FEM and Constitutive Modeling in Geomechanics

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Lecture – 19
Force Vector Due to Surface Traction

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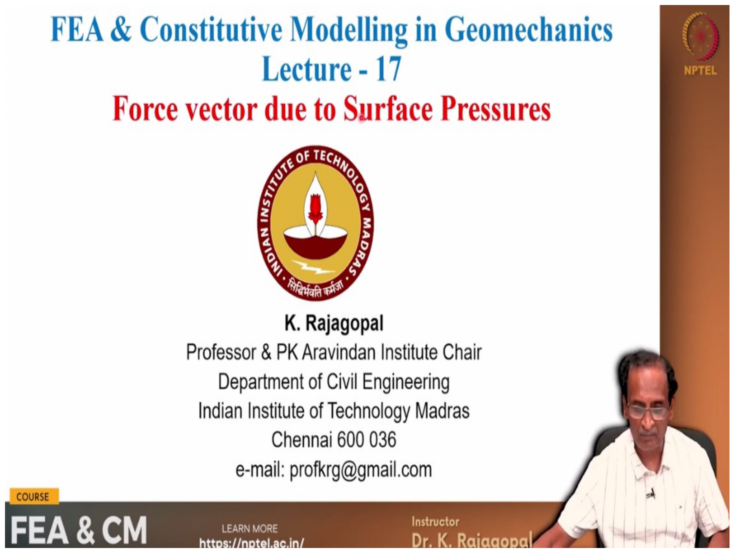

Hello students. In the previous class we had seen three different types of isoparametric calculations. One is the estimation of this stiffness matrix and the other is the load vector due to the body weight then the load vector due to the stresses B transpose sigma and in this class let us look at how we can calculate the force vector due to surface pressure.

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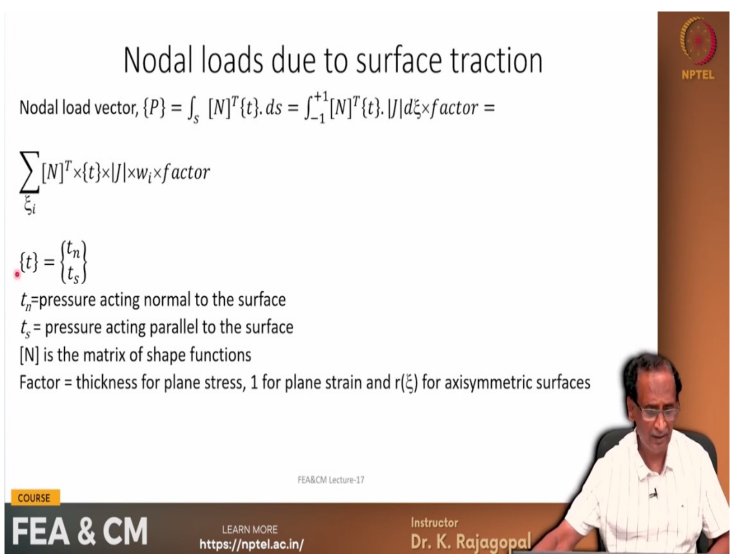

And the equation for this is integral N transpose t ds and this is a surface integral whereas the previous ones are all volume integrals so this is a surface integral and in two dimensional case we have surface means length multiplied by width and so we have only integration 1 direction and then multiplied by either the thickness or unit value or radius because in the case of axisymmetric and so here it is N transpose T and our T is the traction vector.

Nodal loads due to surface traction

Nodal load vector, $\{P\} = \int_s [N]^T\{t\}.ds = \int_{-1}^{+1}[N]^T\{t\}.|J|d\xi \times factor =$

$$\sum_{\xi_i}[N]^T \times \{t\} \times |J| \times w_i \times factor$$

$$\{t\} = \begin{Bmatrix} t_n \\ t_s \end{Bmatrix}$$

t_n=pressure acting normal to the surface
t_s = pressure acting parallel to the surface
[N] is the matrix of shape functions
Factor = thickness for plane stress, 1 for plane strain and r(ξ) for axisymmetric surfaces

J is the Jacobian determinant d Xi and then the factor is the thickness for plane stress and one for the plane strain and then the radius for axisymmetric case and this traction is there could be a normal traction or a shear traction.

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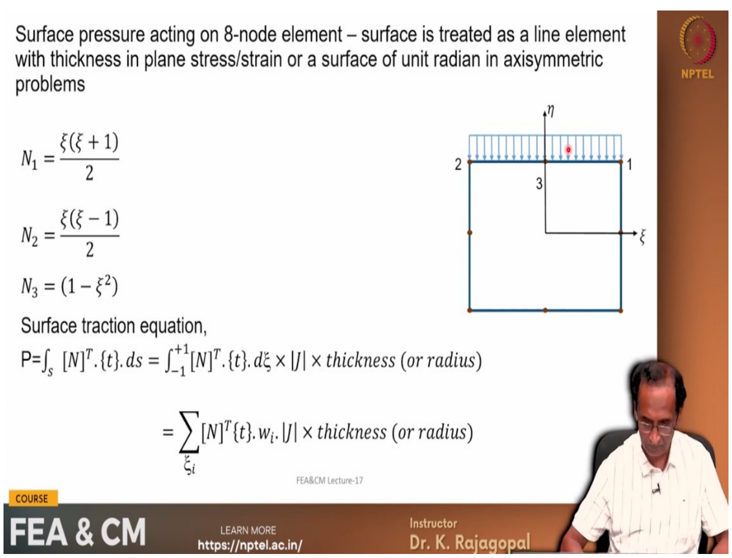


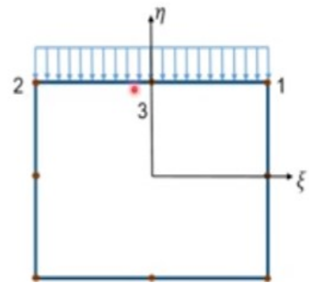
So, let us look at the surface pressure acting at the 8 node element and this we can treat it as a line element with some thickness or unit thickness and N 1 we have already derived them in the context of 1 dimensional isoparametric elements N 1 is Xi times Xi + 1 by 2, N 2 is Xi times Xi – 1 by 2 and N 3 is 1 – Xi square and our equation is N transpose t ds and initially we will do simple calculations.

Surface pressure acting on 8-node element – surface is treated as a line element with thickness in plane stress/strain or a surface of unit radian in axisymmetric problems

$$N_1 = \frac{\xi(\xi+1)}{2}$$

$$N_2 = \frac{\xi(\xi-1)}{2}$$

$$N_3 = (1-\xi^2)$$



Surface traction equation,

$$P = \int_s [N]^T.\{t\}.ds = \int_{-1}^{+1} [N]^T.\{t\}.d\xi \times |J| \times thickness\ (or\ radius)$$

$$= \sum_{\xi_i} [N]^T\{t\}.w_i.|J| \times thickness\ (or\ radius)$$

And then we will not assume any direction for these forces except that we know. If there is a normal pressure acting on the surface it is in the negative y direction or if it is acting on this side it is along the x direction and so on, but we will later we will see how to automate them because we humans we have the brains and we can say that this is horizontal surface and then we have vertical pressure.

But then how do you make a computer know that this is a horizontal surface and this force should be acting in the negative y direction that we will see later.

