FEM and Constitutive Modelling in Geomechanics
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Lecture: 2
Matrix Algebra and Gauss Elimination Method

So, let us look at the brief about Matrix algebra in today's lecture because we need to operate on the matrices for all our finite element calculations.

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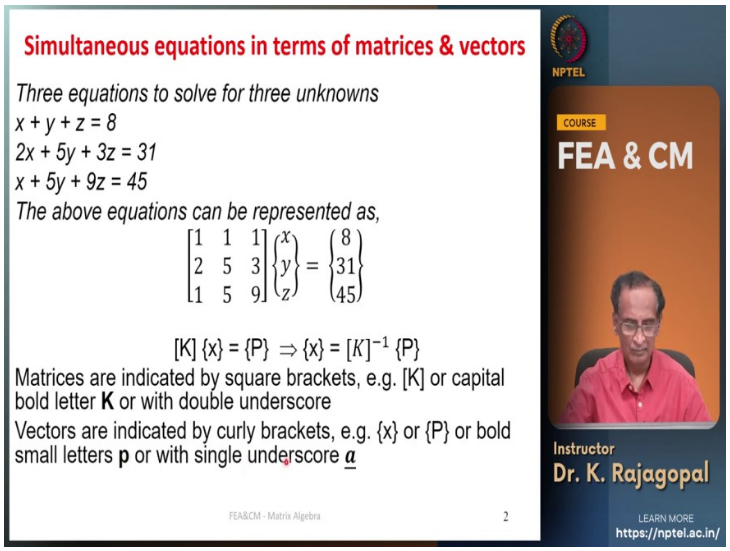

So, why do we require to know about this Matrix operations the reason is we are going to come across number of this type of simultaneous equations to solve for the unknowns. Like for example here we have three equations and three unknowns x, y and z.

Three equations to solve for three unknowns

$x + y + z = 8$

$2x + 5y + 3z = 31$

$x + 5y + 9z = 45$

And so, we need to solve for them and we can represent these equations in Matrix and Vector form like this. So, all the coefficients before x, y and z can be represented as a matrix coefficients like this.

The above equations can be represented as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 3 \\ 1 & 5 & 9 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 8 \\ 31 \\ 45 \end{Bmatrix}$$

And this is a vector of unknowns x, y and z then the right hand side these are the known quantities say 8, 31 and 45. And we can write this as some Matrix K multiplied by x is equal to p and this x Vector can be determined as K inverse of P that is a product of inverse Matrix multiplied and by the by the load vector or P .

$$[K]\,\{x\} = \{P\} \;\Rightarrow\; \{x\} = [K]^{-1}\,\{P\}$$

And in this course the matrices are going to be indicated by square brackets like this : [K] is a matrix or the capital bold letter **K** or with a double underscore.

And then the vectors are in curly brackets like for example here {x} is a vector and the {P} is also a vector or they can also be shown as a bold letter but small letter small **p** or with a single underscore like this .

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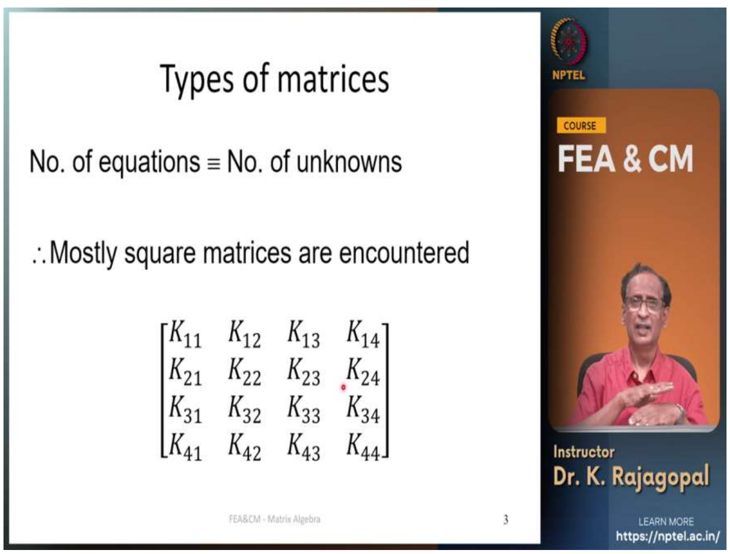

And the type of matrices that we come across in all our finite element analysis will be the square matrices because our number of equations is exactly equal to the number of unknowns. And of course we will have several intermediate matrices that we will see later they need not be square they could be rectangular matrices . And so, like this like for example here we have a four by four Matrix four rows and four columns K 11, K 12 and K 13.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix}$$

The 1 here represents the row number and then the one in the second column represents the column. So, when we say k 11 it is the element in the first row and First Column and similarly K24 means the second element in the second row and the fourth column and so on.

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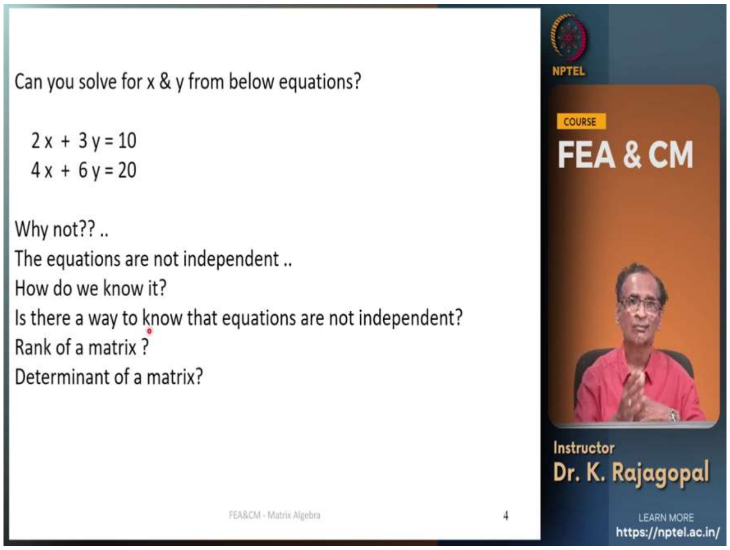

And can we solve for the unknowns if we are given some set of equations that also we need to know like for example here I have two equations

Can you solve for x & y from below equations?

$$2x + 3y = 10$$
$$4x + 6y = 20$$

We have two equations and we have two unknowns x and y can we solve for them? But then we realize that these two are not independent equations is actually the second equation is linearly proportional to the first equation.

So, we cannot solve because we have the thinking capacity we can say that these two are not independent. So, we cannot solve for x and y if we are given this set of equations. But then how does a computer program know? Is there a way of knowing whether these equations are independent or not here that we can say with the rank of a matrix you know that when you have a square Matrix the rank represents the number of independent rows and or the determinant.

So, when you have a matrix if you calculate the determinant that also will tell whether all the equations are independent or not. So, if the determinant of any Matrix is 0 that means that the rank is less than the number of equations that we have. Like and we will not be able to invert that Matrix or do anything.

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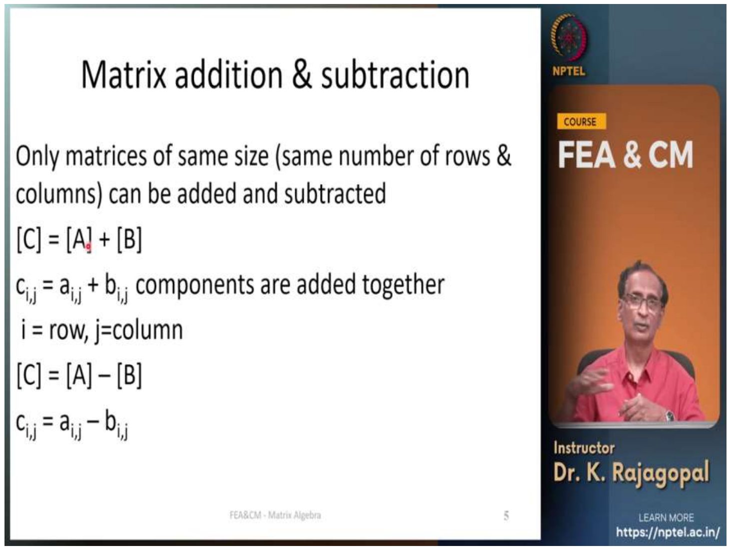

And let us see what is Matrix addition see the only the matrices of the same size that is the same number of rows and same number of columns can be added and subtracted. Matrix addition is very simple our subtraction is also the same. Let us say C is equal to A + B means that we take individual terms from A and B and add them together to get the terms and the C Matrix. Like for example

[C] = [A] + [B]

$c_{i,j} = a_{i,j} + b_{i,j}$ components are added together

i = row, j=column

c ij is equal to a ij + b ij, see these are the components added directly together to get our addition Matrix c ij where i is our row and the j is the column.

And similarly we can subtract C is equal to A - B and once again we can only operate on the same size of matrices.

$$[C] = [A] - [B]$$

$$c_{i,j} = a_{i,j} - b_{i,j}$$

See if A and B are matrices having three rows and three columns The Matrix C also will have the same number of rows three rows and three columns.

