FEM and Constitutive Modeling in Geomechanics
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Lecture – 18
Isoparametric Calculations for Stiffness & Load Vectors

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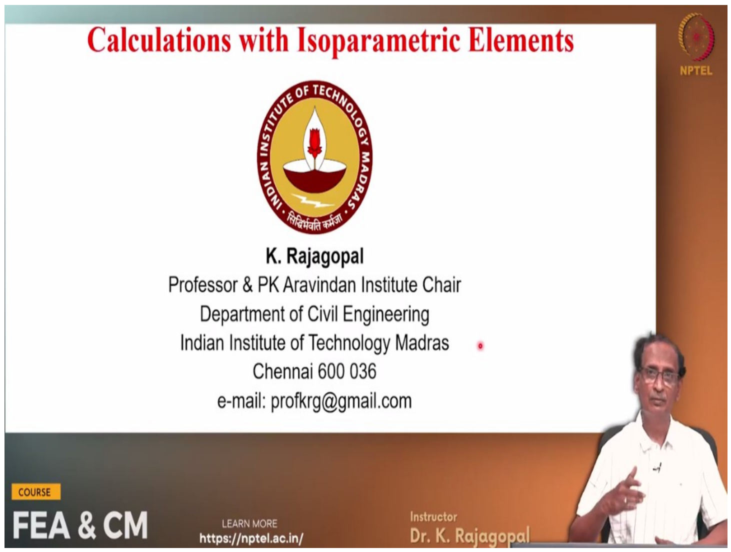

Let us continue our isoparametric calculations.

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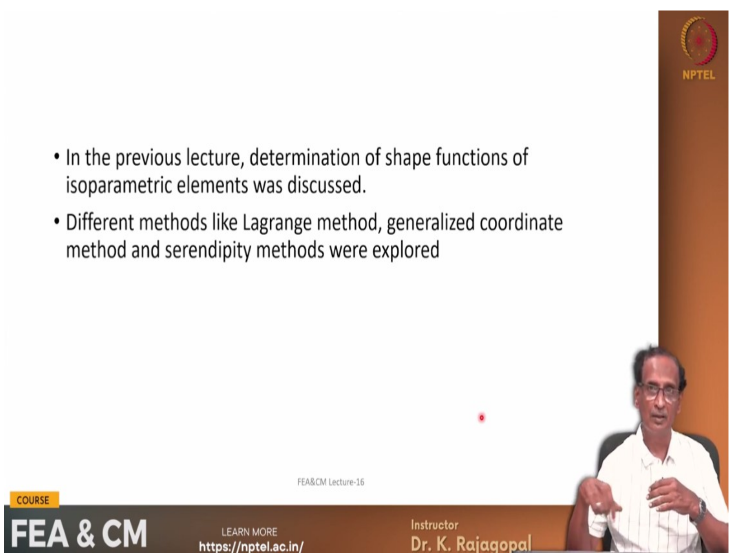

So, in the previous classes we had seen the different methods for determining the shape functions of isoparametric elements using the Lagrange method and then the generalized coordinate method and then also the serendipity method by progressive correction and let us continue with what we had done.

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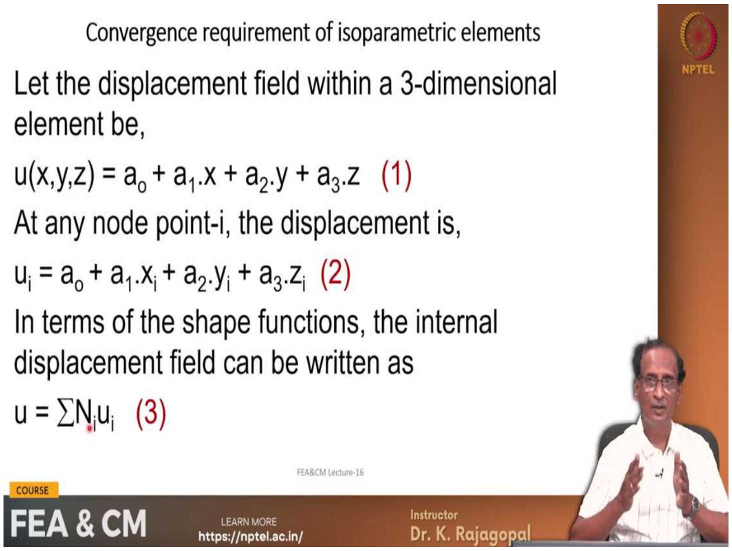
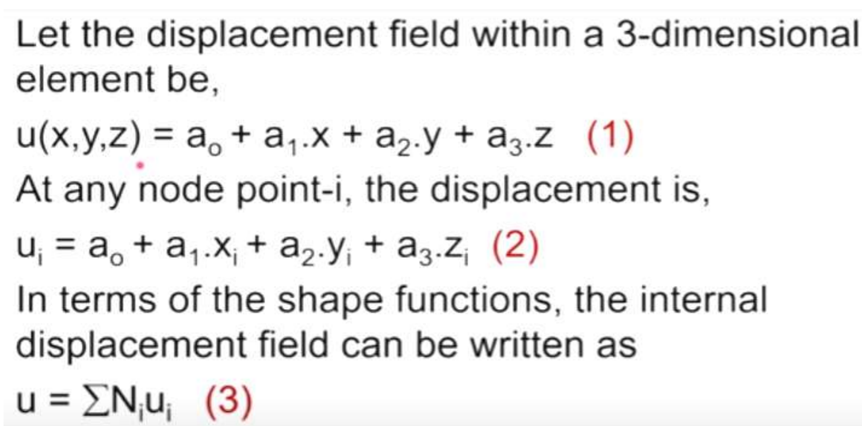


And before we do the calculations let us see what is the requirement for convergence with the isoparametric elements. So, let the displacement field be expressed as a 0 + a 1 x + a 2 y + a 3 z. This is as per our generalized coordinate method and this is particular for three dimensions because we notice that we have three coordinates x, y and z and at any point i the displacement can be obtained by substituting x i, y i and z i the coordinates corresponding to that point i.



So, u i we can write like this a 0 + a 1 x i + a 2 y i + a 3 z i and also in terms of the shape functions the internal displacement field can be written as u is sigma of N i u i where N i are the shape functions at different nodes and u i are the displacements of the different nodes. So, if you have three nodes in the element we have N 1, N 2 and N 3 and u 1, u 2, u 3.

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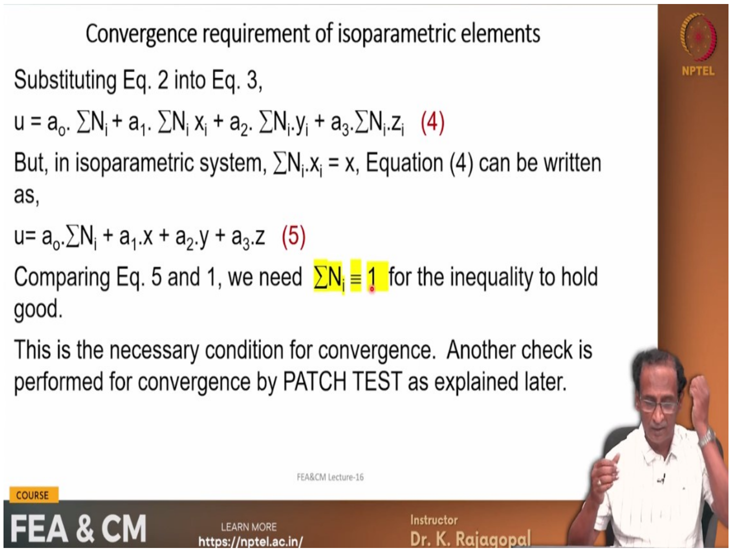
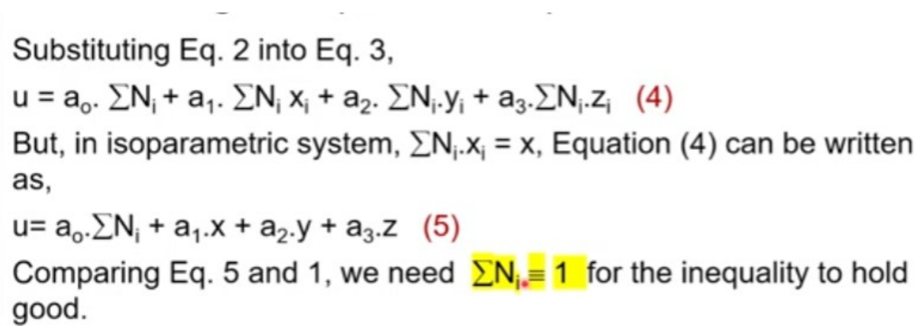


And so if we substitute 2 and 2 into 3 in terms of the shape functions we get the internal displacement as a 0 + sigma N i + a 1 sigma N i x i + a 2 sigma N i y i + a 3 sigma N i z i, but our sigma of N i x i is actually x that is the sum total of shape function value multiplied by the coordinate value, which is x as we have seen earlier through the mapping functions and you realize that in the isoparametric elements we use the same mapping and the shape function.