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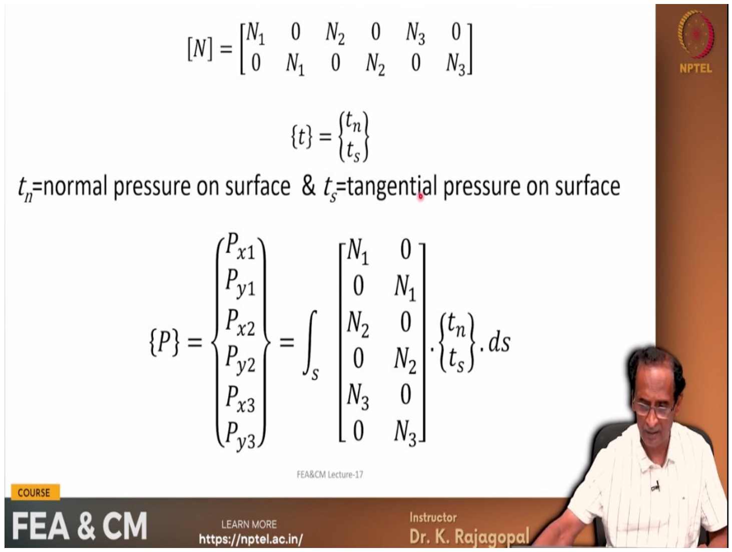

And our shape function matrix will be N 1 0, N 2 0, N 3 0 because we have only 3 nodes and then the traction is t n and t s. The t n is the normal pressure on the surface and t s is the tangential pressure on the surface sorry I think this should be Xi, but somehow because if incompatibility between different computers this happens, but I will correct it later.

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

$$\{t\} = \begin{Bmatrix} t_n \\ t_s \end{Bmatrix}$$

t_n=normal pressure on surface & t_s=tangential pressure on surface

$$\{P\} = \begin{Bmatrix} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \\ P_{x3} \\ P_{y3} \end{Bmatrix} = \int_s \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} . \begin{Bmatrix} t_n \\ t_s \end{Bmatrix} . ds$$

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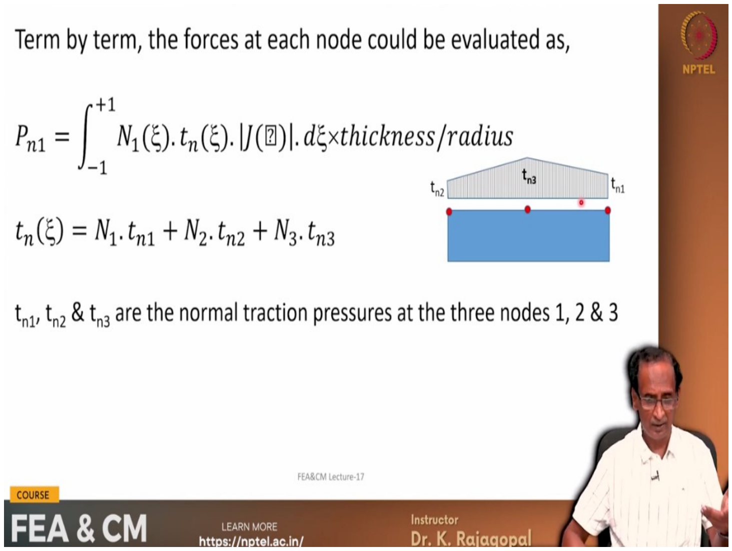

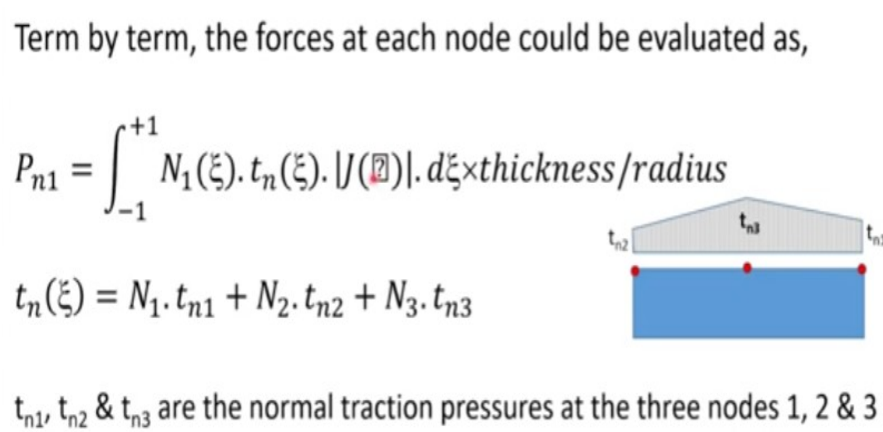

So, our normal force at node 1 is N 1 times t n and then the Jacobian at different Xi and the integrated over – 1 to + 1 and in general the traction at any point could be written as N 1 t n 1 + N 2 t n 2 + N 3 t n 3 where t n 1 is the magnitude of the normal traction at node 1, t n 2 is the magnitude of the normal traction at node 2 and this is at node 3 and this is an input parameter this we have to give as an input as a data.

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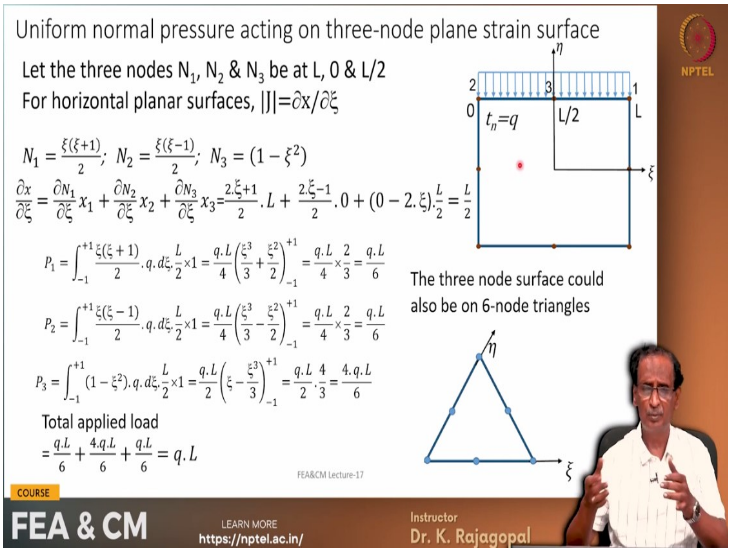
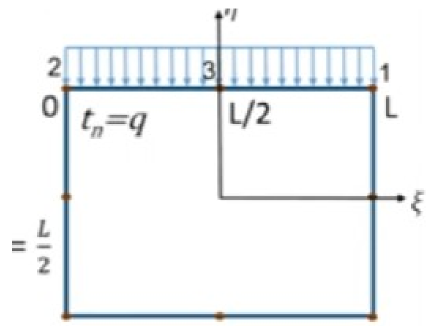


And so our N 1 is Xi times Xi + 1 by 2, N 2 is Xi times Xi – 1 by 2, N 3 is 1 – Xi square and the determinant of the Jacobian matrix just simply dou x by dou Xi because we have only one coordinate Xi and then this being horizontal surface there is no dou y by dou Xi it will be 0. So, our dou x by dou Xi comes out as L by 2 because the center node is exactly at the mid length.



$$N_1 = \frac{\xi(\xi+1)}{2};\ N_2 = \frac{\xi(\xi-1)}{2};\ N_3 = (1-\xi^2)$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi}x_1 + \frac{\partial N_2}{\partial \xi}x_2 + \frac{\partial N_3}{\partial \xi}x_3 = \frac{2.\xi+1}{2}.L + \frac{2.\xi-1}{2}.0 + (0-2.\xi).\frac{L}{2} = \frac{L}{2}$$

So, the Xi's cancel out and then we are left with only the L by 2. If this is the length L dou x by dou Xi is L by 2 and the loaded node 1 is N 1 times q; q is the normal traction acting on that surface d Xi times mod J determinant is L by 2 multiplied by unit thickness. So, P 1 is q L by 6 and then P 2 is also q L by 6 if you do this integration then P 3 is 1 – Xi square that is the shape function q d Xi L by 2 times 1.