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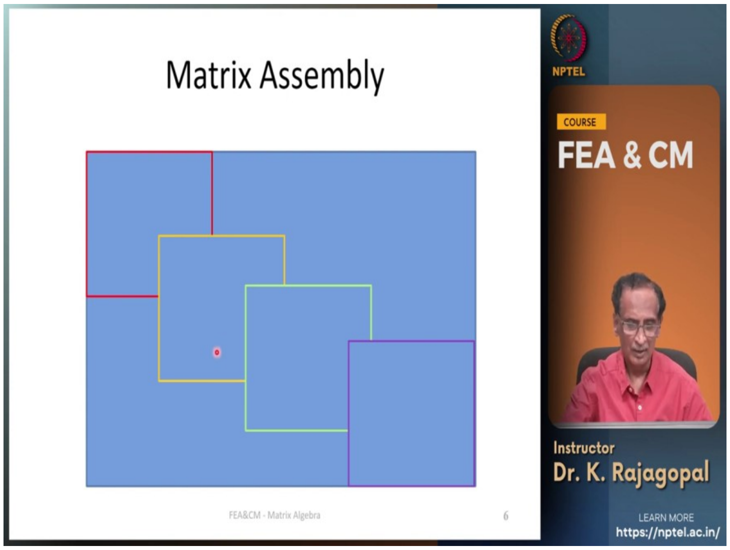

But then we have another thing called as Matrix assembly that we are going to see. From smaller matrices we form a big Matrix like for example we have a big Matrix that is probably representing the entire structure. And in this we have added together smaller matrices may be contributed by each of the elements or each of the components like this and this we call is assembly.

We are not really adding together one Matrix and the other Matrix but we are placing them in the appropriate place within the larger Matrix depending on the on the assembly that we will see later when we deal with structural analysis.

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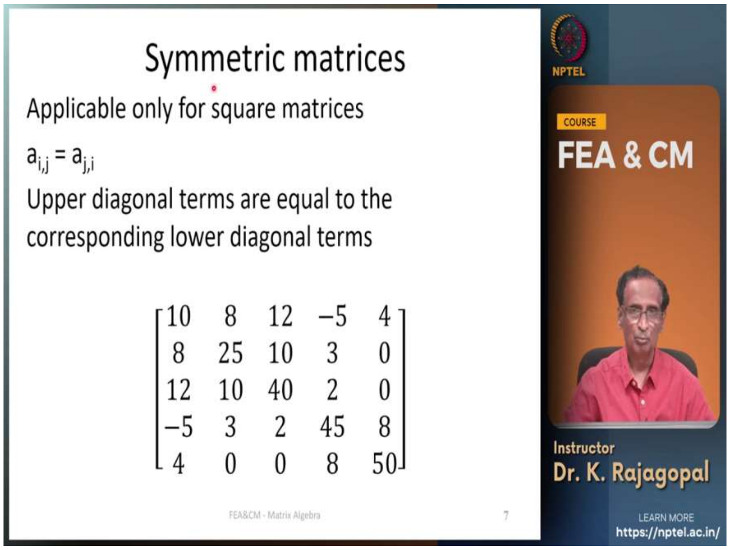

And we will frequently come across symmetric matrices and the Symmetry is only applicable for square matrices. So, the symmetric Matrix means

$$a_{i,j} = a_{j,i}$$

a ij term is exactly equal to a ji.

So, the upper diagonal terms are exactly equal to the corresponding lower diagonal terms say a13 will be exactly equal to a31. And so, that is the basic essence of symmetry but then towards the later part of the course we will also see matrices that are unsymmetric and that is unique to geotechnical engineering that we will see later.

So, here is an example of a symmetric Matrix we have a matrix with five rows and five columns and you see here.

$$\begin{bmatrix} 10 & 8 & 12 & -5 & 4 \\ 8 & 25 & 10 & 3 & 0 \\ 12 & 10 & 40 & 2 & 0 \\ -5 & 3 & 2 & 45 & 8 \\ 4 & 0 & 0 & 8 & 50 \end{bmatrix}$$

You see here 8 and 8, 12, 12 -5 -5 10 and 10 and so on. That is the upper diagonal terms are exactly equal to the lower diagonal terms and in a way that reduces our computational effort. If you find if you determine any component in the upper diagonal upper diagonal part the same thing can be used even for the lower diagonal term we do not need to compute.

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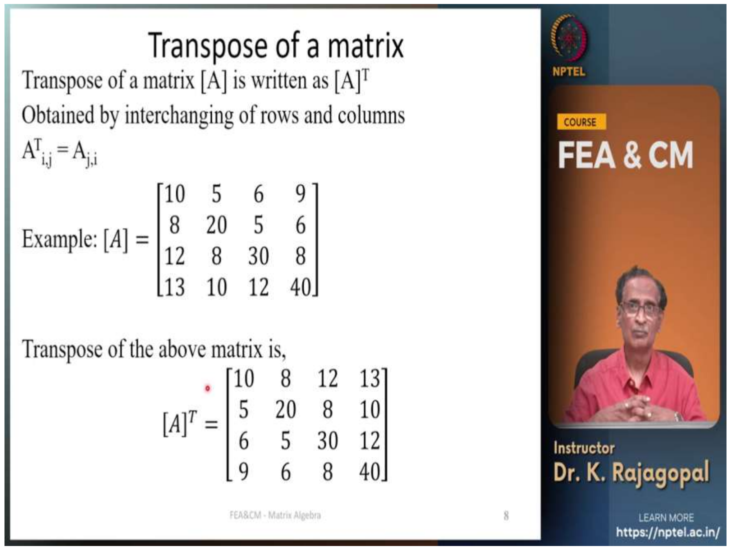

It once again the transpose of a matrix see we can obtain the transverse by interchanging the rows and columns and if we have A matrix, A we can write the transpose as A transpose with a superscript T and we obtain this by interchanging the rows and columns.

$$A^{T}_{i,j} = A_{j,i}$$

And so, for example if our Matrix is like this the transpose of the Matrix will be like this.

$$\text{Example: } [A] = \begin{bmatrix} 10 & 5 & 6 & 9 \\ 8 & 20 & 5 & 6 \\ 12 & 8 & 30 & 8 \\ 13 & 10 & 12 & 40 \end{bmatrix}$$

And this transpose is applicable even for non rectangular matrices.

Transpose of the above matrix is,

$$[A]^T = \begin{bmatrix} 10 & 8 & 12 & 13 \\ 5 & 20 & 8 & 10 \\ 6 & 5 & 30 & 12 \\ 9 & 6 & 8 & 40 \end{bmatrix}$$

They do not need to be symmetric or Square. So, if you have a matrix with the two rows and four columns if you take a transpose we will have four columns four rows and two columns. So, that is the essence of this transposition. And here we have a an example say the first row this Matrix is 10 5 6 9 and when we take a transpose of this and that first row becomes the First Column second row become the second column third row becomes the third column and so on.

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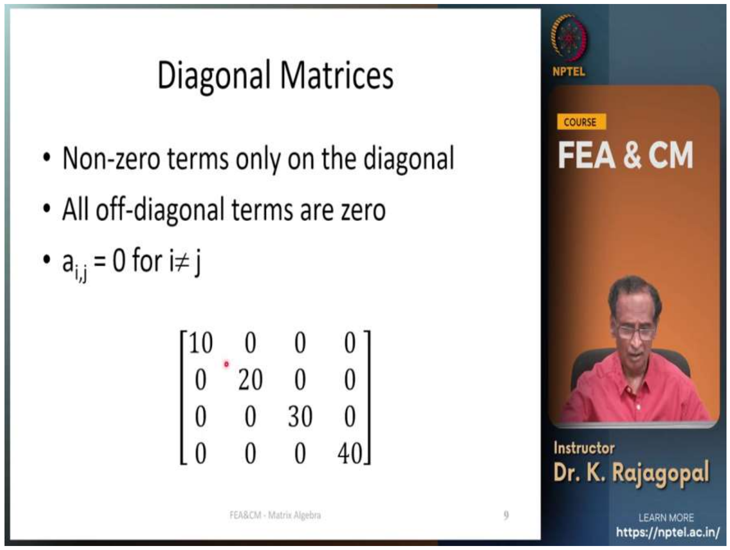

And sometimes we may end up with diagonal matrices. See we may have a huge Matrix but only the diagonal terms are non-zero and all the other terms are zero and that we call as a diagonal matrix and all half diagonal terms are 0 a ij is 0 for i not equal to 0.

$$a_{i,j} = 0 \text{ for } i \neq j$$

one an example for this and select this all the diagonal terms are non-zero and the half diagonal terms are zero and this we call as a diagonal matrix.

$$\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 40 \end{bmatrix}$$

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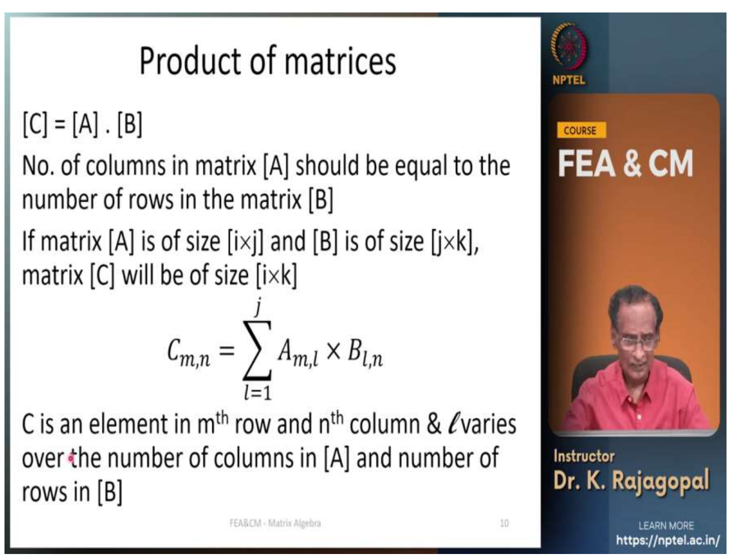

And sometimes we may have to take a product of two matrices say for example say we have two matrices A and B and we want to take a product and determine the Matrix C.

$$[C] = [A] \,.\, [B]$$

See for us to be able to take a product the number of columns in Matrix A should be equal to the number of rows in Matrix B.