So, this is true for our conventional systems. So, our u can be written as a 0 times sigma N i + a 1 x + a 2 y + a 3 z, but then this u is the same as what we had in the equation 1. So, if you compare this equation 5 with the equation 1 that is so in equation 1 we had u as a 0 + a 1 x + a 2 y + a 3 z. So, if you compare both of these equations we have only sigma of N i as an additional term and that should be equal to 1 for equation 5 to be equal to equation 1 because both are referring to the same displacement so we cannot have different values.

So, that can be true only when the sum total of all the shape functions is equal to 1 and this is the necessary condition for convergence and that is monotonic convergence and we will also see or we will also perform some other test called as a patch test that is referring to constant strain conditions to assure that whatever finite elements that we have developed they do meet the requirements of the convergence.

So, actually our sigma of N i of 1 is required so that we can simulate the rigid body displacements without straining the body and also the constant term stay in state within the element.

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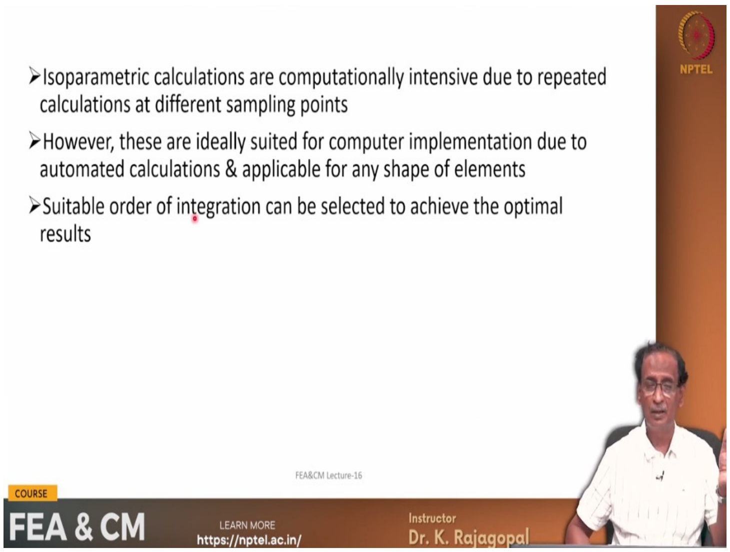

So, now let us see what or how we can do the calculations using the isoparametric elements and as such our finite elements calculations are going to be too intensive like we cannot obviously do by hand and all these calculations are most ideal for computers because they are high speed computers and then they can do the repeated calculations without any numerical mistakes.

So, if you and I do then every second the calculation that we do we might make a mistake, but computer does not do these mistakes and it can do very fast and these isoparametric calculations are good for computer implementation because they are most ideal it is a repeated calculations and these elements are also applicable for any shape of element whatever maybe the shape in the Cartesian space we convert to some standard shape in the natural space and do the calculations.

And then we can choose the order of numerical integration based on the order of polynomial that we have.

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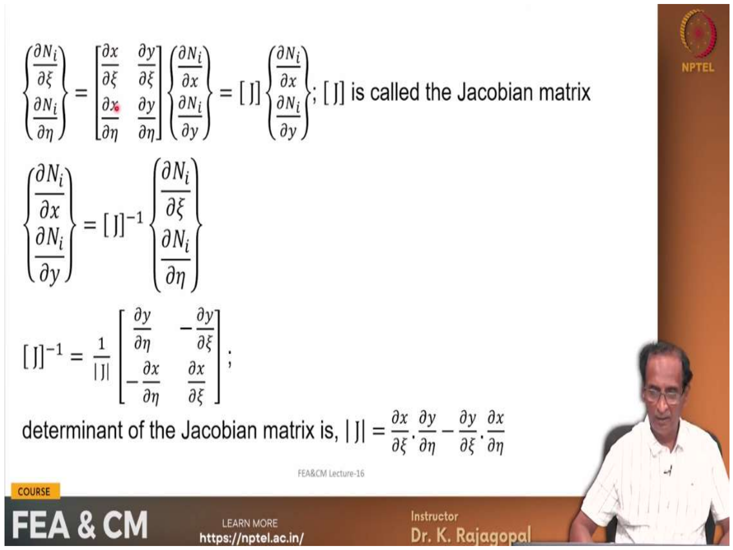

And the relation between our Cartesian system and then the isoparametric system is coming through our Jacobian matrix as we had seen in the previous lecture and we can express our derivatives of the shape functions in terms of Xi and eta through the Jacobian matrix and then the shape function derivatives with respect to the Cartesian coordinates x and y and these terms within the bracket there we can think of them as mapping functions dou x by dou Xi dou y by dou Xi, dou x by dou eta dou y by dou eta and so this is J.

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [\,J\,] \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}; \text{ [J] is called the Jacobian matrix}$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [\,J\,]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

And J is the very important parameter in our finite element calculations and we call this as the Jacobian matrix and the J inverse is 1 divided by determinant of the Jacobian matrix multiplied by dou y by dou eta – dou y by dou Xi – dou x by dou eta dou x by dou Xi and our determinant of this Jacobian matrix is this cross product minus this cross products.

$$[\,J\,]^{-1} = \frac{1}{|\,J\,|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix};$$

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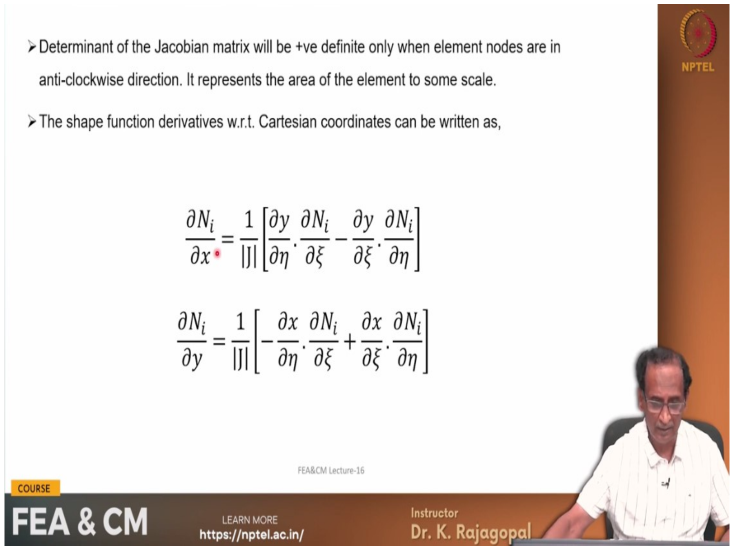

And that represents the area of the element to some scale and this will be positive only when the element nodes are numbered in the anti clockwise direction that is what we had seen even with the three node triangular elements. We numbered the nodes in the anti clockwise direction to get the determinant of that coordinate matrix as positive and it represents the area of the element to some scale and the shape function derivatives with respect to Cartesian coordinates are this, this we have seen earlier.

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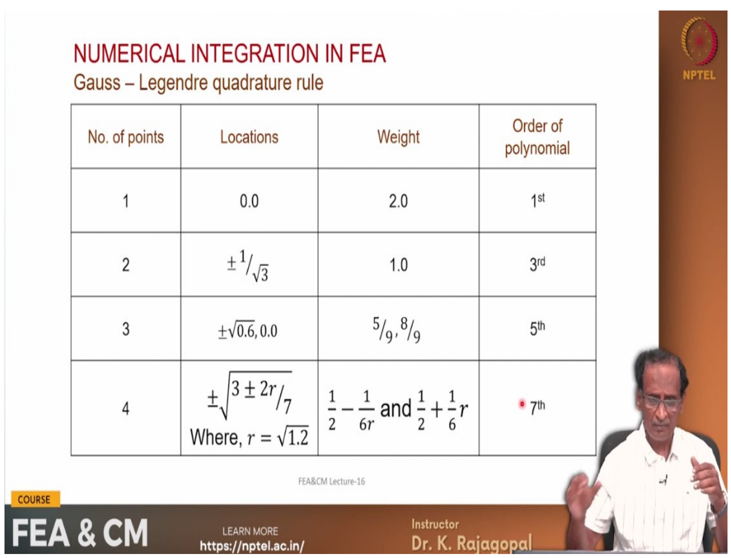

And so once we have this we can form the b matrix and do all the calculations and so these Gauss Legendre quadrature rules we have seen earlier and in fact we had derived the sampling point locations and then the corresponding weight factor for the first three cases point 1, 2 and 3 and similarly we can also derive for 4 point numerical integration and the main advantage this is we know exactly up to what order of polynomial they can exactly integrate.

So, if you have 1 point integration we can exactly integrate up to first points, 2 points means 2 times 2 that is 4 – 1 is 3rd order polynomial. Similarly, 4 points means 7th order polynomial and in general once we have the element and then the type of problem whether it is plane stress plane strain are axisymmetric and then whether it is regular shaped rectangular element or a distorted element we can estimate the order of polynomial that we will have then accordingly we can choose the order of integration.