$$P_1 = \int_{-1}^{+1} \frac{\xi(\xi+1)}{2}.q.d\xi.\frac{L}{2}\times 1 = \frac{q.L}{4}\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)_{-1}^{+1} = \frac{q.L}{4}\times\frac{2}{3} = \frac{q.L}{6}$$

$$P_2 = \int_{-1}^{+1} \frac{\xi(\xi-1)}{2}.q.d\xi.\frac{L}{2}\times 1 = \frac{q.L}{4}\left(\frac{\xi^3}{3}-\frac{\xi^2}{2}\right)_{-1}^{+1} = \frac{q.L}{4}\times\frac{2}{3} = \frac{q.L}{6}$$

$$P_3 = \int_{-1}^{+1} (1-\xi^2).q.d\xi.\frac{L}{2}\times 1 = \frac{q.L}{2}\left(\xi-\frac{\xi^3}{3}\right)_{-1}^{+1} = \frac{q.L}{2}.\frac{4}{3} = \frac{4.q.L}{6}$$

And this comes out as 4 q L by 6. We see that at the two corner nodes the load is q L by 6 whereas at the center node at the mid side node it is 4 q L by 6 it is 4 times the load at the two corner nodes and similar response we have also seen when we calculated N transpose b or B transpose sigma. Always the mid side node they have higher contribution compared to the corner nodes and then the center node it might have much higher contribution.

Total applied load

$$= \frac{q.L}{6} + \frac{4.q.L}{6} + \frac{q.L}{6} = q.L$$

And if you add up P 1 + P 2 + P 3 it comes to q L and we are considering unit length in the perpendicular direction because this is a plane strain case so it is just simply q L and this surface could also be a part of 6 node triangle, it need not be of 8 node quadrilateral or 9 node quadrilateral.

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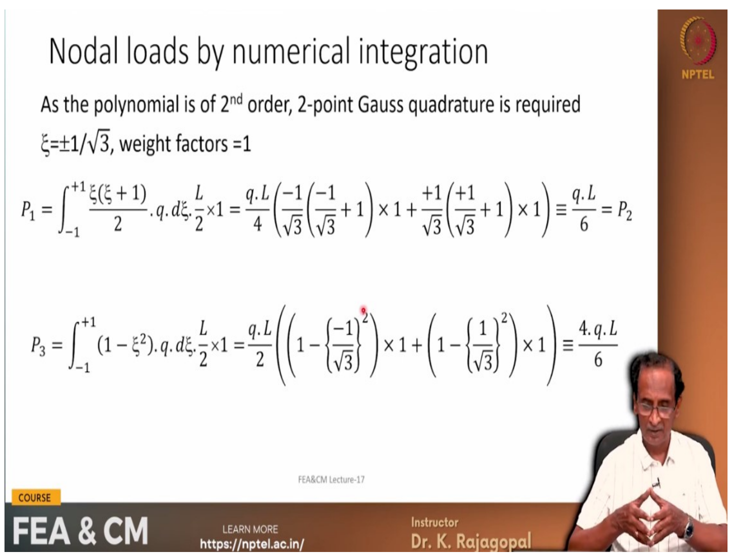

And we can do the same thing even with the numerical integration. So, previously we have done the analytical integration, but now let us do with a numerical integration. Let us use a 2 point integration because we have a second order polynomial.

Nodal loads by numerical integration

As the polynomial is of 2nd order, 2-point Gauss quadrature is required

$\xi = \pm 1/\sqrt{3}$, weight factors =1

$$P_1 = \int_{-1}^{+1} \frac{\xi(\xi+1)}{2}.q.d\xi.\frac{L}{2}\times 1 = \frac{q.L}{4}\left(\frac{-1}{\sqrt{3}}\left(\frac{-1}{\sqrt{3}}+1\right)\times 1 + \frac{+1}{\sqrt{3}}\left(\frac{+1}{\sqrt{3}}+1\right)\times 1\right) \equiv \frac{q.L}{6} = P_2$$

$$P_3 = \int_{-1}^{+1} (1-\xi^2).q.d\xi.\frac{L}{2}\times 1 = \frac{q.L}{2}\left(\left(1-\left\{\frac{-1}{\sqrt{3}}\right\}^2\right)\times 1 + \left(1-\left\{\frac{1}{\sqrt{3}}\right\}^2\right)\times 1\right) \equiv \frac{4.q.L}{6}$$

So, if you go through this product and use the brackets correctly you will get q L by 6 that is the P 2 and then at the mid side node we get 4 q L by 6 like if you go through this entire thing and then simplify we will get 4 q L by 6 that is the same as what we got analytically.

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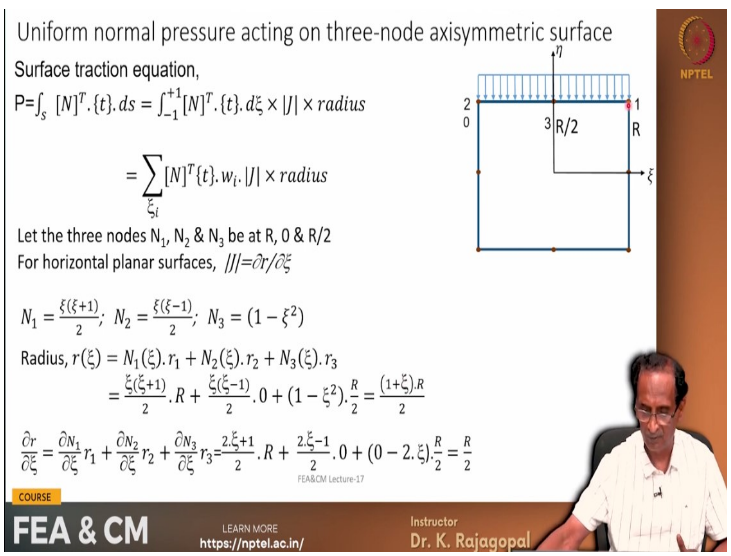
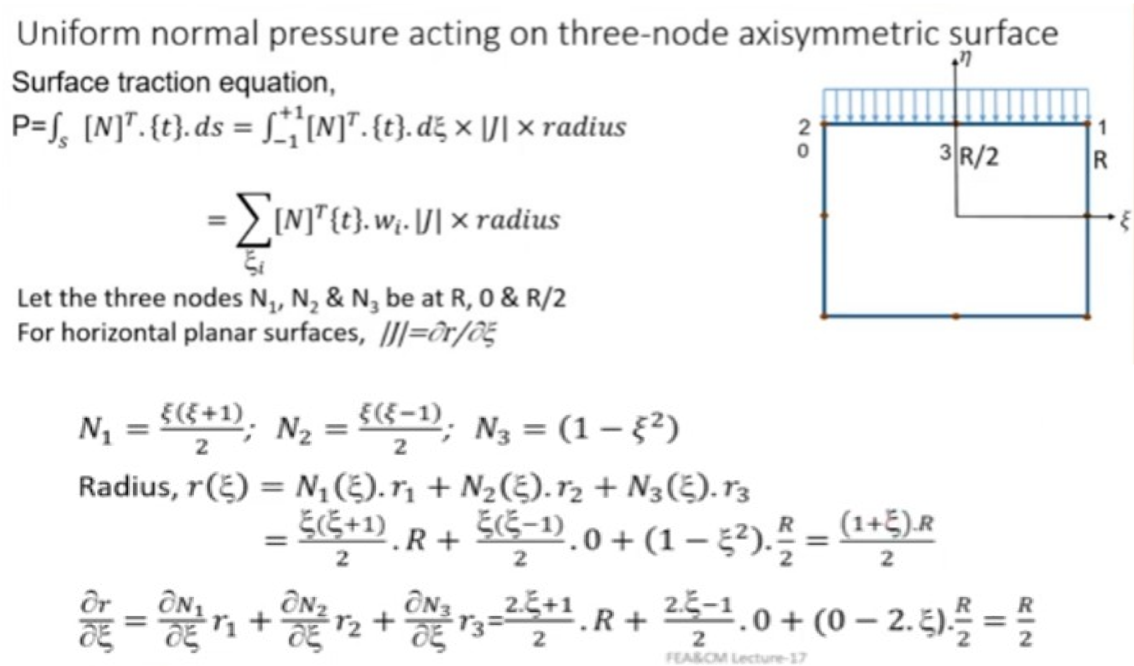