We cannot arbitrarily take any Matrix and multiply with some other Matrix. And so, this is actually if A and B are of the same size like Square matrices.

We can take product or the number of columns in Matrix A should be equal to the number of rows in Matrix B. And say if Matrix A is of size [i × j] that is the i number of rows and j number of columns and B is of size [j × k] j number of rows and i in K number of columns Matrix C will be of size [i × k] that is i number of rows. And the K number of columns and the elements in this Matrix C now determined like this.+

$$C_{m,n} = \sum_{l=1}^{j} A_{m,l} \times B_{l,n}$$

C mn the element in the mth and nth column in Matrix C can be obtained by taking a product of A ml times B ln and l varying from 1 to j where our j is the number of columns in A are the number of rows in B. So, for example if you want a C 23 your A 21 times B 12 22 and then 2 1 and so on like we can do the product like this. And this C is an element in the mth row and the nth column and l varies over the number of columns in [A] and then the number of rows in [B] and this l is the variable.

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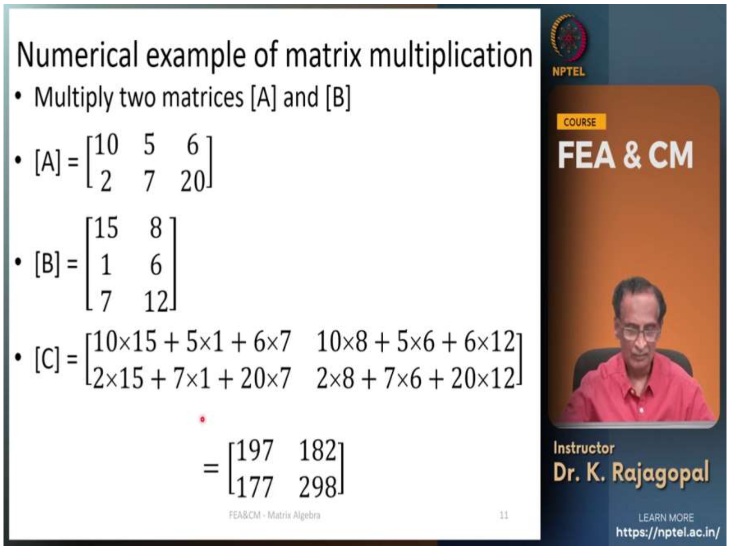

And let us look at two matrices [A] and [B], A has two rows and three columns and B has three rows and the two columns and the product A and B will result in a matrix of two rows and two columns corresponding to two rows in Matrix A and two columns in Matrix B and the elements in C can be obtained by multiplying the first row and First Column first row and second column second row First Column second row second column like this and.

$$[A] = \begin{bmatrix} 10 & 5 & 6 \\ 2 & 7 & 20 \end{bmatrix} \qquad [B] = \begin{bmatrix} 15 & 8 \\ 1 & 6 \\ 7 & 12 \end{bmatrix}$$

So, the C is a two by two Matrix like this.

$$[C] = \begin{bmatrix} 10\times15 + 5\times1 + 6\times7 & 10\times8 + 5\times6 + 6\times12 \\ 2\times15 + 7\times1 + 20\times7 & 2\times8 + 7\times6 + 20\times12 \end{bmatrix}$$

$$= \begin{bmatrix} 197 & 182 \\ 177 & 298 \end{bmatrix}$$

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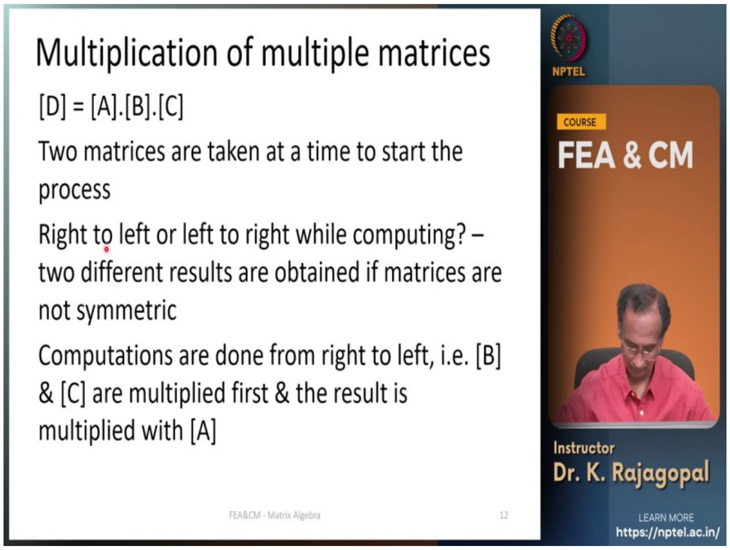

And sometimes we may have more than two matrices like say we have Matrix D is equal to A B C.

$$[D] = [A] . [B] . [C]$$

So, how do we multiply? So, we have to obviously take two matrices at a time and then do the product and then go for the go for the third Matrix and the process starts from right to left or left to right how do we do? And will we get the same result whether we multiply first Matrix A and B and then the resultant we multiply with C are multiply the two matrices B and C first and then the resultant is Multiplied with A.

So, it is if our matrices are not symmetric we may end up with the two different matrices. Like for example for unsymmetric matrices if you multiply from left to right or right left you might end up with the two different results. So, we should be consistent and we always start from the right and go towards the left. And if we have symmetric matrices it does not matter whether we go from left to right or right to left.

And luckily for us we mostly have symmetric matrices. So, it does not matter whether we do it from right to left or right or left to right . So, here our computations are done from right to left B and C are first multiplied and then the resultant is Multiplied with A to get our resultant Matrix B A D.

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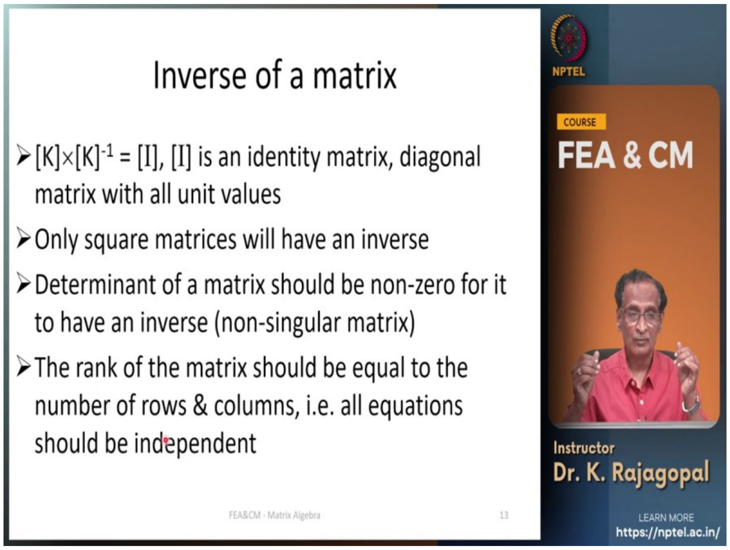

And we also need to evaluate the inverse of a matrix. See previously we have seen that Kx is equal to p and where x is the vector of unknowns and x can be written as K inverse of p and for that we require the inverse of that Matrix K and the inverse is defined such that The Matrix multiplied by its own inverse is equal to an identity Matrix.

$$[\mathbf{k}] \times [\mathbf{k}]^{-1} = [\mathbf{I}]$$

See it is an identity Matrix is a diagonal matrix with all unit values. So, along the diagonal we have only one one one one one and so on and all the others are zeros and that is called as a diagonal matrix. And only the square matrices will have an inverse and the determinant of a matrix should be non-zero for it to have an inverse and we call that type of matrices as non-singular.

And if the determinant is zero we call that as a singular Matrix and see there should not be any rank deficiency. The rank should be equal to the number of rows and columns that means that all the equations that we have should be independent equations so, that we can solve that system of equations by either inversion or by some other process.

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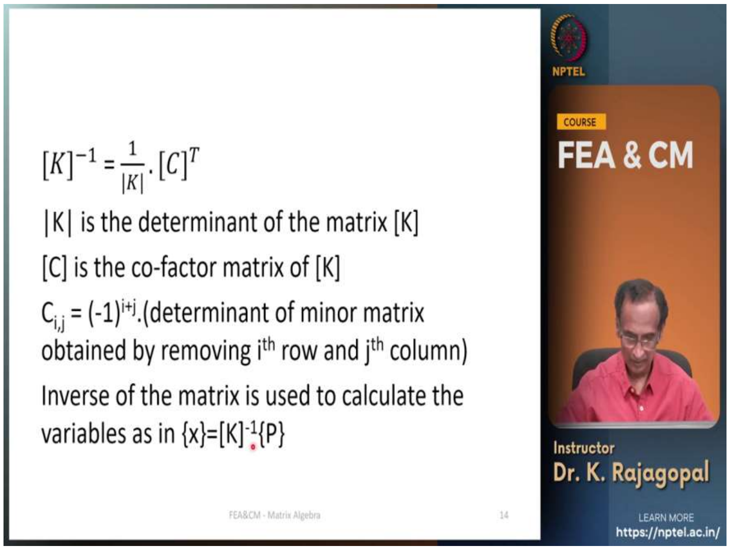

So, our K inverse we can write as a C transpose divided by determinant of K and this symbol is the determinant and the C is actually I will explain what that C is.

$$[K]^{-1} = \frac{1}{|K|}.[C]^T$$

|K| is the determinant of the matrix [K]

[C] is the co-factor matrix of [K]

It this mod K is the determinant of the Matrix K and C is the cofactor Matrix of K and that is a C transpose and C ij that is the cofactor determine the cofactor Matrix is equal to -1 to the power i + j multiplied by determinant of the minor Matrix obtained by removing the ith row and jth column.

$$C_{i,j} = (-1)^{i+j}.$$

And that is a very simple one like say if you have a two by two Matrix very simple because the cofactor Matrix will you are left with only one term. If you eliminate the row and column but if we have more number of rows and columns then it becomes difficult. And the inverse of the Matrix can be used to calculate the variables as in x is equal to K inverse P.

$$\{x\}=[K]^{-1}\{P\}$$

So, our K times x is equal to P. So, our x can be written as K inverse of P.