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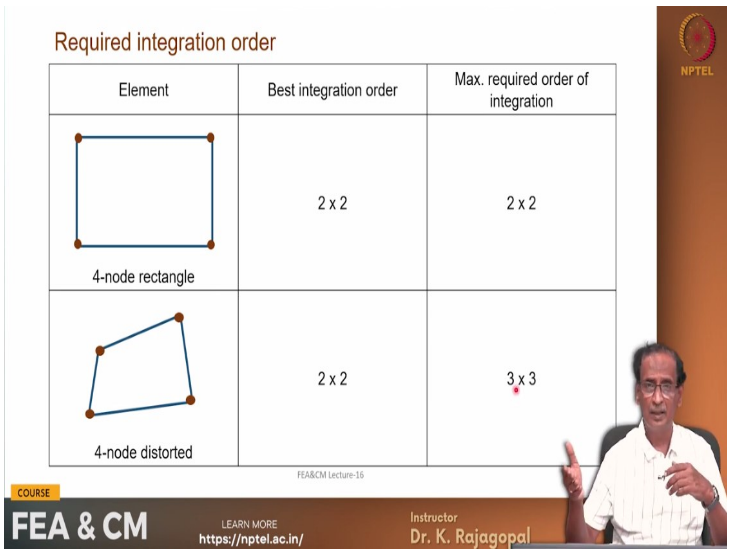

And if we have a four node rectangle the best order of integration is 2 by 2 and the maximum required order is 2 by 2 like we do not need more than that and if it is a four node distorted the best order is 2 by 2 and the maximum order is 3 by 3. Actually the difference between these two is see theoretically you may require 3 points in each direction for exact integration of the quantities for estimating this stiffness matrix.

But then when we are evaluating the stiffness of continuum since we are considering only a finite number of degrees of freedom we are over estimating the stiffness and to partially compensate for that we use a lower order of integration and so if we use a lower order of integration we can partially compensate for the over stiffness of the continuum then get better results.

So, that is the difference between the best integration order and then the maximum integration order and by using the maximum integration order you may not get any better solution, but definitely you will spent more computational effort.

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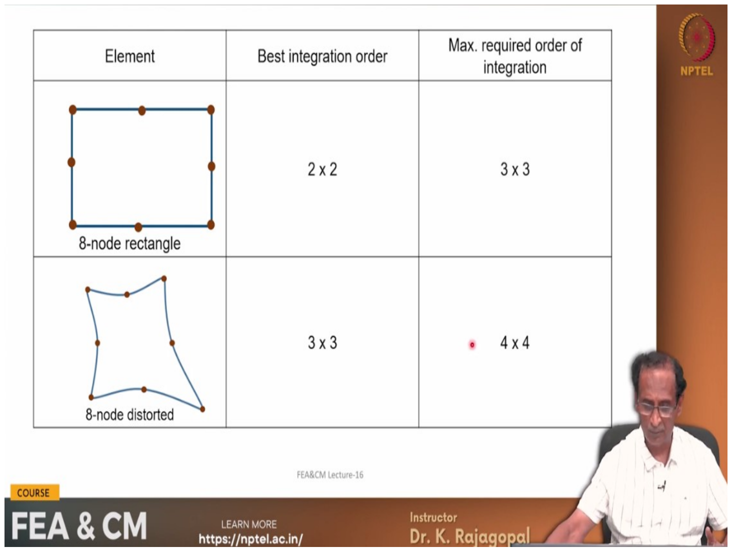

And if you have 8 node quadrilateral like this the best order is still 2 by 2 maximum order is 3 by 3 and if you have a 8 node distorted element the best order is 3 by 3 and maximum order is 4 by 4.

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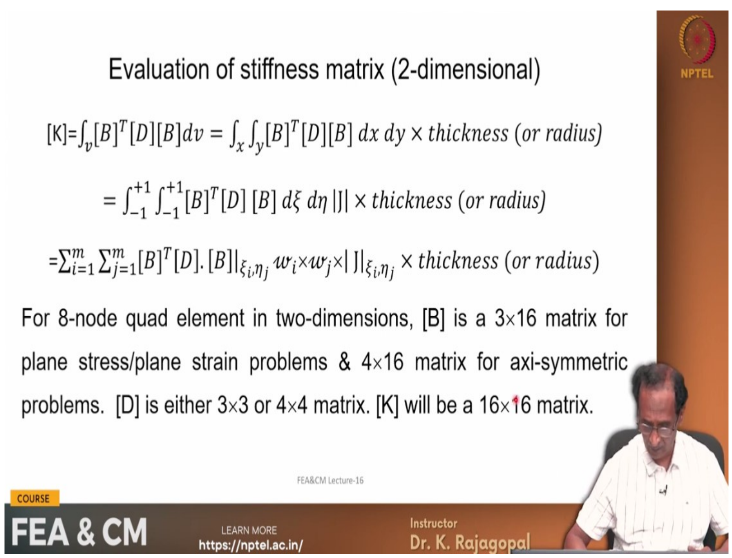

And let us look at this stiffness equation once again and look at the steps required for doing the evaluation. Our stiffness matrix is calculated as this product B transpose D B integrated over the volume of the element and in two dimensions we will do the integration in x and y.

So, we have B transpose D B dx dy multiplied by thickness in the case of plane stress and plane strain we have the thickness and in the case of axisymmetry we will have the radius.

$$[K]=\int_v [B]^T[D][B]dv = \int_x \int_y [B]^T[D][B]\, dx\, dy \times thickness\ (or\ radius)$$

$$= \int_{-1}^{+1}\int_{-1}^{+1} [B]^T[D]\,[B]\, d\xi\, d\eta\, |J| \times thickness\ (or\ radius)$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{m}[B]^T[D].[B]|_{\xi_i,\eta_j}\, w_i \times w_j \times |J|_{\xi_i,\eta_j} \times thickness\ (or\ radius)$$

So, here it is put thickness or radius that will give you the incremental volume and we also know the relation between the Cartesian coordinates and then the natural coordinates the dx dy is equal to the determinant of the Jacobian matrix multiplied by d Xi d eta. So, we can write like this integral – 1 to + 1 – 1 to + 1 B transpose D B d Xi d eta multiplied by the determinant of the Jacobian matrix.

And in terms of summation we can write this summation form in the numerical form by evaluating these quantities at different sampling points Xi i delta j and then multiplying with the corresponding weight factor w i w j and then the determinant and if you have a irregular shape or a distorted shape the Jacobian value will not be constant. It might be varying a different sampling points and so this calculations repeated several times.

So, if you are using 2 by 2 integration in two dimensions we are going to do this repeat the calculations 4 times and then if you have a 3 by 3 integration we will have 9 times and so on. So, our B matrix has a size of 3 by 16 in the plane stress plane strain because we have 3 strain components and then 16 degrees of freedom and in the case of axisymmetric we have 4 strains. So, our K will be a 16 by 16 matrix for 8 node quadrilateral.

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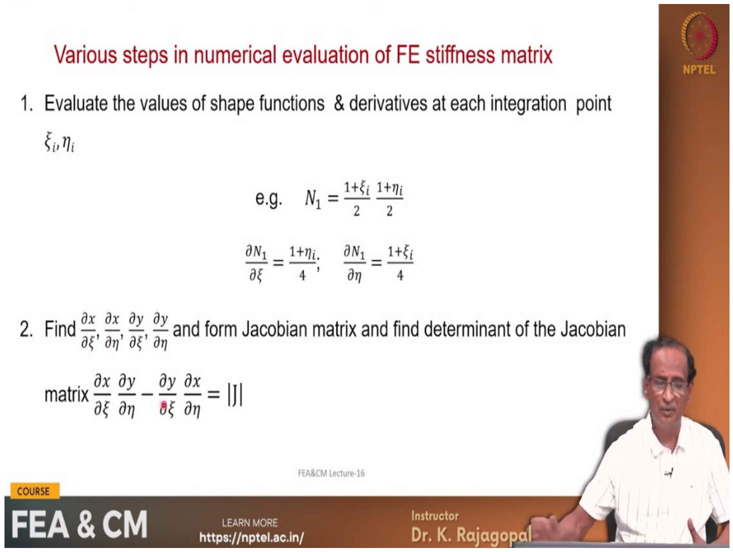

So, these are the steps in our calculations at each integration points Xi i eta i we need to get the shape function and then we get the shape function derivatives with respect to Xi and eta and then we form the Jacobian matrix by determining dou x by dou Xi dou x by dou eta and so on and then we find the determinant of the Jacobian matrix.