And now let us look at an axisymmetric surface and the only difference between the plane strain and axisymmetric is the radius. So, in the plane strain case we have a unit thickness, but in the case of axisymmetric we have the radius that is varying along the length along Xi and our N 1 is Xi times Xi + 1 by 2, N 2 is Xi times Xi – 1 by 2 and N 3 is this and our radius can be written as 1 in general N 1 r 1 + N 2 r 2 + N 3 r 3 and that comes out as 1 + Xi r by 2.



So, when Xi is – 1 we are at this point node 2 that is 0 and at Xi of + 1 you have at node 1 that is R 1 + 1 by 2 times R then at the mid side node Xi is 0 so that is R by 2. So, dou r by dou Xi comes out as r by 2 just as how we got as L by 2 for the plane strain case we will get the same thing r by 2.

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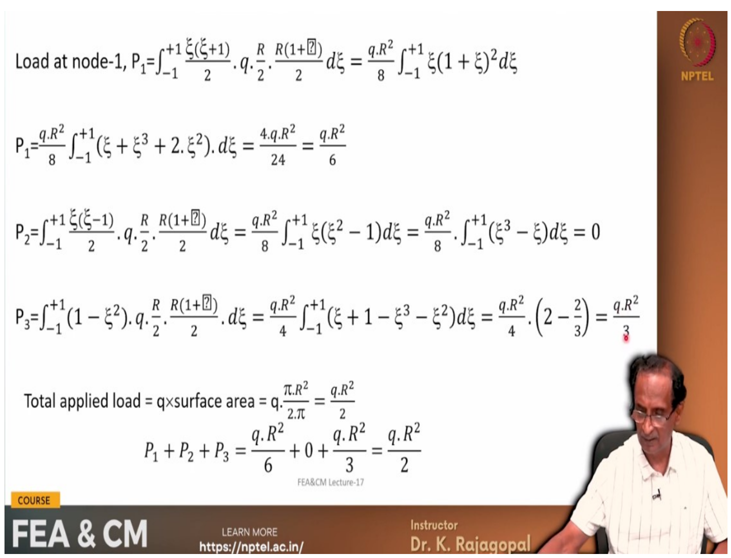

And the load at node 1 is P 1 is q R square by 6 and P 2 it is coming out as 0 and P 3 is q R square by 3 and so we have 3 nodes; node 1, node 2 and node 3. At node 2 there is no load actually you can imagine this is a circle and the center point is the point of center of gravity and so it should not attract any load whereas the other nodes are attracting load and the mid side load is attracting higher load compared to the outside node.

$$\text{Load at node-1, } P_1=\int_{-1}^{+1}\frac{\xi(\xi+1)}{2}.q.\frac{R}{2}.\frac{R(1+\xi)}{2}d\xi=\frac{q.R^2}{8}\int_{-1}^{+1}\xi(1+\xi)^2 d\xi$$

$$P_1=\frac{q.R^2}{8}\int_{-1}^{+1}(\xi+\xi^3+2.\xi^2).d\xi=\frac{4.q.R^2}{24}=\frac{q.R^2}{6}$$

$$P_2=\int_{-1}^{+1}\frac{\xi(\xi-1)}{2}.q.\frac{R}{2}.\frac{R(1+\xi)}{2}d\xi=\frac{q.R^2}{8}\int_{-1}^{+1}\xi(\xi^2-1)d\xi=\frac{q.R^2}{8}.\int_{-1}^{+1}(\xi^3-\xi)d\xi=0$$

$$P_3=\int_{-1}^{+1}(1-\xi^2).q.\frac{R}{2}.\frac{R(1+\xi)}{2}.d\xi=\frac{q.R^2}{4}\int_{-1}^{+1}(\xi+1-\xi^3-\xi^2)d\xi=\frac{q.R^2}{4}.\left(2-\frac{2}{3}\right)=\frac{q.R^2}{3}$$

And the total applied load is the q times surface area and we are considering only unit radian. See pi R square is the total area of the circle multiplied by q divided by 2 pi is our applied load q R square by 2 and our P 1 + P 2 + P 3 is also q R square by 2. So, our calculations at the different nodes are correct like if you compare with the total applied load.

$$\text{Total applied load} = q\times\text{surface area} = q.\frac{\pi.R^2}{2.\pi}=\frac{q.R^2}{2}$$

$$P_1+P_2+P_3=\frac{q.R^2}{6}+0+\frac{q.R^2}{3}=\frac{q.R^2}{2}$$

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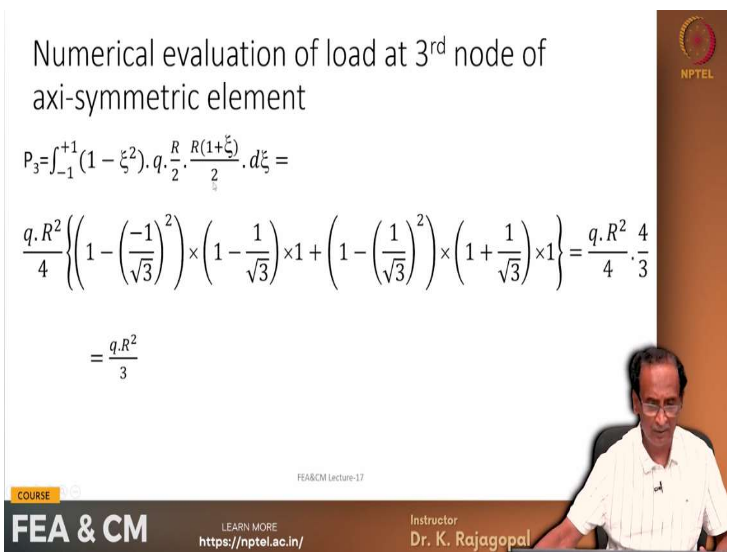
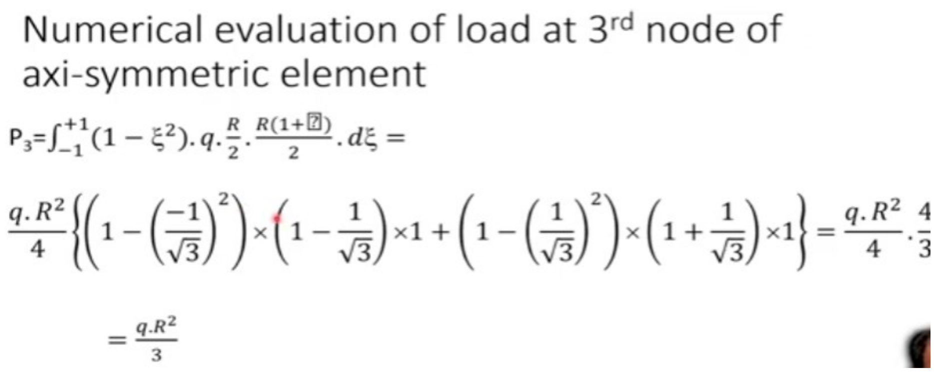


And we can also apply our numerical integration and our polynomial order is of 3. So, our R at any Xi is R times 1 + Xi by 2 and since we have a third order polynomial we can use 2 point integration and then we get exactly the same as what we got analytically q R square by 3.