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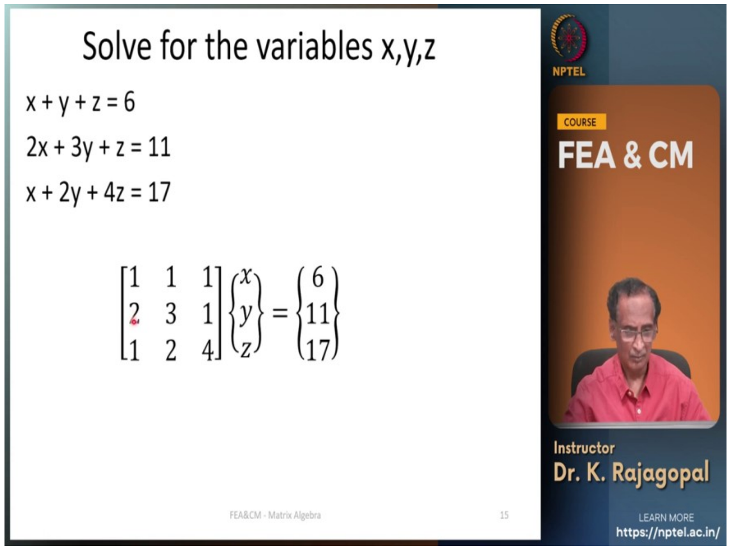

Let us look at an example let us solve for the variables x y and z we have three equations x + y + z is 6, 2x + 3y + z is 11. Then the third equation is x + 2y + 4z is 17. And so, we need to solve for x, y and z and we can write this in a matrix and Vector form like this. So, when we look at this Matrix on the right hand side vector we notice that these three are independent equations there is no direct proportionality between each row or each column.

So, we should be able to invert this Matrix and find a unique solution for x y and z and the determinant of the Matrix is actually it is we can easily solve it I hope you know how to find the determinant of this Matrix that will be 1 times the determinant of the smaller Matrix and -1 times the determinant of the smaller Matrix 2 1 and 1 4 and +1 times the determinant of the smaller 2 by 2 Matrix.

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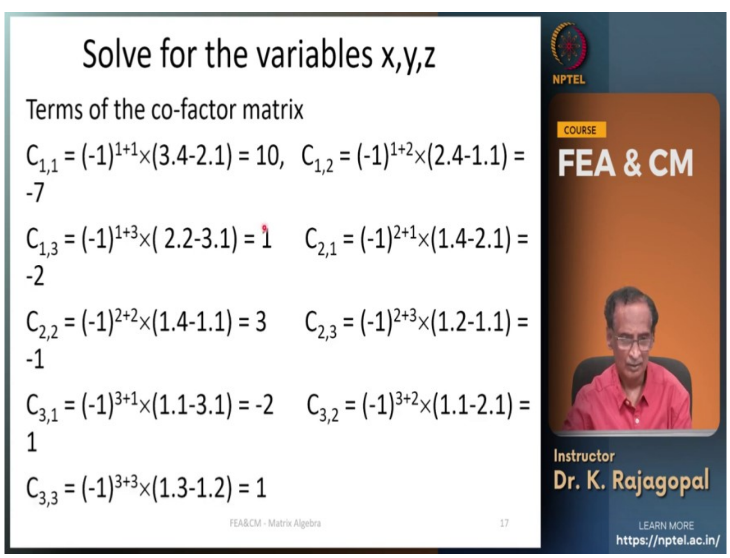

Determinant of the matrix,

$$|A| = (-1)^{1+1}\times(3.4 - 2.1) + (-1)^{1+2}\times (2.4 - 1.1) + (-1)^{1+3}\times(2.2 - 1.3) = 4$$

And the determinant is a is -1 to the power i + j that is we are dealing with the first row and first column and then 1 + 2 first row and second column one + three means first row and third column and the determinant is 4 and that is positive definite. So, we will definitely have the inverse of this Matrix and the terms of the cofactor Matrix we need to get and C 11 is -1 to the power 1 + 1 that is number of the first row and the first column multiplied by the determinant of the Matrix the minor Matrix obtained by eliminating the first row and first column.

Terms of the co-factor matrix

$C_{1,1} = (-1)^{1+1}\times(3.4-2.1) = 10$, $C_{1,2} = (-1)^{1+2}\times(2.4-1.1) = -7$

$C_{1,3} = (-1)^{1+3}\times(2.2-3.1) = 1$ $C_{2,1} = (-1)^{2+1}\times(1.4-2.1) = -2$

$C_{2,2} = (-1)^{2+2}\times(1.4-1.1) = 3$ $C_{2,3} = (-1)^{2+3}\times(1.2-1.1) = -1$

$C_{3,1} = (-1)^{3+1}\times(1.1-3.1) = -2$ $C_{3,2} = (-1)^{3+2}\times(1.1-2.1) = 1$

$C_{3,3} = (-1)^{3+3}\times(1.3-1.2) = 1$

Like for example the minor Matrix when we eliminate the first row and first column is 3 1 2 4 and the determinant of this is 3 times 4 - 2 times 1. And so, that is 10 and then similarly C 12 C 13 C 21 and so on. Totally we have 9 quantity C 33.

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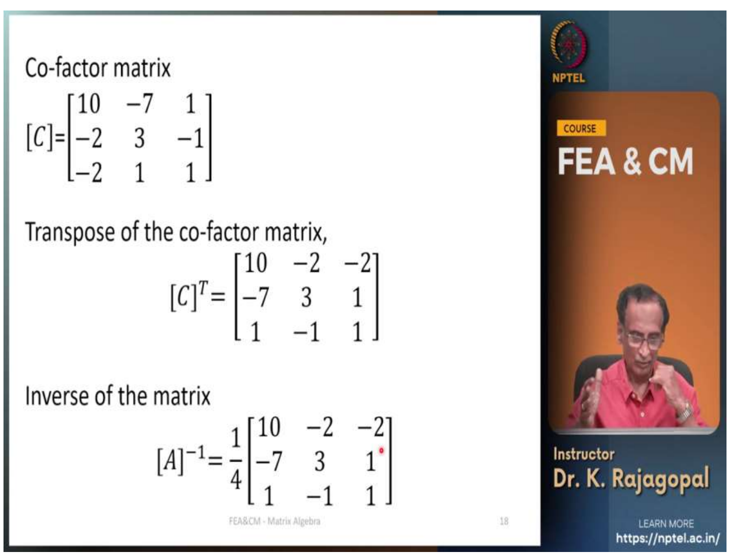

So, these are all the terms and the cofactor Matrix then the cofactor Matrix C is this

$$[C]=\begin{bmatrix}10 & -7 & 1\\ -2 & 3 & -1\\ -2 & 1 & 1\end{bmatrix}$$

and the transpose of this; is this.

Transpose of the co-factor matrix,

$$[C]^T=\begin{bmatrix}10 & -2 & -2\\ -7 & 3 & 1\\ 1 & -1 & 1\end{bmatrix}$$

So, our inverse is an inverse is one fourth of this the transpose of the cofactor Matrix.

Inverse of the matrix

$$[A]^{-1}=\frac{1}{4}\begin{bmatrix}10 & -2 & -2\\ -7 & 3 & 1\\ 1 & -1 & 1\end{bmatrix}$$

And so, if we want to know whether this inverse is correctly calculated or not we can find out by taking a product of A and A inverse and if we have done the calculations correctly we should get diagonal unit Matrix.

We can either take A times A inverse or A inverse times A both will end up with unit diagonal matrix.

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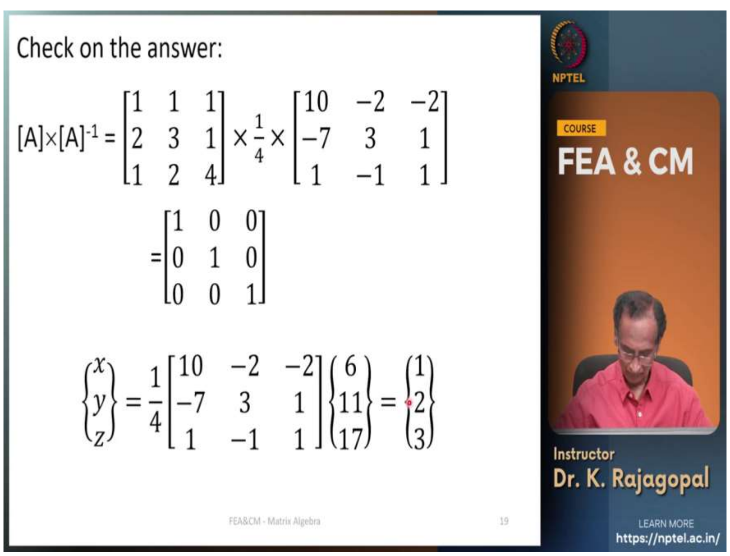

So, the check on the answer is Matrix multiplied by this inverse is exactly equal to a unit diagonal matrix. So, that means that our inverse is correct. So, now we can solve for the three unknowns x y z as this Matrix multiplied by the right hand side Vector 6 11 and 17 and the value is one two three x is 1 y is 2 and z is 3.