1. Evaluate the values of shape functions & derivatives at each integration point ξ_i, η_i

$$\text{e.g.} \quad N_1 = \frac{1+\xi_i}{2} \frac{1+\eta_i}{2}$$

$$\frac{\partial N_1}{\partial \xi} = \frac{1+\eta_i}{4}, \quad \frac{\partial N_1}{\partial \eta} = \frac{1+\xi_i}{4}$$

2. Find $\frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \eta}, \frac{\partial y}{\partial \xi}, \frac{\partial y}{\partial \eta}$ and form Jacobian matrix and find determinant of the Jacobian matrix $\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} = |J|$

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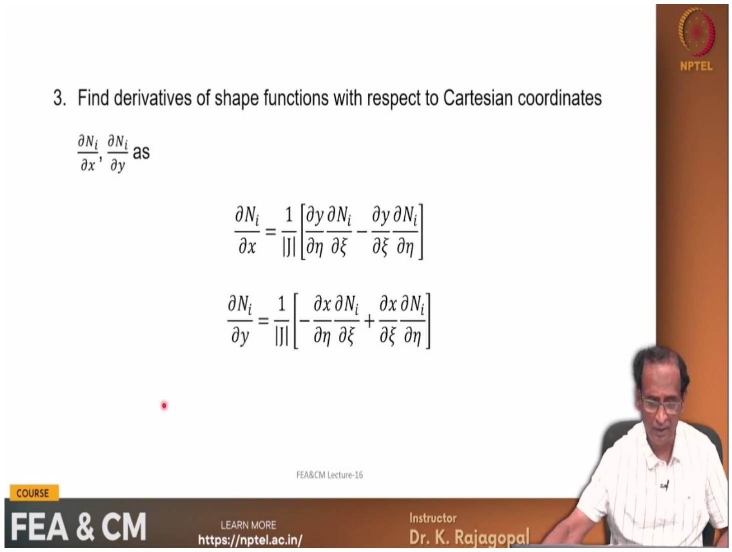

And then after that we can find the shape function derivatives with respect to x and y or the Cartesian coordinates.

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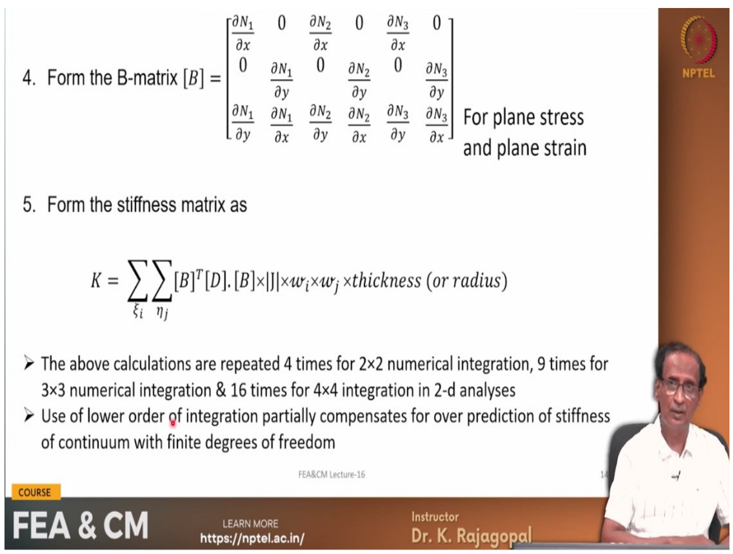

And then we can write our B matrix in terms of the shape functions derivatives and this particular one is for plane stress and plane strain dou N by dou x dou N by dou y and so on and then we can form the stiffness matrix by this product B transpose D B where B is our strain displacement matrix. D is our constitute matrix relating the stress and strain and this term we will see it in more detail when we look at the constitutive models.

4. Form the B-matrix $[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$ For plane stress and plane strain

5. Form the stiffness matrix as

$$K = \sum_{\xi_i} \sum_{\eta_j} [B]^T [D].[B] \times |J| \times w_i \times w_j \times thickness\ (or\ radius)$$

And then this our determinant of our Jacobian matrix. So, these calculations are repeated several times say for a 2 by 2 integration we repeat these calculations 4 times and we repeat this 9 times if we use a 3 by 3 integration so almost double the times and then if we use 4 by 4 it is even larger number of time 16 times. So, actually it is most important that we do not use a higher order of integration than necessary.

We should always go for the best order of integration and as a general rule we use the least integration order, we do not use 1 point integration because it is mostly insufficient and unless you have a 3 node triangle for that we can use 1 point integration, but for any general say 4 node quadrilateral or 8 node quadrilateral the general thumb rule is use the lowest order of integration first.

And then you also try with higher order of integration and then see whether the results are similar or not so that we have confidence in the result that we get and as I mentioned earlier use of lower order of integration is better for us because we partially compensate for the over stiffness of the continuum.

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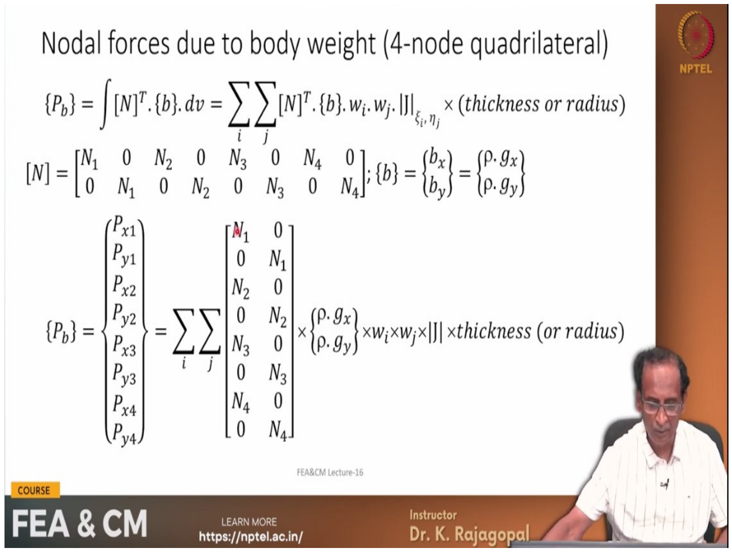

So, now let us see the other calculations that is related to the estimation of the right hand side force vector and one of the terms on the right hand side is the body force vector and that is written as N transpose b dv and here I am going to illustrate only for a 4 node quadrilateral so that I can write out all the matrices and vectors and our P b is N transpose b d v integrated over the volume.

$$\{P_b\} = \int [N]^T.\{b\}.dv = \sum_i \sum_j [N]^T.\{b\}.w_i.w_j.|J|_{\xi_i,\eta_j} \times (thickness\ or\ radius)$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}; \{b\} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \begin{Bmatrix} \rho.g_x \\ \rho.g_y \end{Bmatrix}$$

And in terms of summation we can write like this N transpose b w i w j the determinant or the Jacobian matrix multiplied by thickness or radius and our shape function matrix for a 4 node quadrilateral is like this N 1 0 N 2 0 N 3 0 N 4 0 and so on and our b is the body force vector and it can have two components one is in the x direction and the other is y direction and that is simply rho g x rho g y rho is the mass density.

G is the gravitational constant and our force vector has 8 components because we are 4 nodes and at each node we have a component in the x direction and the another component in the y direction. So, we have 4 nodes 1, 2, 3, 4. So, this product it can be written like this double summation and then N transpose b w i w j and then the determinant of the Jacobian matrix and so on.

$$\{P_b\} = \begin{Bmatrix} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \\ P_{x3} \\ P_{y3} \\ P_{x4} \\ P_{y4} \end{Bmatrix} = \sum_i \sum_j \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix} \times \begin{Bmatrix} \rho . g_x \\ \rho . g_y \end{Bmatrix} \times w_i \times w_j \times |J| \times thickness\ (or\ radius)$$

And it is actually although we write like this, but inside a computer program we are going to evaluate each of these separately and then assemble them in a matrix form or in vector form. So, do not be too confused by looking at this matrix we are going to evaluate only term by term.

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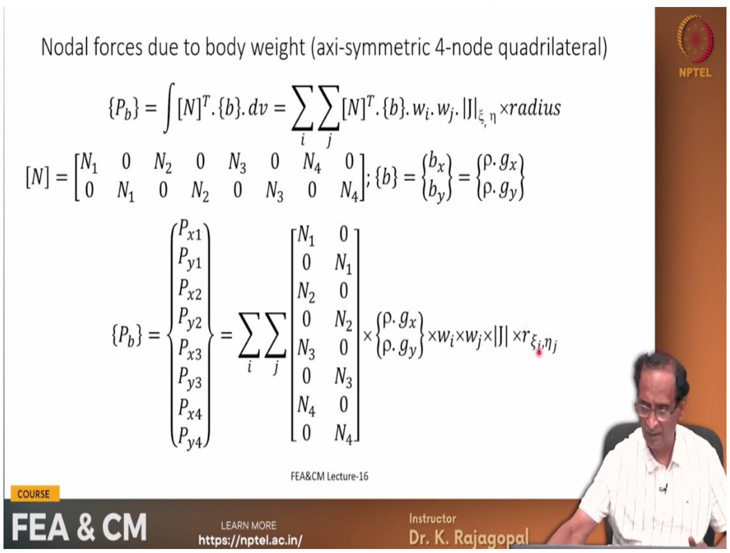

And for axisymmetric 4 node quadrilateral will have the radius and this radius also could be varying with Xi. So, when we estimate the polynomial order we have to also look at the polynomial order that the radius could not have. So, our equation will be like this everything is the same except here we have the radius evaluated at Xi i and eta j.

