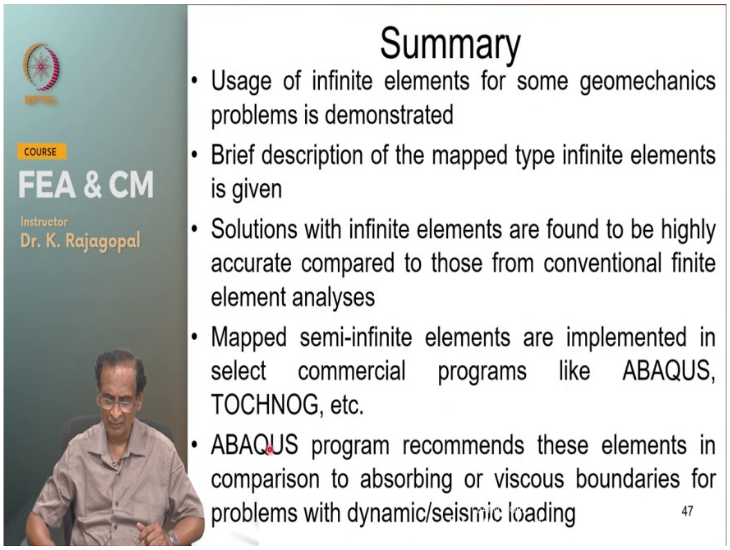

So, just to summarize the use of infinite elements for some geomechanics problems is demonstrated we have seen for footing problems circular footing and then the wall footing or a strip footing and then we have seen for pile foundations. And the mapped infinite elements are very similar to our isoparametric elements except that they have singularity in their mapping functions.

And they are highly accurate and these mapped functions that we have discussed they are implemented in some programs like Abacus and TOCHNOG. Abacus is a is a very good commercial program that is used by not only civil engineers by mechanical engineers electrical chemical Engineers everybody. TOCHNOG is another geotechnical specific program that also can be used for several things.

And in the Abacus manual they recommend the use of infinite elements for all the in place of our viscous boundaries because they usually give better results. That we will see later I will show you one example that was obtained with viscous boundary and then with these infinite elements. So, that is the end of my lecture. So, if you have any questions please send an email to this address profkrg@gmail.com. So, thank you very much.