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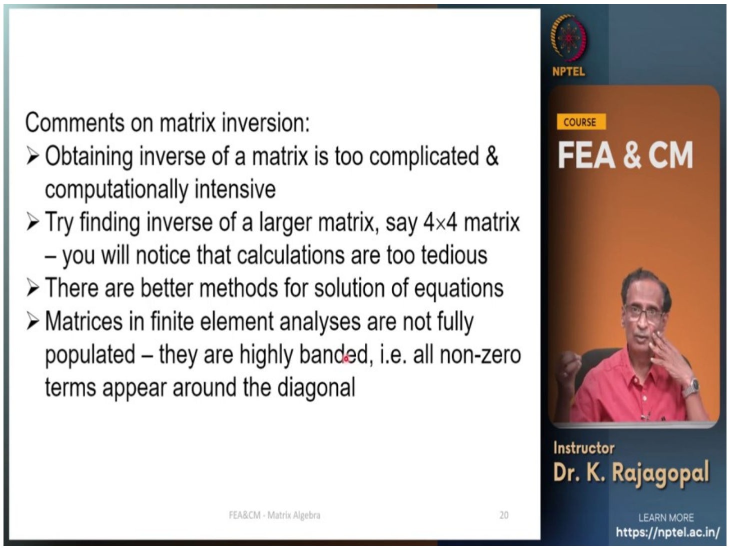

So, actually this Matrix inversion is a good procedure no doubt we can solve for the unknowns. By doing this x is equal to A inverse times the right hand side vector P. So, here we can get the variables x is equal to K inverse P that is the purpose of finding the inverse Matrix. So, we can find the three values x y z as one two three and but actually if you look back obtaining the inverse of a matrix is too complicated and it is very very computationally intensive.

So, for a three by three Matrix it was very simple but let us say I give you a 10 by 10 matrix what do we do? So, if we eliminate one row and one column we are left with nine rows and then you have to split that so, many times to get to get to our two by two Matrix so, that we can find the determinant. So, it is not easy it is not very simple and the calculations are too tedious and too many.

And so, there are different methods for finding the coefficients x y z and there are several varieties and the one method that we are going to solve that we are going to see is the Gauss elimination method and that is especially suitable for finite element analysis because the matrices that we come across are not fully populated. They are highly banded that is actually in our diagonal matrix is also abandoned Matrix but only with one row of elements or one diagonal elements.

And so, in this abandoned matrices we have the non-zero terms only within a small band and all the elements outside of the band are zeros and they do not have much of significance that is during the our competitions these zeros will remain like zeros.

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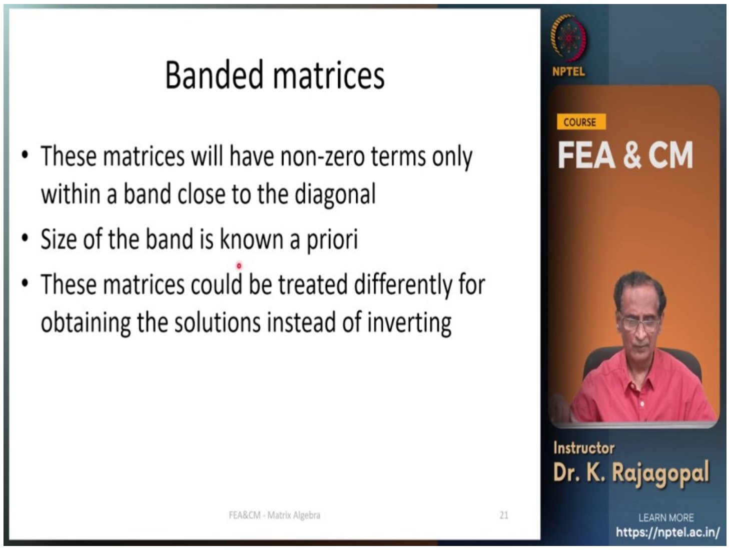

And a banded Matrix is like this and these banded matrices they have non-zero terms only within a band close to the to the diagonal and the size of the band can be estimated beforehand based on the structure and then the node numbering that we assign we can determine the size of the band. And these matrices we can treat them differently for obtaining the solutions instead of our brute method like the inversion method.

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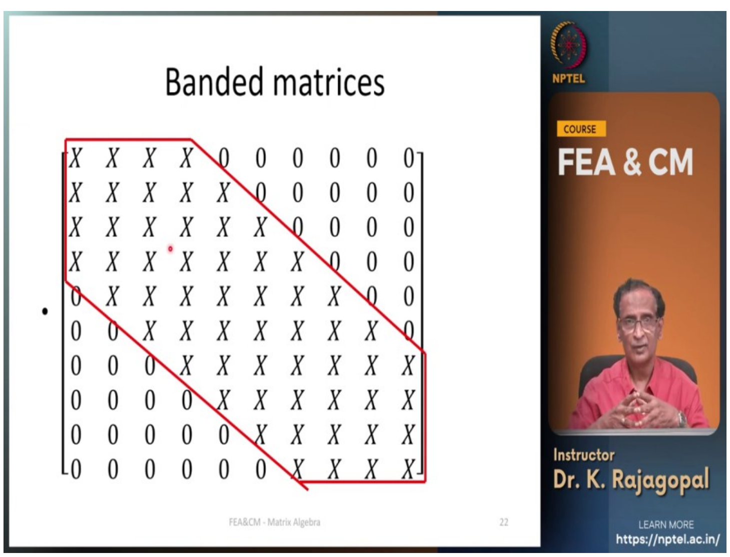

Say a banded Matrix may look something like this. So, within this red band we have all the x axis they have some non-zero values and then we have zeros all beyond your banded Matrix.

And what we are going to do is when we calculate the stiffness matrix of any structure you are not going to evaluate all the elements. Say for example if you have a hundred by hundred element 100 by 100 Matrix 100 times 100 means 10000. So, this Matrix will have 10000 elements but only say some two or three thousand of them could be non-zero and could be within this band.

So, what we do is instead of storing all the ten thousand we will be operating only on the values within this band and the other thing that we need to notice is that our matrices are symmetric. So, whatever you have in the upper diagonal all the upper diagonal terms are also equal to the lower diagonal terms. So, we efficiently save only a small part of the Matrix. So, if you have 100 by 100 Matrix maybe the size of your stiffness Matrix could be just hardly 1500 to 2000 depending on the the type of connections that we have.

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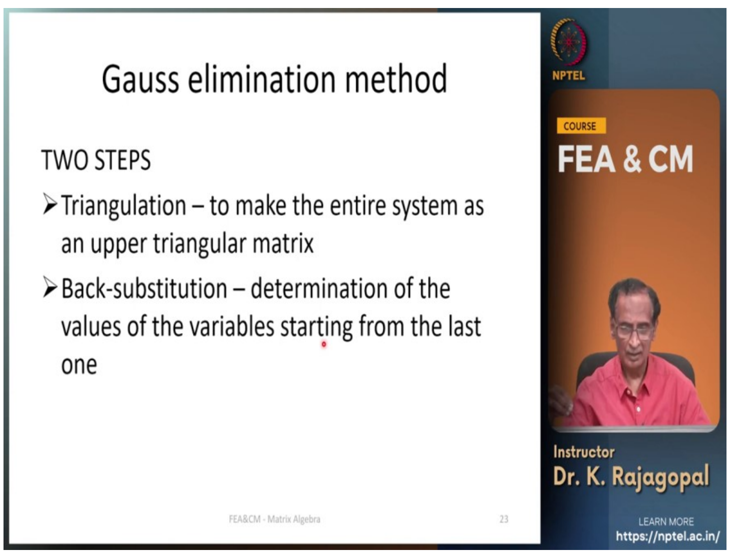

And say one good method that we can use is the Gauss elimination method and this is specially suitable for the type of matrices that we come across which are highly banded. And there are two steps in this Gauss elimination procedure the first one is the triangulation we make the entire system as an upper diagonal matrix by doing some manipulation systematic manipulations. Then once we convert our Matrix into an upper diagonal matrix then we back substitute that is we determine the values of the variables starting from the last value.

Because in the bottommost row will have only one value and one unknown and we determine that and then go backwards. So, it is called as a back substitution.

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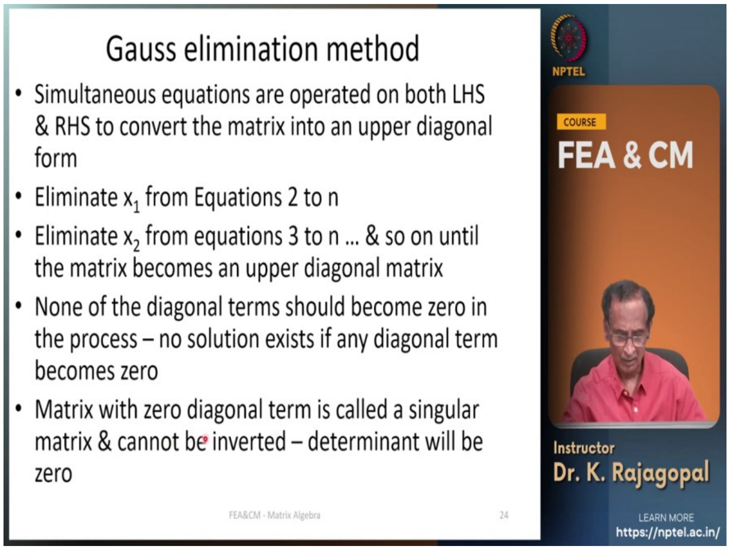

I will explain it in more detail. So, we do some manipulations or arithmetic operations and both left hand side and also on the right hand side to convert the matrix into an upper diagonal matrix and by eliminating let us say x1 from all the equations 2 to n. Like for example x1 is the first term variable that we have and we eliminate that from row 2 to till the last one in the first column.