Nodal forces due to body weight (axi-symmetric 4-node quadrilateral)

$$\{P_b\} = \int [N]^T.\{b\}.dv = \sum_i \sum_j [N]^T.\{b\}.w_i.w_j.|J|_{\xi,\eta} \times radius$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}; \{b\} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \begin{Bmatrix} \rho.g_x \\ \rho.g_y \end{Bmatrix}$$

$$\{P_b\} = \begin{Bmatrix} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \\ P_{x3} \\ P_{y3} \\ P_{x4} \\ P_{y4} \end{Bmatrix} = \sum_i \sum_j \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix} \times \begin{Bmatrix} \rho.g_x \\ \rho.g_y \end{Bmatrix} \times w_i \times w_j \times |J| \times r_{\xi_i,\eta_j}$$

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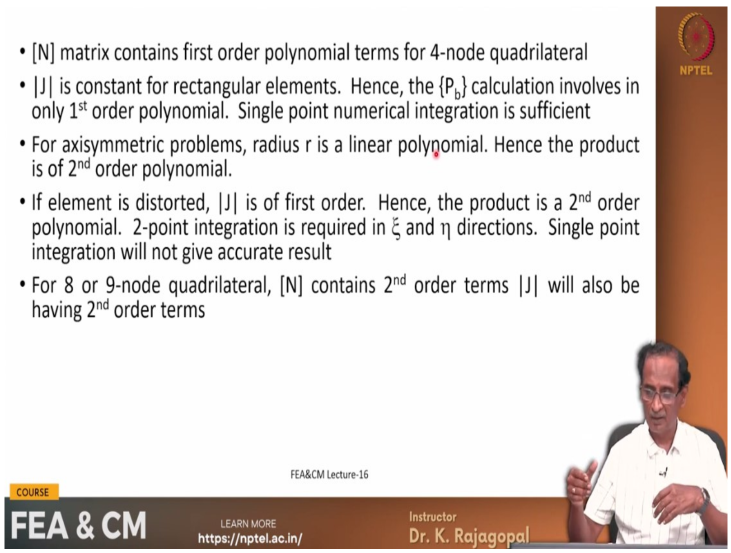

And so the N matrix is a first order polynomial for the 4 node quadrilateral and our element is rectangular the J is constant. So, the calculation for the forced vector is only a first order polynomial and 1 point integration is sufficient that is for plane stress and plane strain, but for axisymmetric problems the radius r is a linear polynomial because it is varying with the Xi and eta.

So, our product will be a second order polynomial so we need a higher order of integration and if element is distorted J could be of first order that we will see with examples numerical examples and then accordingly our order of polynomial will change and for 8 and 9 node quadrilateral our shape function will be of second order and our J will also be having second order terms.

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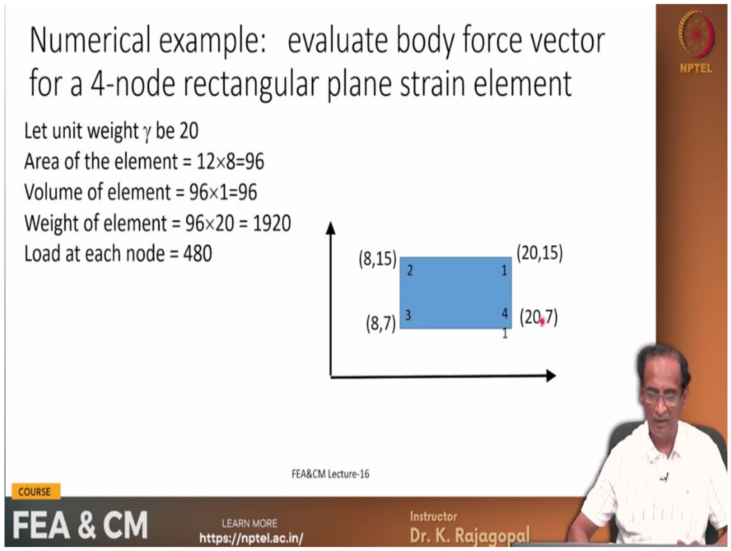


Let us calculate the body force vector for a rectangular plane strain element and as you recall the plane strain element has unit thickness in the outer plane direction and this element has 4 nodes; node 1, node 2, node 3 and node 4 and please note that all these 4 nodes are numbered in the anti clockwise direction and these are the coordinates node 1 as coordinates 20 and 15, node 2 8 and 15, node 3 8 and 7, node 4 20 and 7.

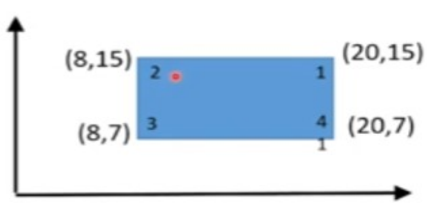
Let unit weight γ be 20

Area of the element = 12×8=96

Volume of element = 96×1=96

Weight of element = 96×20 = 1920

Load at each node = 480



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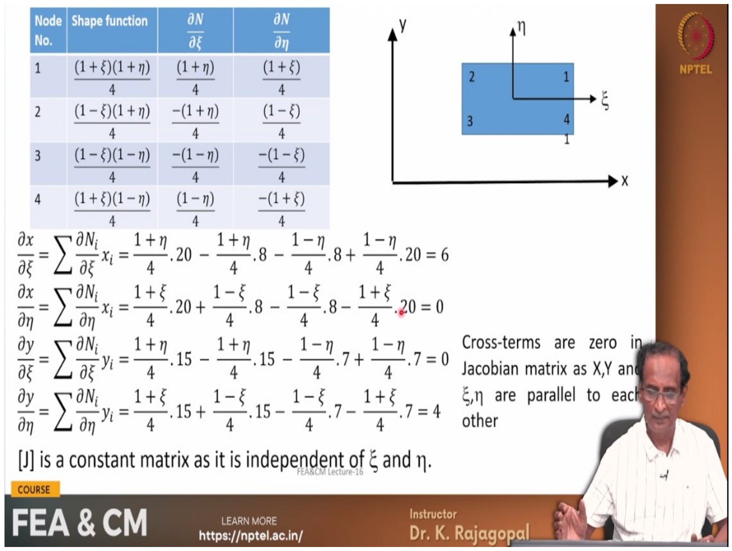
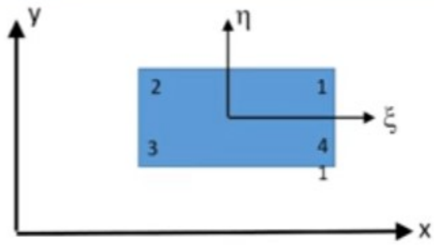


And we can evaluate the shape functions; shape function values are different integration points and also we need to calculate our determinant of the Jacobian matrix. So, these are the four shape functions and these are the derivatives of the shape functions with respect to Xi and eta and dou x by dou Xi is the summation of dou N by dou Xi times x i and if you substitute the dou N by dou Xi.

Node No.	Shape function	$\frac{\partial N}{\partial \xi}$	$\frac{\partial N}{\partial \eta}$
1	$\frac{(1+\xi)(1+\eta)}{4}$	$\frac{(1+\eta)}{4}$	$\frac{(1+\xi)}{4}$
2	$\frac{(1-\xi)(1+\eta)}{4}$	$\frac{-(1+\eta)}{4}$	$\frac{(1-\xi)}{4}$
3	$\frac{(1-\xi)(1-\eta)}{4}$	$\frac{-(1-\eta)}{4}$	$\frac{-(1-\xi)}{4}$
4	$\frac{(1+\xi)(1-\eta)}{4}$	$\frac{(1-\eta)}{4}$	$\frac{-(1+\xi)}{4}$



And then the different coordinate values we will see that dou x by dou Xi is 6 and it is constant it is not varying with Xi and dou x by dou eta we will see that it is 0 and dou y by dou Xi is also 0 and dou y by dou eta is 4. So, actually if you see the x and y coordinates are coinciding with Xi and eta coordinates, they are parallel to each other and they are just coinciding because we can literally move the location of these coordinates.

$$\frac{\partial x}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} x_i = \frac{1+\eta}{4}.20 - \frac{1+\eta}{4}.8 - \frac{1-\eta}{4}.8 + \frac{1-\eta}{4}.20 = 6$$
$$\frac{\partial x}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} x_i = \frac{1+\xi}{4}.20 + \frac{1-\xi}{4}.8 - \frac{1-\xi}{4}.8 - \frac{1+\xi}{4}.20 = 0$$
$$\frac{\partial y}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} y_i = \frac{1+\eta}{4}.15 - \frac{1+\eta}{4}.15 - \frac{1-\eta}{4}.7 + \frac{1-\eta}{4}.7 = 0$$
$$\frac{\partial y}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} y_i = \frac{1+\xi}{4}.15 + \frac{1-\xi}{4}.15 - \frac{1-\xi}{4}.7 - \frac{1+\xi}{4}.7 = 4$$

[J] is a constant matrix as it is independent of ξ and η.