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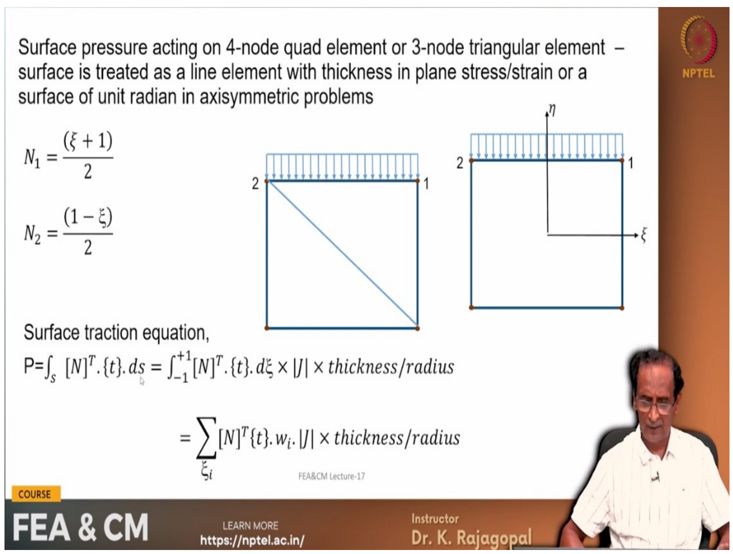
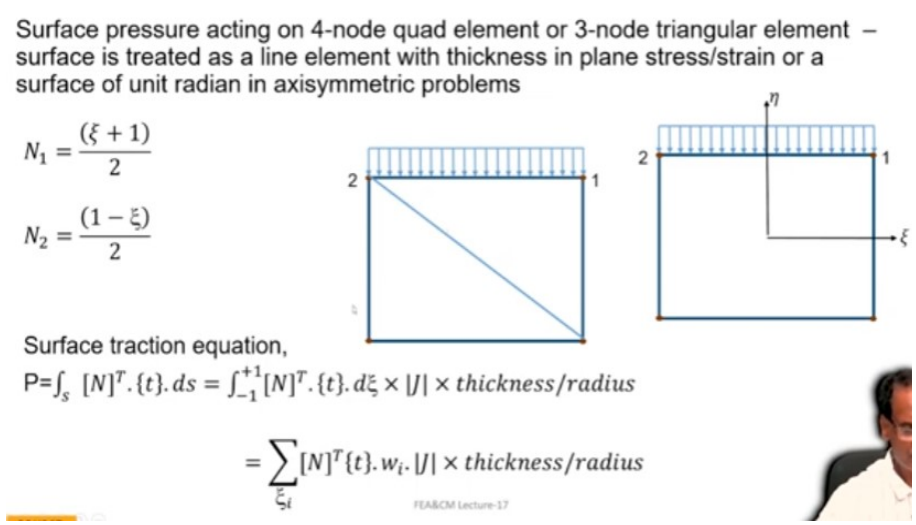


And the surface pressure on 3 node triangles or 4 node quadrilaterals are much simpler.



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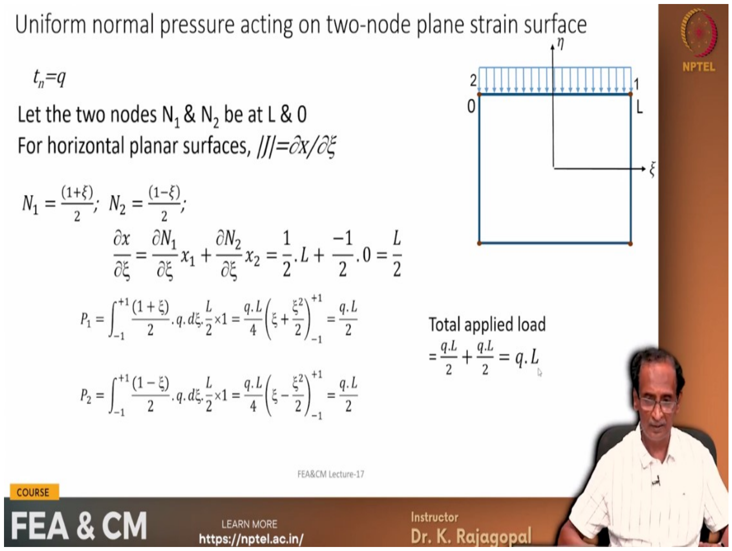
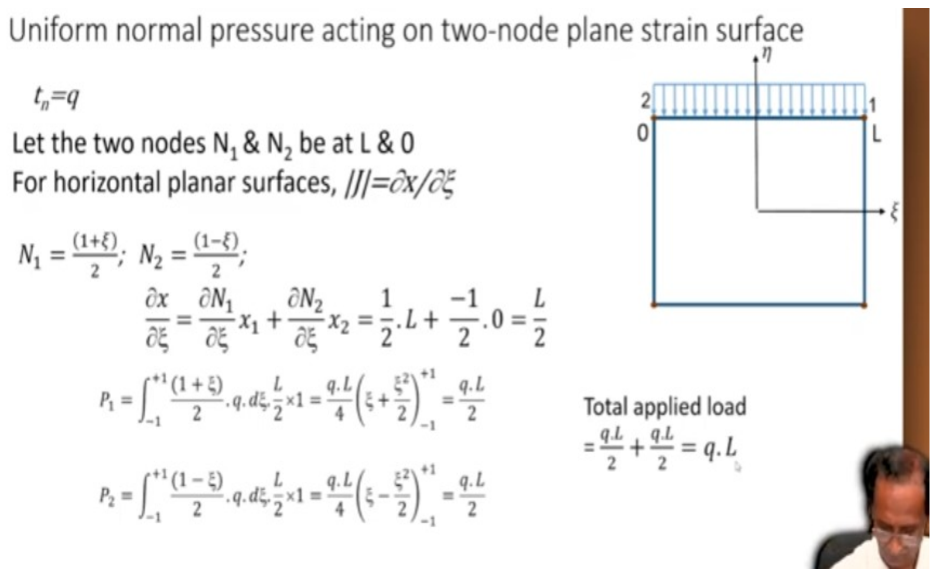


We can follow the same procedure and we will get equal load at these two nodes q L by 2 q L by 2 and the total load is q L because we are considering a unit thickness without a plane direction.



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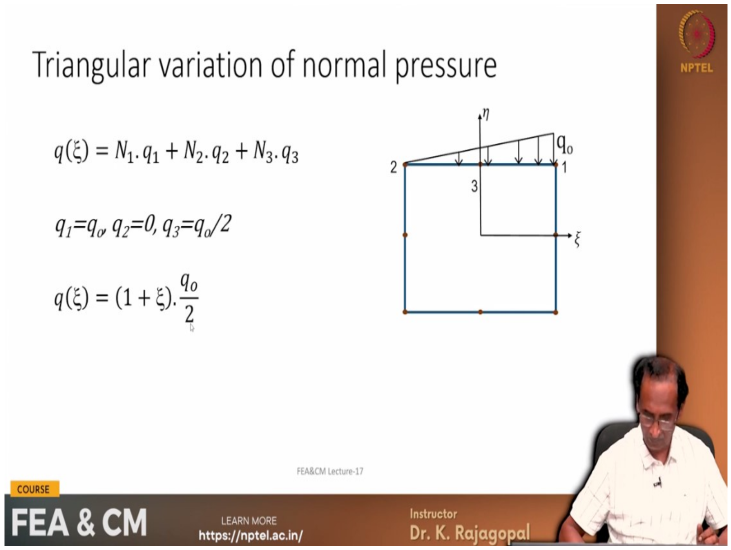

Then we could apply our numerical integration procedure for any distribution like our q need not be constant. Let us say we have a triangular distribution like this q is q 0 at Xi of + 1 0 at Xi of – 1 and then q by 2 at node 3.

$$q(\xi) = N_1.q_1 + N_2.q_2 + N_3.q_3$$

$$q_1 = q_o,\ q_2 = 0,\ q_3 = q_o/2$$

$$q(\xi) = (1+\xi).\frac{q_o}{2}$$

The q at any Xi can be written as 1 + Xi times q 0 by 2 and we can use this q in our equation and then do the integration either by analytically or by numerical integration then we can get the solution.