And x1 is the is the value in the first column and x2 is the value in the second column and so on. So, in the second column we eliminate all the x2 from row 3 to n, n is the total number of equations that we have until our Matrix becomes an upper diagonal matrix. And in the process of making the lower diagonal times zero our Matrix should not become zero should not none of the diagonal terms in our Matrix should become zero.

And if any of the diagonal terms becomes zero then there is no solution there is no unique solution and we cannot find any values for the variables that we have. And a matrix with a zero diagonal term we call it as a singular Matrix and we cannot invert or we cannot even find the determinant of that type of Matrix.

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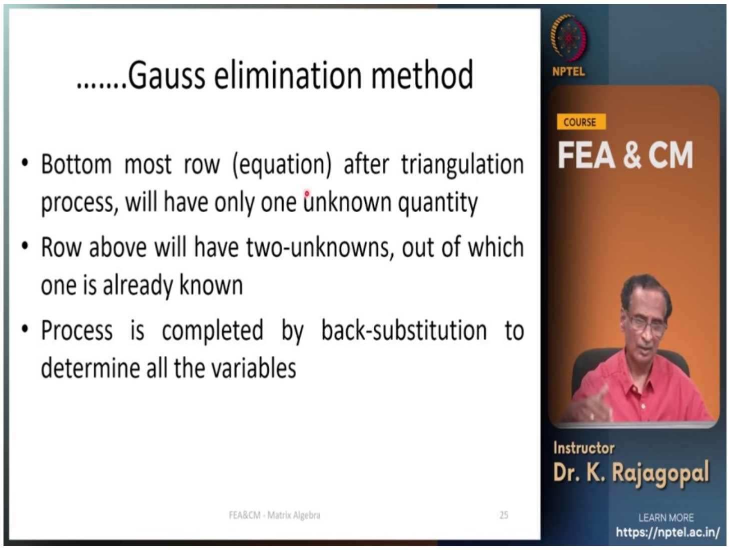

So, the bottom most row of the system after triangulation will have only one unknown and the row above will have two unknowns and above that will have three unknowns and so on. And we start the back substitution by determining the last value because we have only one unknown in that last equation we can solve it and then go for the now the upper rows and so on.

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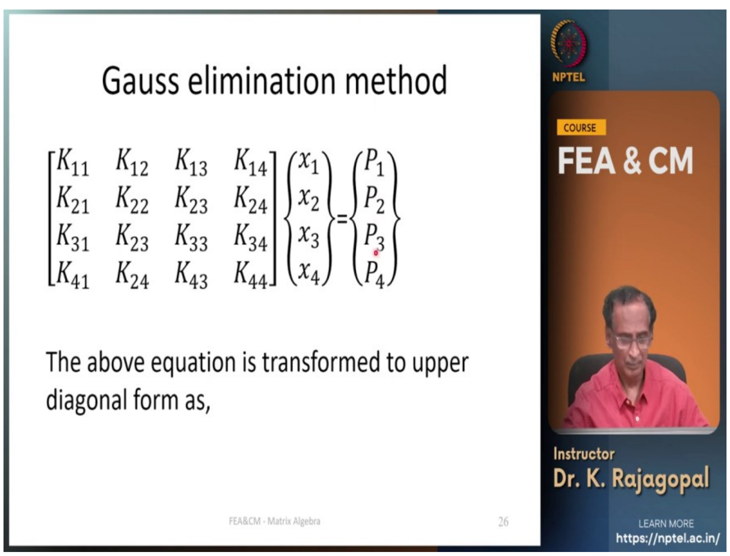

So, for example say we have four unknowns x1 x2 x3 and x4 and the right hand side we haveP1 P2 P3 P4 then these are the coefficients that we have K 11 K 12 and so on.

And these are the known quantities and the right hand side also we have the known quantities P1 P2 P3 and P4 and we need to find the unknowns x1 x2 x3 and so on.

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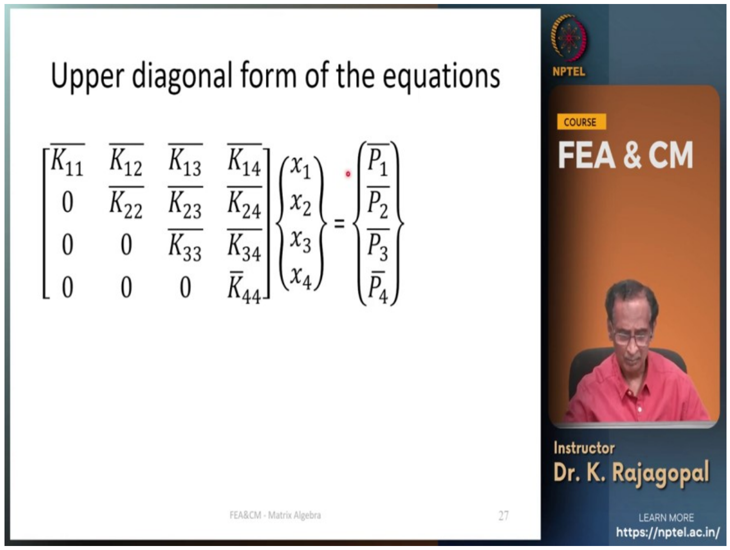

So, after making the Matrix into an upper diagonal form we will have something like this. So, all the lower diagonal terms are zero and K 11 bar K 12 bar and so on.

$$\begin{bmatrix} \overline{K_{11}} & \overline{K_{12}} & \overline{K_{13}} & \overline{K_{14}} \\ 0 & \overline{K_{22}} & \overline{K_{23}} & \overline{K_{24}} \\ 0 & 0 & \overline{K_{33}} & \overline{K_{34}} \\ 0 & 0 & 0 & \overline{K_{44}} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} \overline{P_1} \\ \overline{P_2} \\ \overline{P_3} \\ \overline{P_4} \end{Bmatrix}$$

The right hand side we have P 1 bar P 2 bar P 3 bar because actually I put a bar because these are not the same as what we heard in the earlier Matrix and this P 1 bar need not be equal to P 1.

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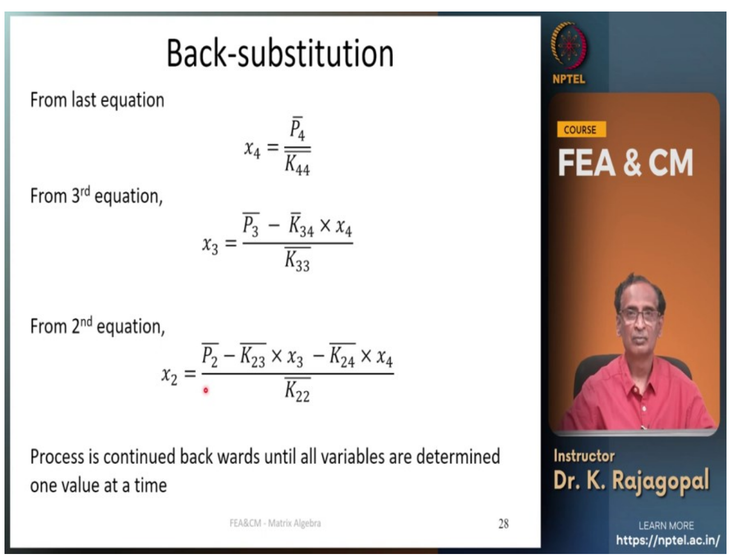

So, from the last equation that we have x4 can be determined as P 4 bar by K 44 bar and then the third equation.

So, let me just see sorry I think I cannot draw it oops I think I will just show with this mouse pointer it is x4 can be written as P4 bar by K 44 bar and once we get x4 the equation Above This will be K 33 times x 3 + K 34 times x 4 is P3 and. So, we can get x 3 as P 3 bar - K Bar 34 times x 4 divided by K 33 bar and so on x2 and so on. We continue until all our unknowns are determined.

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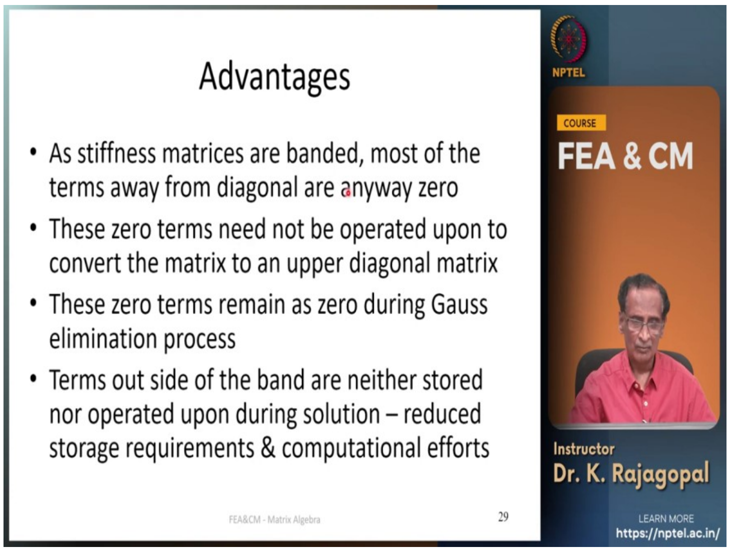

And the advantage is because our stiffness Matrix matrices are banded and most of the terms are away from the diagonal or anywhere zero. So, we do not do any operations to turn them at zero and these zeros they need not be operated upon to convert the matrix into an upper diagonal matrix and these zero terms they remain as zeros during the Gauss elimination process.

So, actually we intelligently use this nature of the banded Matrix and we do not store the terms outside of the band or do any operations and this will reduce our storage requirements and then the computational efforts.