And because they are parallel to each other as you are moving along Xi your y is remaining constant because our element shape is rectangular. So, our Xi and eta they are going to be oriented towards the length and height of the element and as you are moving along eta your x is going to remain constant. So, our dou x by dou eta is 0 and as you are moving along Xi y is constant so dou y by dou Xi is 0.

And then dou x by dou Xi is 6 and dou y by dou eta is 4 and we can see we can relate them to the length of the element; length of the element is 20 – 8 is 12 and the height is 15 – 7 is 8 and 12 is the length and divided by 2 is your dou x by dou Xi and height is 8 and dou y by dou eta is 4 8 by 2.

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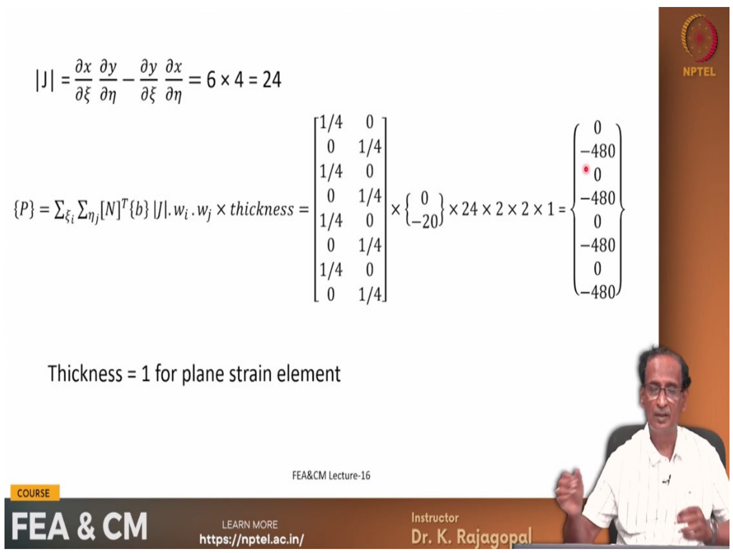

And our determinant of the Jacobian matrix is 24 and our area of the element is 12 times 8 that is 96 and your Jacobian determinant is 24. So, we see that the determinant of Jacobian matrix is a by 4 96 by 4 and our force vector P is the summation over Xi and eta N transpose b times determinant of the Jacobian matrix then w i w j times thickness and for this because we have a first order polynomial we can just use 1 point integration.

$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} = 6 \times 4 = 24$$

$$\{P\} = \sum_{\xi_i}\sum_{\eta_j}[N]^T\{b\}\,|J|.\,w_i\,.\,w_j \times thickness = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \\ 1/4 & 0 \\ 0 & 1/4 \\ 1/4 & 0 \\ 0 & 1/4 \\ 1/4 & 0 \\ 0 & 1/4 \end{bmatrix} \times \begin{Bmatrix} 0 \\ -20 \end{Bmatrix} \times 24 \times 2 \times 2 \times 1 = \begin{Bmatrix} 0 \\ -480 \\ 0 \\ -480 \\ 0 \\ -480 \\ 0 \\ -480 \end{Bmatrix}$$

And that 1 point integration Xi and eta are 0. So, if you substitute Xi and eta of 0 we get one forth for all the shape functions. So, you have this N transpose as 1 / 4 0 0 1 / 4 and so on and our body force vector our unit weight is 20 and if you take the y axis is vertical upwards the gravity is going to act downwards so it is – 20 0 and – 20 and the determinant of the Jacobian matrix is 24 and the weight factors are 2 and 2 thickness is 1.

So, we see this the force is equally distributed between all the 4 nodes – 480 -480 -480 – 480 and the thickness for the plane strain element is 1 that is what we used here.

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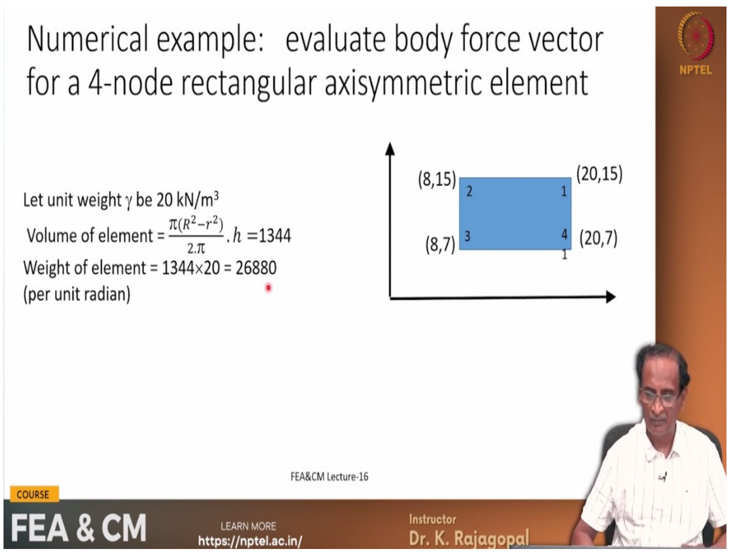
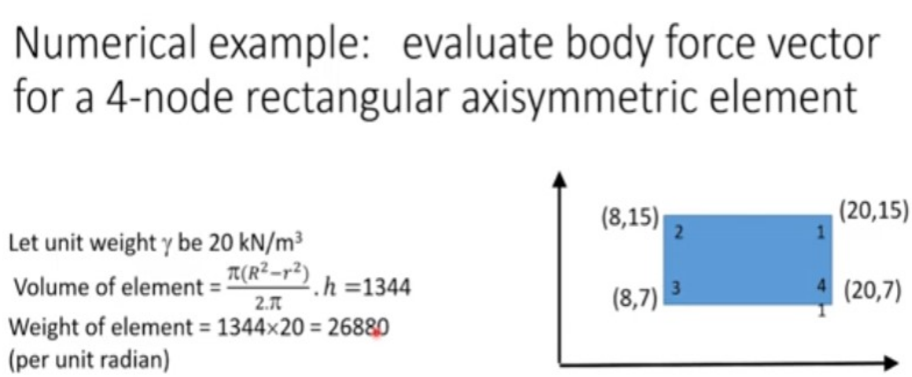


And now let us do the same calculations for axisymmetric element. Let us take the same dimensions, the same coordinates except that now we have an axisymmetric geometry. So, the volume of this element for a unit radian is 1344 and then the weight of the element is the volume multiplied by 20 that is 26,880 and this weight is per unit radian as we have discussed earlier the 2 pi is common both on the left hand side, on the right hand side.



So, the 2 pi gets cancelled out and we work only in terms of unit radian and so if you use a 1 point integration for the axisymmetric problem and our radius at Xi and eta of 0 that is at the center point will be the average radius 20 + 8 is 28 by 2 is 14.

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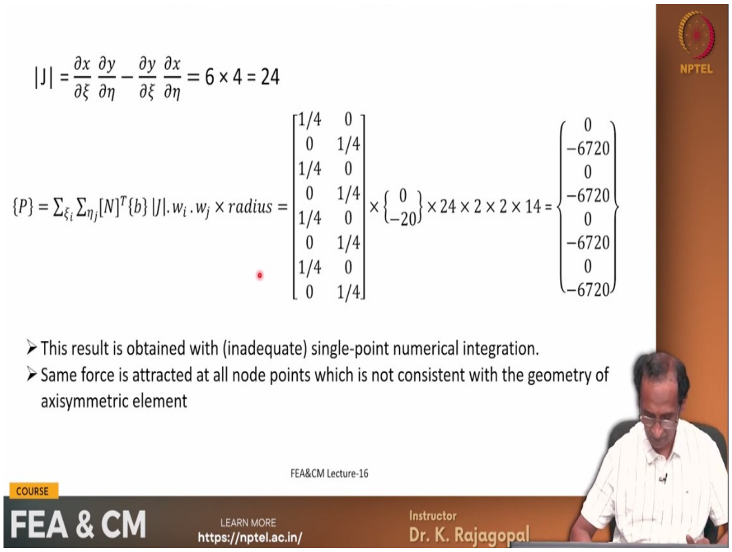

And so the entire thing is the same as we had seen for the plane strain case except for the radius here. So, now our force is 6720 equally distributed at all the 4 nodes and we can realize that this is not correct because our order of polynomial is 2. So, our order of polynomial in the shape functions is 1 and then for the radius is 1. So, the total order of polynomial is 2.

$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} = 6 \times 4 = 24$$

$$\{P\} = \sum_{\xi_i}\sum_{\eta_j}[N]^T\{b\}\,|J|.w_i.w_j \times radius = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \\ 1/4 & 0 \\ 0 & 1/4 \\ 1/4 & 0 \\ 0 & 1/4 \\ 1/4 & 0 \\ 0 & 1/4 \end{bmatrix} \times \begin{Bmatrix} 0 \\ -20 \end{Bmatrix} \times 24 \times 2 \times 2 \times 14 = \begin{Bmatrix} 0 \\ -6720 \\ 0 \\ -6720 \\ 0 \\ -6720 \\ 0 \\ -6720 \end{Bmatrix}$$

So, that means that you need minimum 2 point integration for getting good result. So, this distribution is not consistent with the type of element, but then if you look at the total weight 6720 times 4 will give you the same weight as 26,880 at least and we are able to calculate the exact total weight because our area is actually it is a first order polynomial and we can evaluate the area exactly and so we need to go in for higher integration.