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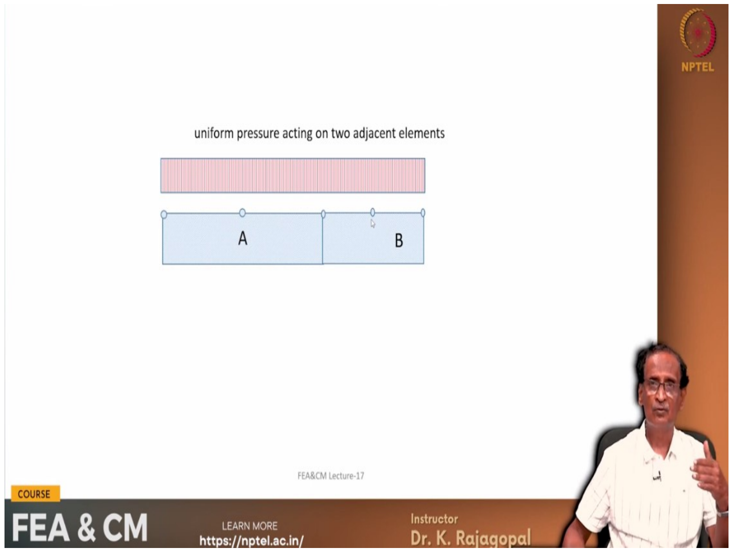

So, if we have a situation like let us say we have two elements A and B and what we do is we calculate the nodal forces separately on each of these elements. Element A we are going to calculate and element B we are going to calculate separately and then assemble and this particular node is common for both element A and element B. So, the force at this node is going to get added up from element A and B.

Whereas for other nodes it is just simply at the load from its own element just as how we are assembling this stiffness matrices we have to also assemble the load vectors.

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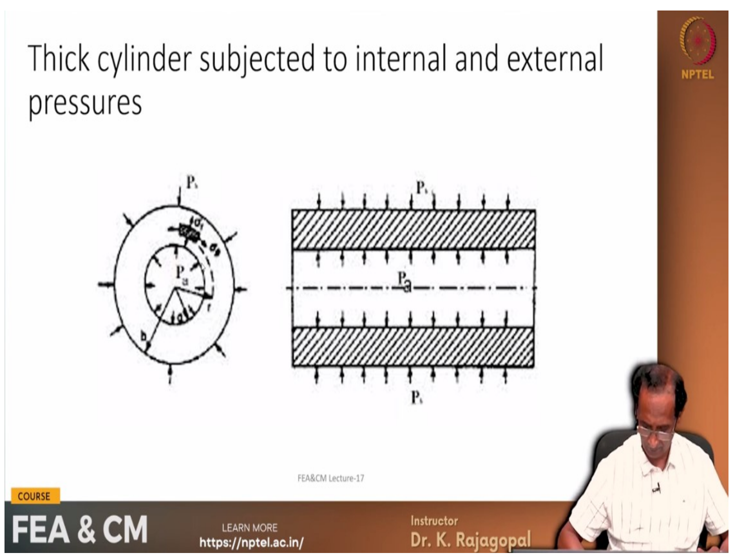

See till now we considered only plane surfaces, but then let us say we have thick cylinder subjected to internal pressure. So, now our surface is not a planar surface it is a curved surface because it is so complicated shape and how do we deal with it? So, for that we need to completely change our approach because till now we considered only a single surface, but then we may have a situation where our slope itself is changing like instead of being horizontal or vertical we could have a curved surface.

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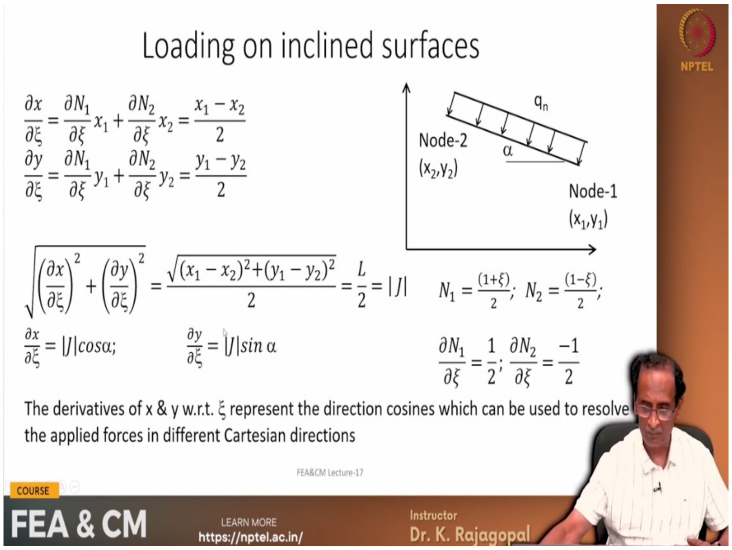
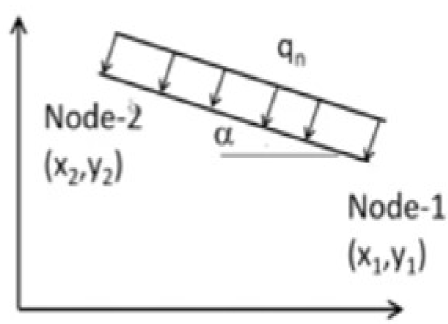


And before we generalize let us do the calculation for simple two cases like this. Let us take an inclined case like this wherein our surface is inclined at an angle of alpha and we have node 1 and node 2 and once again we number in the anti clockwise direction. Node 1 is x 1 y 1 node 2 has coordinates x 2 y 2 and dou x by dou Xi is x 1 – x 2 by 2 and dou y by dou Xi is y 1 – y 2 by 2.



$$\frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 = \frac{x_1 - x_2}{2}$$

$$\frac{\partial y}{\partial \xi} = \frac{\partial N_1}{\partial \xi} y_1 + \frac{\partial N_2}{\partial \xi} y_2 = \frac{y_1 - y_2}{2}$$

$$\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2} = \frac{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}{2} = \frac{L}{2} = |J|$$

$$\frac{\partial x}{\partial \xi} = |J| cos\alpha; \qquad \frac{\partial y}{\partial \xi} = |J| sin\,\alpha$$

And the length of this element we can get as this dou x by dou Xi whole square + dou y by dou Xi whole square this whole thing under root and this is L by 2 and that is J and what is dou x by dou Xi it is x 1 – x 2. So, we can write this as this Jacobian determinant multiplied by cosine alpha whereas dou y by dou Xi is y 1 – y 2 by 2 that is the vertical component. So, we can write this diagonal length multiplied by sin alpha.

$$N_1 = \frac{(1+\xi)}{2}; \quad N_2 = \frac{(1-\xi)}{2};$$

$$\frac{\partial N_1}{\partial \xi} = \frac{1}{2}; \frac{\partial N_2}{\partial \xi} = \frac{-1}{2}$$

The diagonal length is our J and our dou y by dou Xi is mod J times sin alpha and actually our dou x by dou Xi and dou y by dou Xi they are not only giving us the determinant of the Jacobian, but also the direction cosine; cosine alpha sin alpha as we can see here represent the direction cosines we can use them for resolving our applied forces in different directions.