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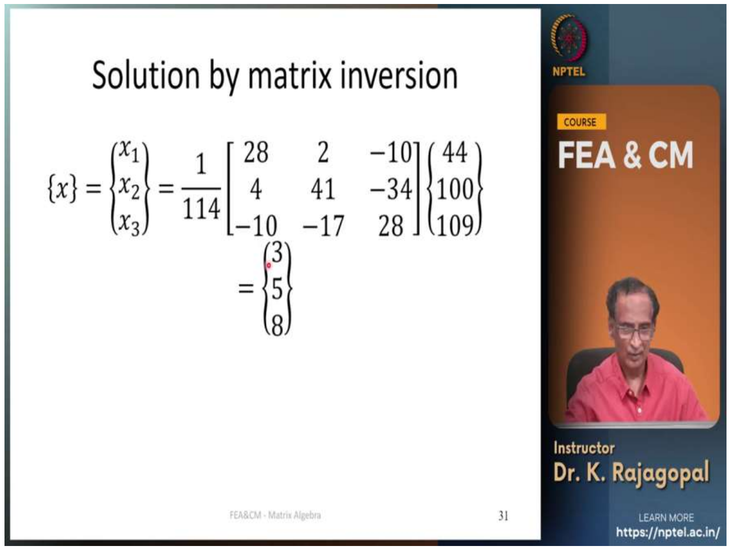

So, let us take three questions with the three unknowns x 1, x 2 and x 3 and solve for x 1 x2 x3 and one method is by inverting and A inverse is this and the x values are invert that Matrix multiplied by the right hand side vector. So, that is x 1 is 3 x2 is 5 and x3 is 8.

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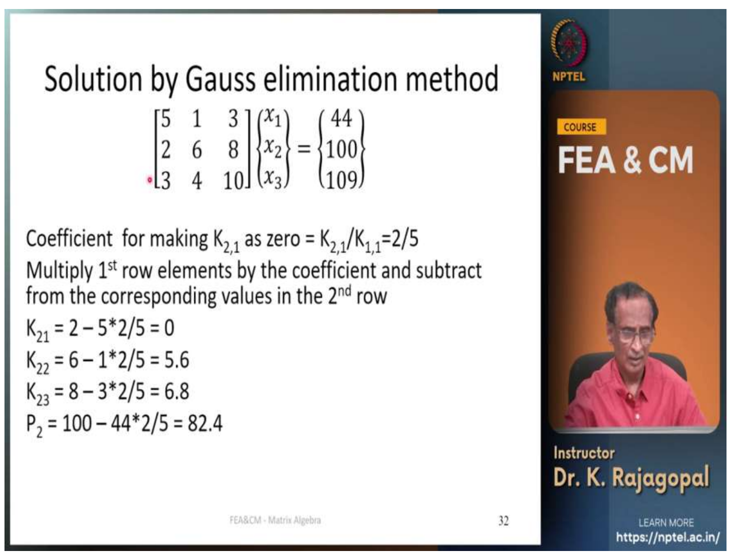

And let us solve the simultaneous equations by Gauss elimination method let us take the same set of equations. And so, our procedure is to gradually turn all the lower diagonal terms to zeros we first turn the terms in the lower diagonal in the first column as zeros and then in the second column is 0 and then the third column anyway it is the third column is the diagonal term that we do not make it as zero.

Coefficient for making $K_{2,1}$ as zero = $K_{2,1}/K_{1,1}$=2/5

And our object is to first convert this K 21 as 0. So, for doing that we can find the coefficient as K 21 divided by K 11 in this particular case it is 2 by 5 that is equal to 0.4.

Then the next step is multiplied by this coefficient with all the terms in the first row and subtract from the corresponding rows in the second row.

$$K_{21} = 2 - 5*2/5 = 0$$
$$K_{22} = 6 - 1*2/5 = 5.6$$
$$K_{23} = 8 - 3*2/5 = 6.8$$
$$P_2 = 100 - 44*2/5 = 82.4$$

So, for example we have determined this coefficient as a two by five k two one will be K 21 - 5 multiplied by this coefficient 2 by 5 that is 0. And K 22 is 6 - 1 times 2 by 5 so, that is 5.6 then K 23 is 8 - 3 times 2 by 5 and that will be 6.8 . So, K 23 is also 0 and P 2 the modified the right hand side Vector is 100 - 44 times 2 by 5 that is equal to 82.5. And then for eliminating this K 13 we go through the same process. So, we take this three by five that is our coefficient but then we multiply all the terms and the first row and subtract them from the third row.

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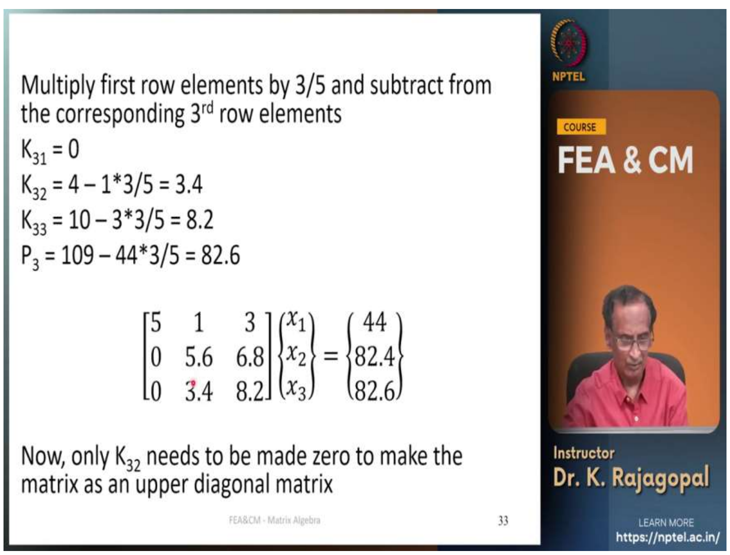

To get our K 31 is 0 K32, K33 and P 3 is a previously we changed that as 109 and we um.

$$K_{31} = 0$$
$$K_{32} = 4 - 1*3/5 = 3.4$$
$$K_{33} = 10 - 3*3/5 = 8.2$$
$$P_3 = 109 - 44*3/5 = 82.6$$

So, we take this coefficient multiplied by the by the P 1 and get our value. And so, we have eliminated all the lower diagonal terms in the first column and we need to eliminate only one more value K 32 and for that the value the coefficient is 3.4 divided by 5.6.

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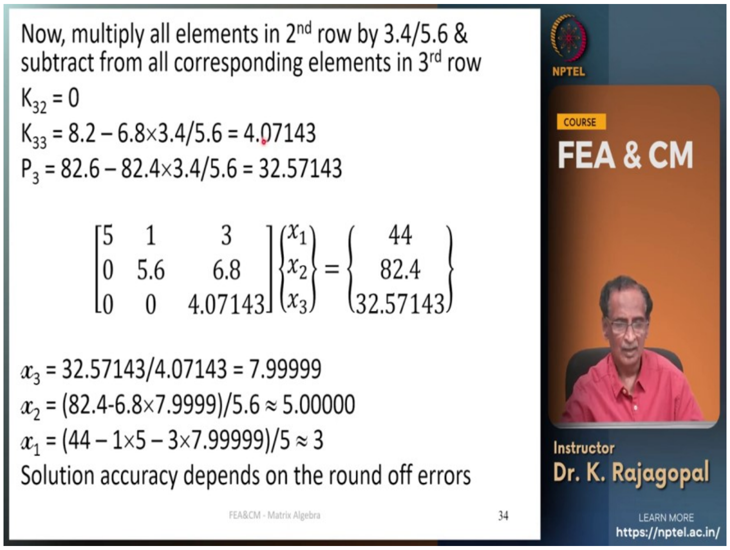

And then by subtracting this coefficient with the with the values in the second row and subtracting from the corresponding elements we get our K 32 as 0 K 33 has some 4.07 and so on. P 3 is 82.5 minus all this and.

Now, multiply all elements in 2nd row by 3.4/5.6 & subtract from all corresponding elements in 3rd row

$$K_{32} = 0$$
$$K_{33} = 8.2 - 6.8\times3.4/5.6 = 4.07143$$
$$P_3 = 82.6 - 82.4\times3.4/5.6 = 32.57143$$

So, this is the upper diagonal matrix that we have on the right hand side vector and then by back substituting we can get x3 as 32.57143 divided by this and that is approximately equal to 8 almost equal to 8 but because of round off errors we are getting this as 7.999.

$$\begin{bmatrix}5 & 1 & 3\\0 & 5.6 & 6.8\\0 & 0 & 4.07143\end{bmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix} = \begin{Bmatrix}44\\82.4\\32.57143\end{Bmatrix}$$

And our x2 can be determined because we have the x3 so, 5.6 times x2 + 6.8 times x3 is equal to 82.4 . So, our x2 is 82.4 - 6.8 times x3 that is while evaluated here divided by 5.6.

$$x_3 = 32.57143/4.07143 = 7.99999$$
$$x_2 = (82.4-6.8\times7.9999)/5.6 \approx 5.00000$$
$$x_1 = (44 - 1\times5 - 3\times7.99999)/5 \approx 3$$

So, our x3 x2 and x1 we can determine and here even in the matrix inversion procedure the accuracy of the solution depends on the round of errors. Say round of errors is let us say I write this 4.07. So, on as 4.1 and P 3 we write it as a 32.6 instead of 32.57 then that we call as round rounding off and if you round off then our solution may not be accurate.