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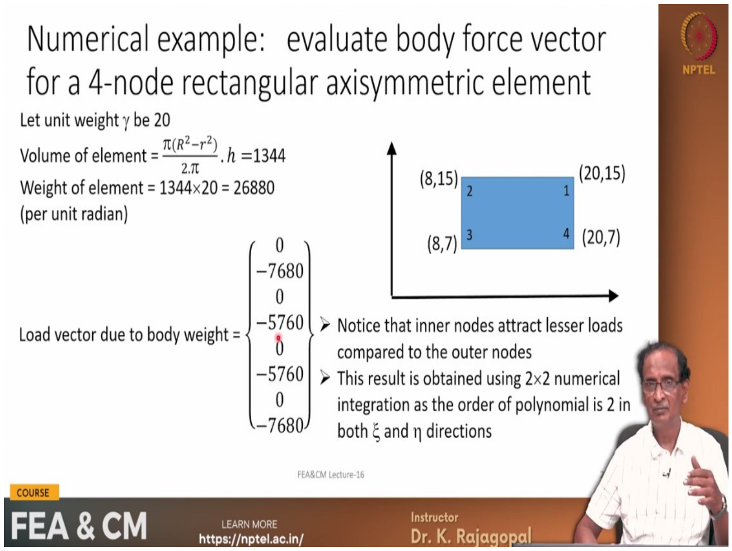

$$\text{Load vector due to body weight} = \begin{Bmatrix} 0 \\ -7680 \\ 0 \\ -5760 \\ 0 \\ -5760 \\ 0 \\ -7680 \end{Bmatrix}$$

And this result is obtained with 2 point integration 2 by 2 integration and you see that the distribution of loads has changed between the nodes on the two outer nodes we have a force of 7,680 whereas in the inner nodes we have 5,760 and if you total up all these loads it will add up to 26,880 that you can easily see and this is with 2 by 2 integration I will show a computer program that you can utilize or otherwise we do the previous calculation four times and then add them up.

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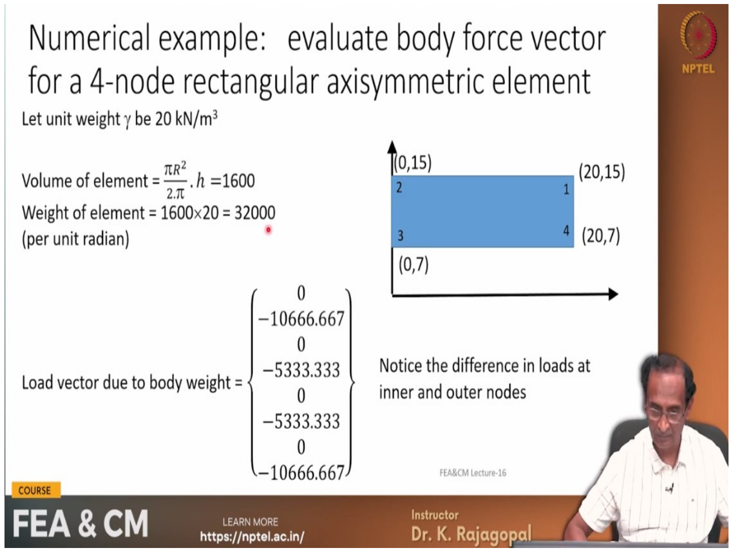
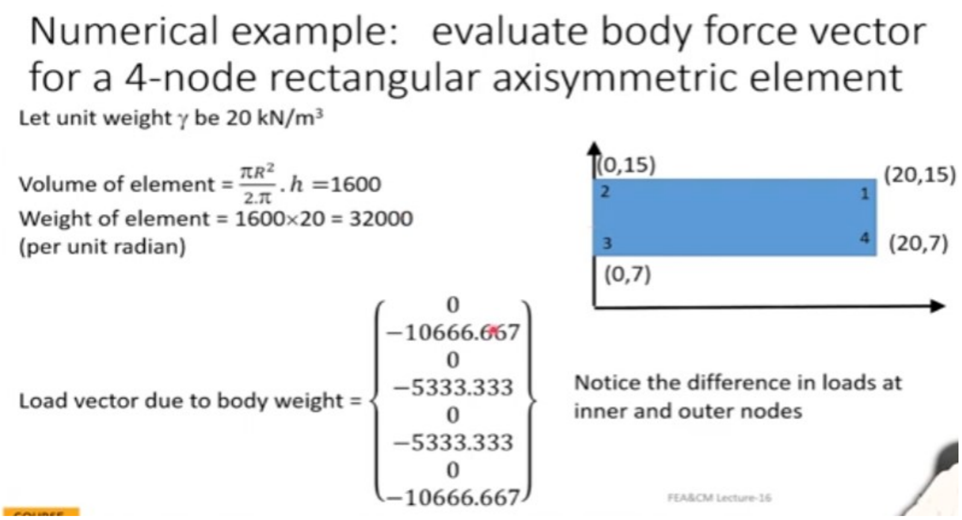


And the same problem with slightly different coordinates here we have the axisymmetric elements starting from the origin and now our total weight is 32,000 and the distribution is like this 10,666 on the outer nodes and inside nodes 5,333 and the sum total for all these four forces will be equal to 32,000.



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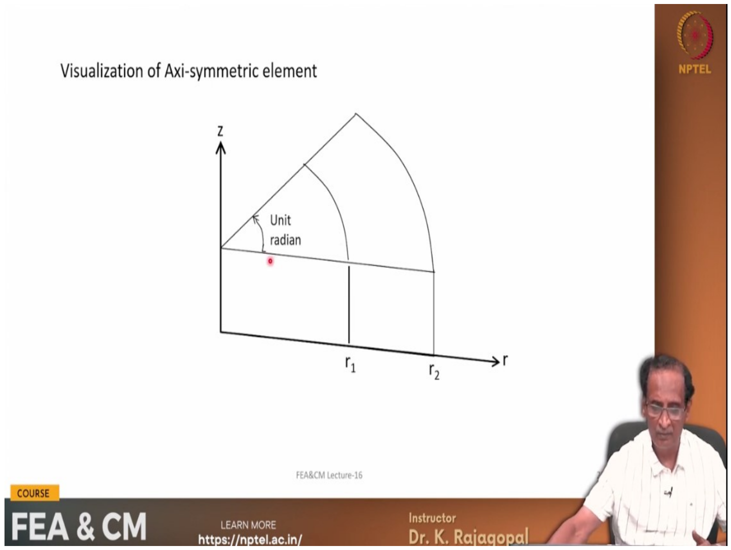

And what we are doing within the axisymmetric element is we are doing the calculations only for unit radian. We are not doing for the full 2 pi and this is what we are doing like this is the volume that we are dealing with.

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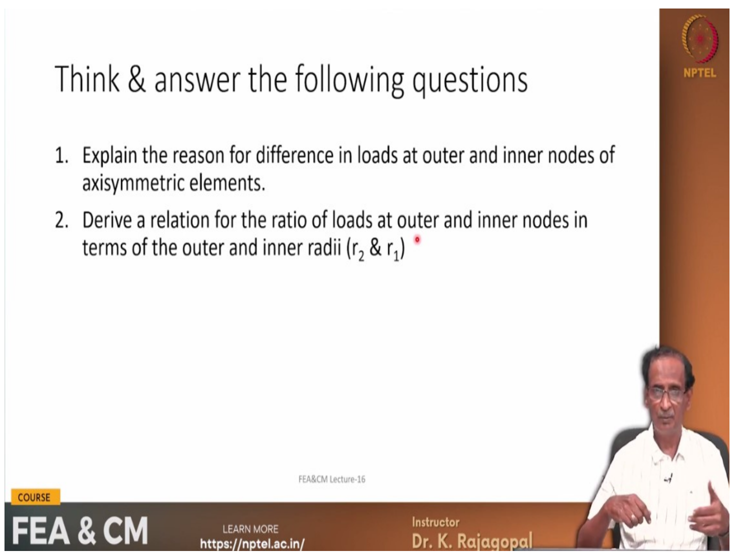

And some questions for you to think about. See what is the reason for difference the loads at outer and inner nodes in the axisymmetric element. See this is what I mean. See at the outer nodes we have 10,666 whereas in the inner node we have 5,333. In this case it is exactly 2 times the difference like 2 times of this is this, but in the previous case it is not 2 times it is some value 5,760 and the outer node is 7,680.

See this distribution depends on the geometry and one should think about an answer what is the reason for difference and then can you derive a relation for the ratio of the loads in terms of r 2 and r 1. In terms of the loads at the outer nodes and the inner node anyway through the finite element calculations we get the exact loads, but whether you can think differently and get some analytical equation in terms of r 2 and r 1 you please try them out and you can send me the solutions.