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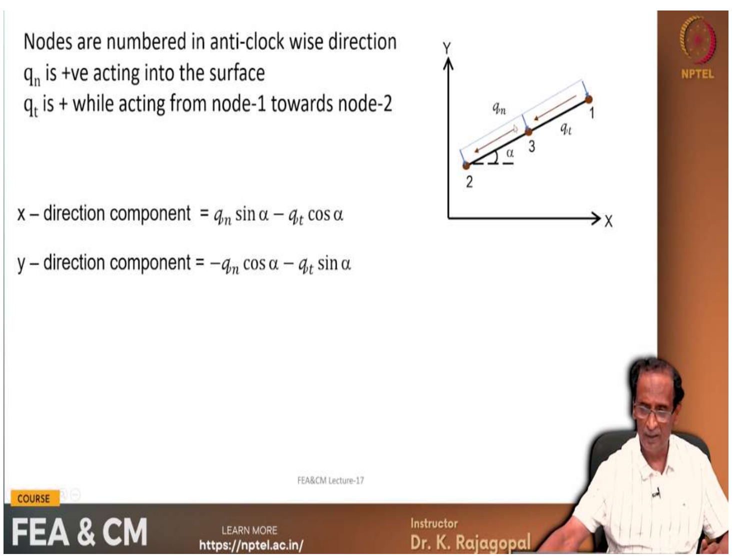
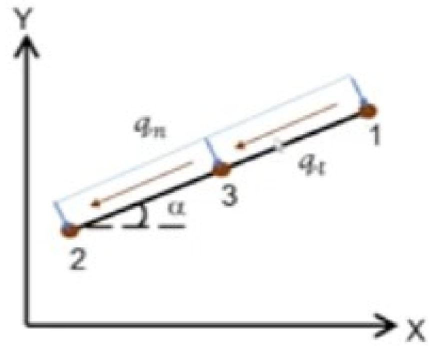


And so if you have any surface like this node 1, node 2 and node 3 and let us take the positive normal component as a pressure acting into the surface. So, we need to have a consistent scheme so that we can program them easily and the constant scheme is anti clockwise numbering and then the normal pressure acting into the surface and then the shear force or the shear stress acting from node 1 towards node 2.



So, the q n is positive acting into the surface and q t is positive acting from node 1 to node 2 and if that is so we can write the x direction component as the q n times sin alpha – q t times cosine alpha. So, the normal force multiplied by sin alpha – tangent force multiplied by cosine alpha. Similarly, y direction component is – q n times cosine alpha – q t sin alpha. Basically we are just resolving them this q normal and q t by using the trigonometry into x and y directions.

Nodes are numbered in anti-clock wise direction
q_n is +ve acting into the surface
q_t is + while acting from node-1 towards node-2

x – direction component $= q_n \sin\alpha - q_t \cos\alpha$

y – direction component $= -q_n \cos\alpha - q_t \sin\alpha$

While our x and y are the coordinate directions and the q n and q t they are referred to the element surface.

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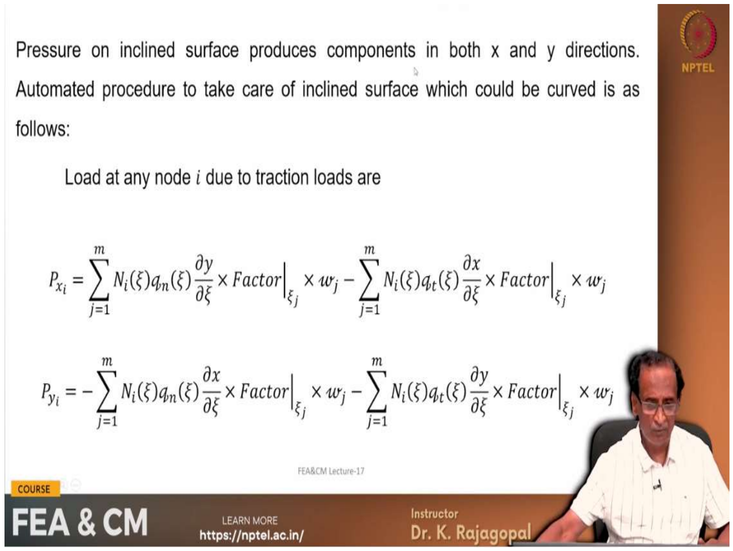

So, by combining the whole thing we can write our load at any node like this P x in terms of our sin and cosine terms because our sin and cosine are related to the dou x by dou Xi dou y by dou Xi like this. So, we can write using our derivatives of x and y with respect to Xi and this is our x components. This minus this like here you see the q n sin alpha – q t cosine alpha and sin alpha is with dou y by dou Xi term, cosine alpha is with dou x by dou Xi term.

Load at any node i due to traction loads are

$$P_{x_i} = \sum_{j=1}^{m} N_i(\xi) q_n(\xi) \frac{\partial y}{\partial \xi} \times Factor\Big|_{\xi_j} \times w_j - \sum_{j=1}^{m} N_i(\xi) q_t(\xi) \frac{\partial x}{\partial \xi} \times Factor\Big|_{\xi_j} \times w_j$$

$$P_{y_i} = -\sum_{j=1}^{m} N_i(\xi) q_n(\xi) \frac{\partial x}{\partial \xi} \times Factor\Big|_{\xi_j} \times w_j - \sum_{j=1}^{m} N_i(\xi) q_t(\xi) \frac{\partial y}{\partial \xi} \times Factor\Big|_{\xi_j} \times w_j$$

And similarly P y that is minus of this cosine alpha term and – of this sin alpha term the q n and q t. So, if you apply these relations we should be able to take care of any curved surface because at any curved surface the tangent direction is changing. At different Xi you get a different tangent and automatically our dou x by dou Xi dou y by dou Xi if you evaluate them at that point you will automatically take care of this tangent.

Then with these equations we can directly distribute into the correct directions whether it is positive x or positive y and negative x or negative y.

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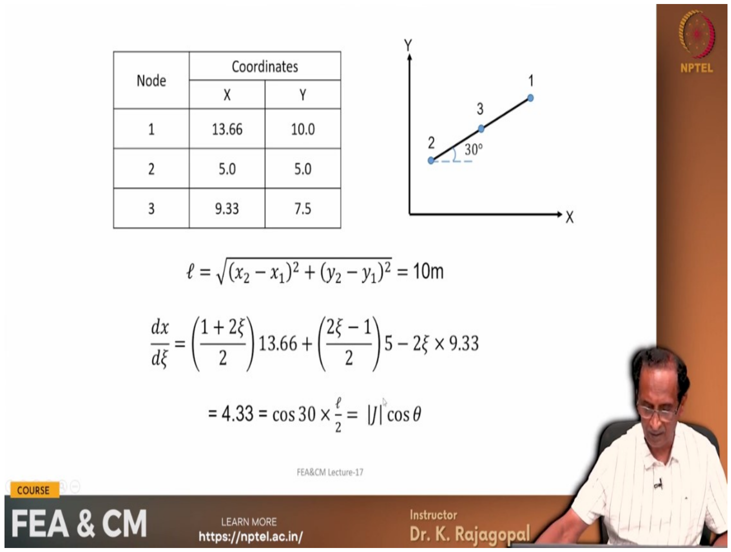
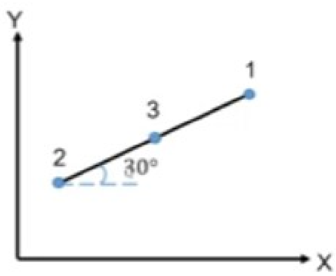


And just one small example let us take an element like this 1, 2 and 3 at an angle of 30 degrees and the coordinates are defined like this node 1, node 2 and node 3 and the length is 10 meters and if you calculate dou x by dou Xi you get 4.33 that is cosine 30 times L by 2; L by 2 is 5, 5 times cosine 30 is 4.33 and 5 is the mod J for the dou x by dou Xi sorry the J L by 2.