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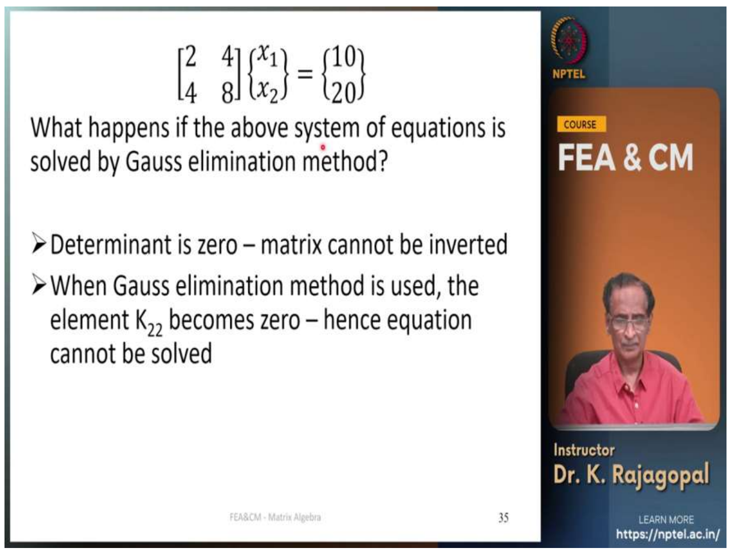

So, if you have an equation like this a system;

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 20 \end{Bmatrix}$$

like this two equations but then they are not independent. So, we cannot solve for the values and so, if you do this. So, in the process of Gauss elimination this K 22 that is 8 will become 0. So, hence we cannot solve the system of equations that we have and so, that is what we call as a singular Matrix or rank deficient Matrix.

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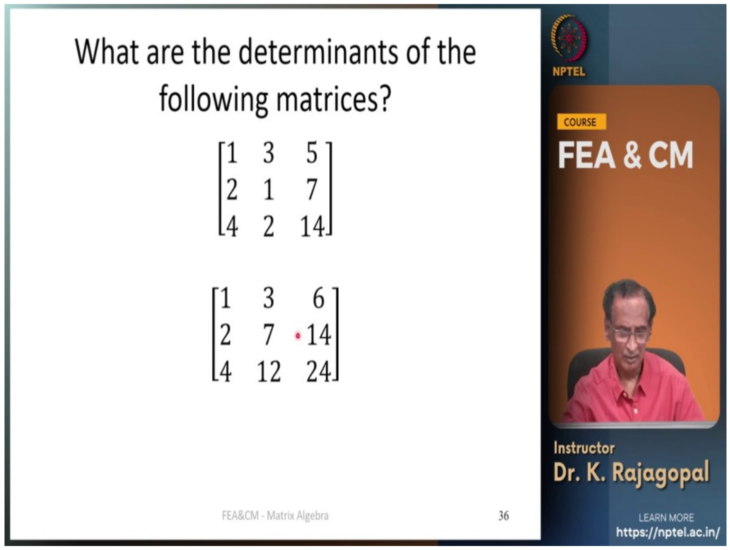
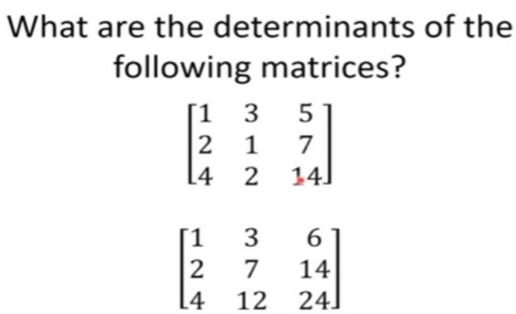


So, let us say I give you these two matrices and what is the determinant of these matrices it is a three by three Matrix.



And the usual procedure for determining the determinant you know but in this particular case if You observe the third row and second row that proportional to each other. So, I can get the third row by taking product of this row by 2, 2 times 2 is 4 2 times 1 is 2 2 times 7 is 14 and so on. So, the determinant of this Matrix is 0.

And the second Matrix see the second column and third column there are proportional to each other. So, once again our determinant is zero. So, we will not be able to invert this system of equations.

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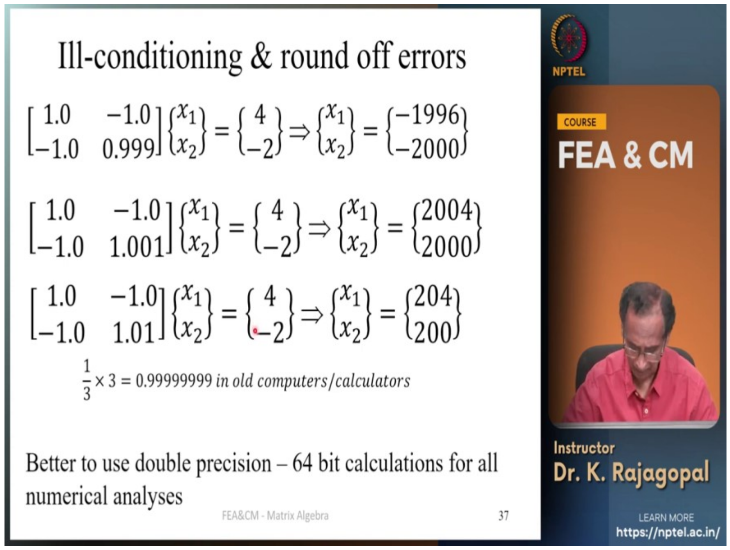

And apart from a round of errors we could have ill conditioning. So, I have two equations here in terms of two quantities x 1 and x 2

$$\begin{bmatrix} 1.0 & -1.0 \\ -1.0 & 0.999 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} -1996 \\ -2000 \end{Bmatrix}$$

$$\begin{bmatrix} 1.0 & -1.0 \\ -1.0 & 1.001 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2004 \\ 2000 \end{Bmatrix}$$

1 - 1 that is x 1 - x 2 is 4 and - x 2 sorry – x 1 + 0.999 times x 2 is equal to -2 and if you solve we will get x 1 and x 2 as these values and say these numbers they can be rounded off like this say here instead of 0.999 we have 1.001 or 1.01. And our equations can be solved and we get in the if you solve the first set of equations you get x 1 of -1996 and x 2 is – 2000.

$$\begin{bmatrix} 1.0 & -1.0 \\ -1.0 & 1.01 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 204 \\ 200 \end{Bmatrix}$$

Whereas from the third set x 1 is 204 and x 2 is 200. is actually they are totally different we just have one small value 0.999 and 1.001 and 1.01 but so, much difference in the value that is because of deal conditioning and then because of this the the non-unique nature of this solution. So, actually it is like say we have two equations and they are both let us say if they are both parallel to each other and we define that the parameters should satisfy both the equations and they will do that at the intersection point.

And so, here we can think of this as almost two parallel lines they will intersect with each other at a very very large distance whereas in this case also they meet at very large distance whereas the third equation they both the both of these lines they have an intersection at some reasonable Point x 1 index 2 like this. So, the reason I am telling you is so, when I was growing up we had some computers and some calculators and in those calculators if you do this operation one divided by 3 times 3 you will get 0.99999999 you will not get one.

$$\frac{1}{3} \times 3 = 0.99999999 \textit{ in old computers/calculators}$$

It is in the modern calculator so, we get one but you will not believe this was the solution that we used to get when we were growing up and so, that means that depending on the number of decimal places that we are working with our solution might completely change. So, it is best to use all our calculations in double precision says 64-bit calculations that is we have more number of decimal places in our calculations that is the only way to get accurate solutions in finite element analysis.

Because all our calculations they involve number of these matrices we have to invert them or we have to do some arithmetic manipulations or to determine these values and at each stage if we go on rounding off then at the end you might get entirely different result. So, we should be careful in all our computations.

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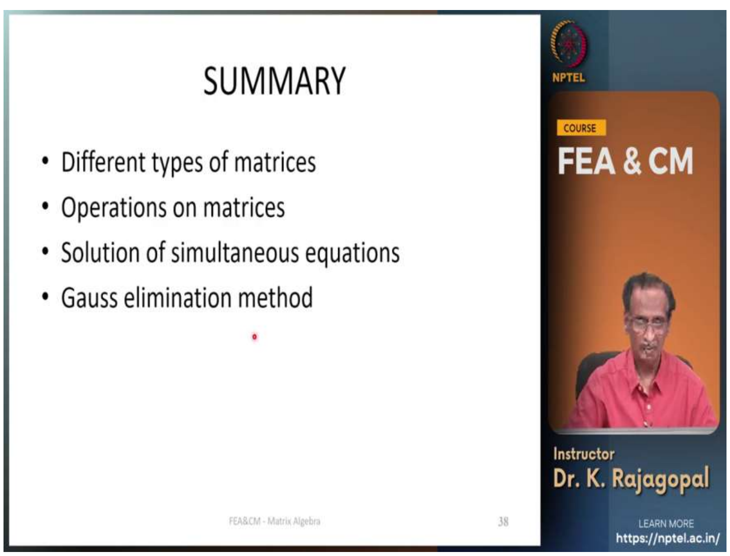

So, that is a brief introduction to Matrix methods and let me just summarize we have seen the different types of matrices the square Matrix symmetric Matrix and then the transpose of a matrix then operations are the matrices addition subtraction multiplication and then solution of simultaneous equations by either taking the inverse of that Matrix or by Gauss elimination method.

And between these two the Gauss elimination method is more optimal especially in the context of finite element analysis because all the matrices that we have they are highly banded just around the diagonal term we have some non-zero terms and beyond that we have all zero terms that we do not need to either store or do any operations. So, that is a brief introduction to Matrix algebra and if you have any questions please send email to me at this address profkrg@gmail.com.

And I will be able to respond back and we will have some tutorial problems later and please try to attempt all these problems and make sure that you understand all these Matrix operations. Because when we are discussing when we are doing the finite element calculations we need to do all these things like the matrix products inversion or Gauss elimination and determining the displacements and so on. So, thank you very much we will meet in the next class.