Node	Coordinates	
	X	Y
1	13.66	10.0
2	5.0	5.0
3	9.33	7.5



$$\ell = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 10\text{m}$$

$$\frac{dx}{d\xi} = \left(\frac{1+2\xi}{2}\right) 13.66 + \left(\frac{2\xi - 1}{2}\right) 5 - 2\xi \times 9.33$$

$$= 4.33 = \cos 30 \times \frac{\ell}{2} = |J| \cos\theta$$

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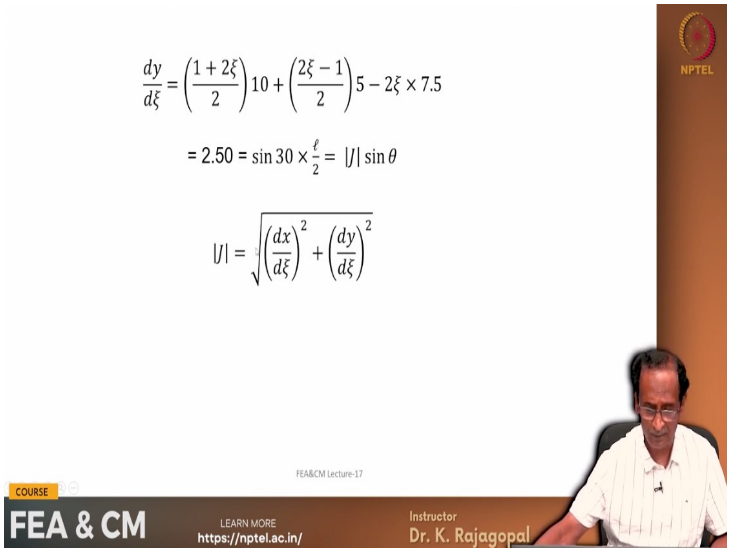

And similarly dou y by dou Xi is 2.5 that is sin 30 times mod J sin theta times mod J and then our mod J is square root of dou x by dou Xi whole square + dou y by dou Xi whole square. So, we can easily apply these equations that we have here to directly calculate our forces in different coordinate directions.

$$\frac{dy}{d\xi} = \left(\frac{1+2\xi}{2}\right)10 + \left(\frac{2\xi-1}{2}\right)5 - 2\xi \times 7.5$$

$$= 2.50 = \sin 30 \times \frac{\ell}{2} = |J| \sin\theta$$

$$|J| = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}$$

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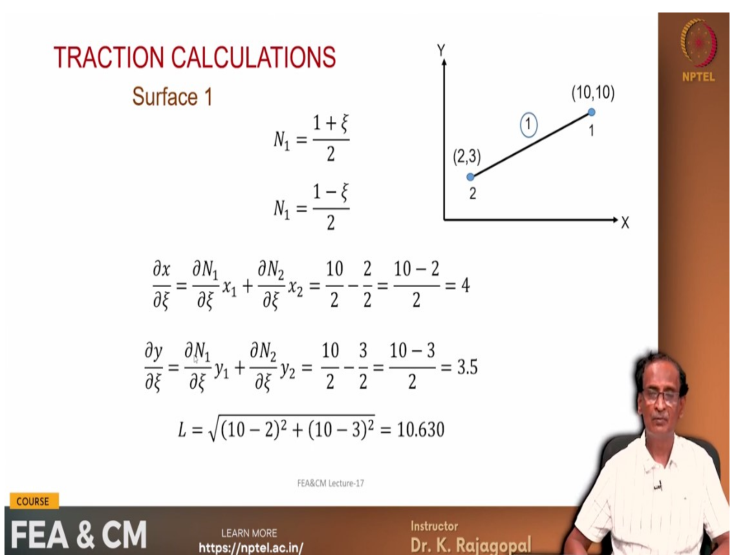

So, these are all different numerical examples that you can follow and then if we implement this algorithm into the computer program it can automatically take care of our pressure acting in different directions.

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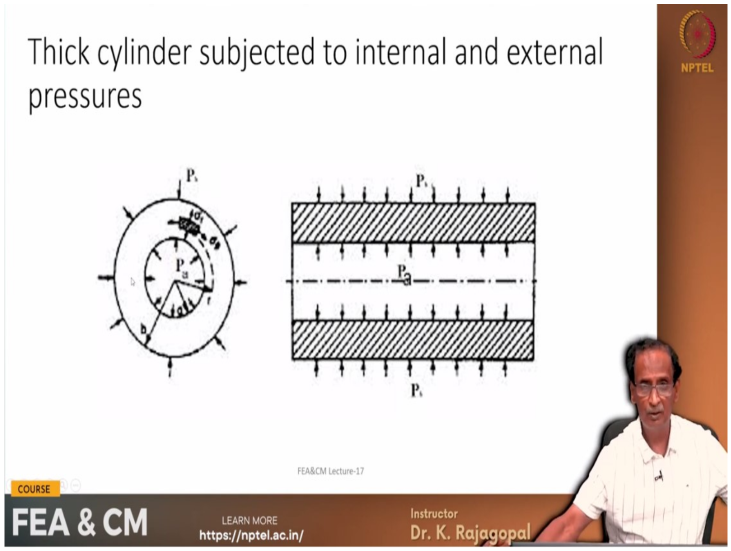

Like if you see the circular surface see the pressure value is the same, but then its component is acting in different directions. At this point it is acting vertically up whereas at this point it is acting horizontal to the right hand side and here it is acting down whereas here it is acting horizontally to the left side and our cosine alpha and sin alpha terms they help us in automatically doing these calculations without resorting to anything and the systematically we have to be consistent.

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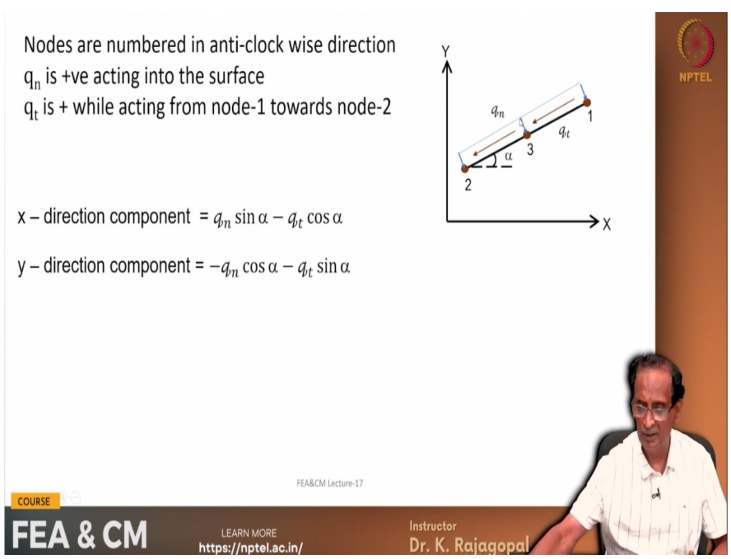

And our consistent notation is our node number is in the anti clockwise direction 1, 2, 3 and then the normal force is normal to the surface tangential force is acting from node 1 to node 2. So, as long as you do that you will get the correct x and y directions. So, I think that is the end of my lecture and if you have any questions please send an email to my email profkrg@gmail.com. So, thank you very much.