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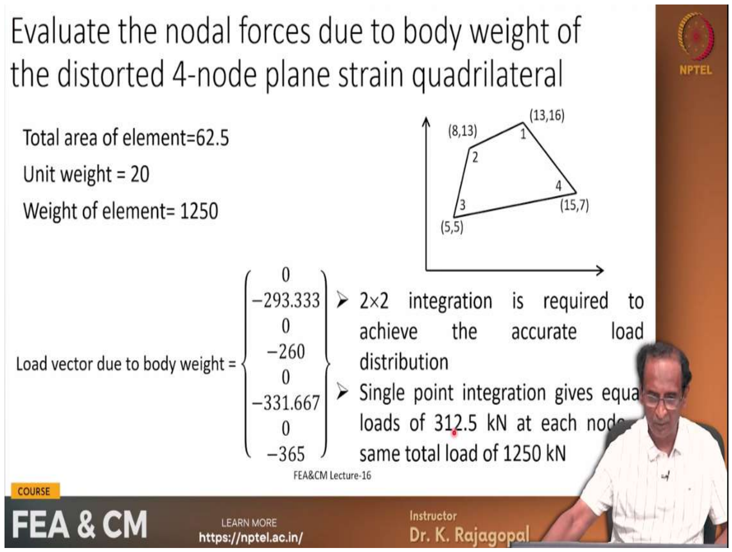
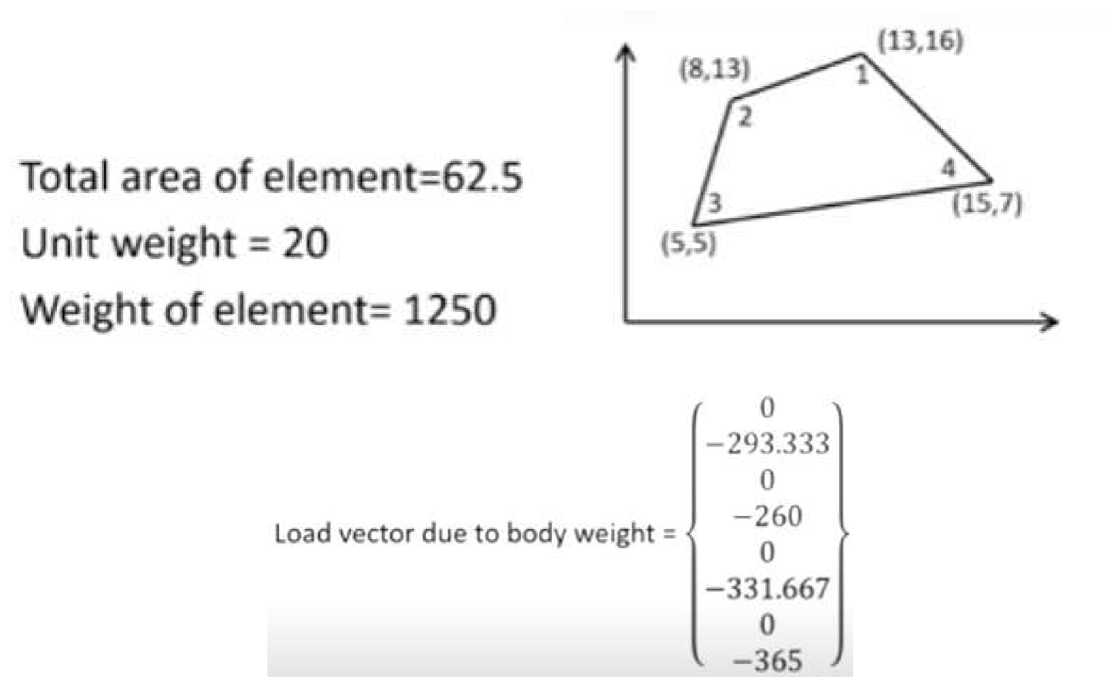


So, now let us do the same calculations for a distorted element a plane strain distorted element our coordinates are like this and with a 2 by 2 integration these are the loads at different nodes and if we use 1 point integration we get the load of 312.5 at each of these 4 points. So adding up to a total of 1,250, but then the nature of distribution is different.



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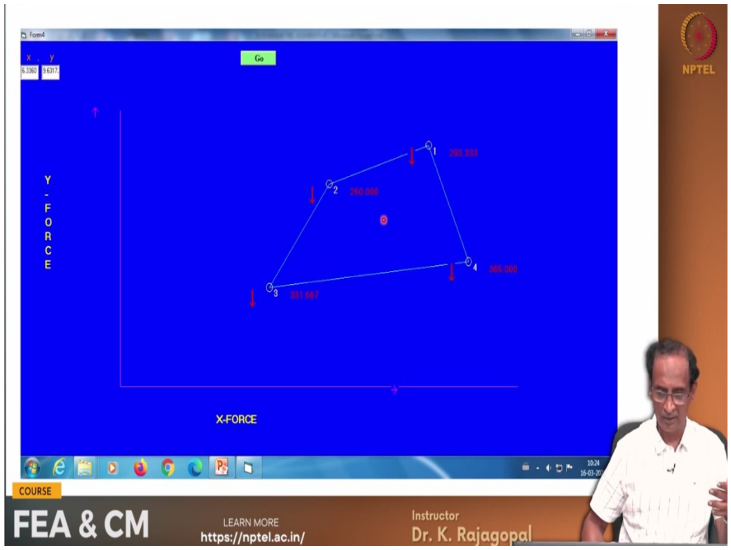

This is with 2 point integration this is the result from the program that you are going to see later. This is node 1, node 2, node 3, node 4 these are the forces that are estimated using the 2 by 2 integration.

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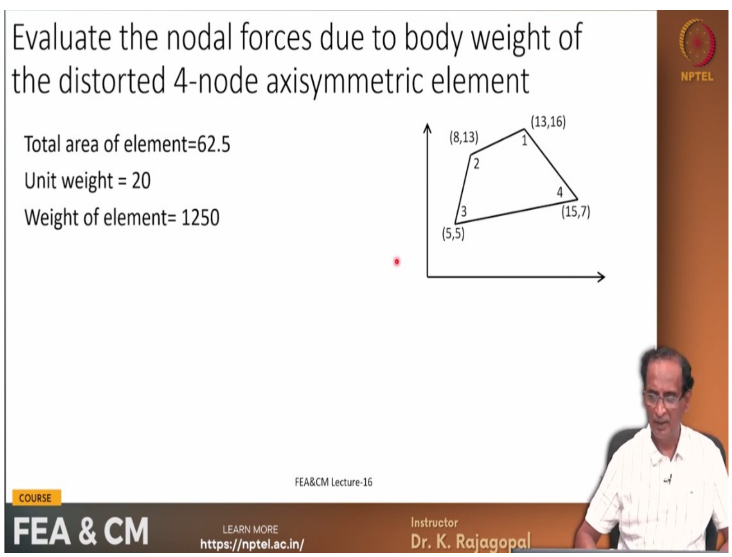
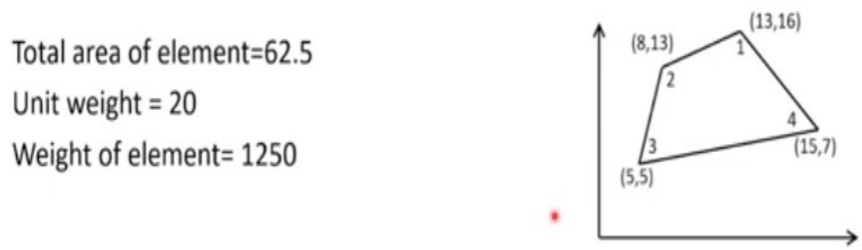


And now let us look at the axisymmetric element the same element, but in the axisymmetric geometry.



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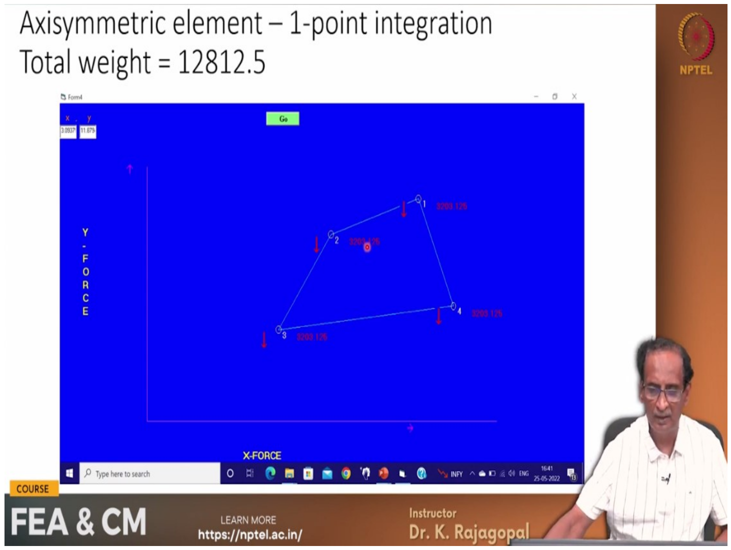

And so with 1 point integration we get the same load at all the 4 nodes.

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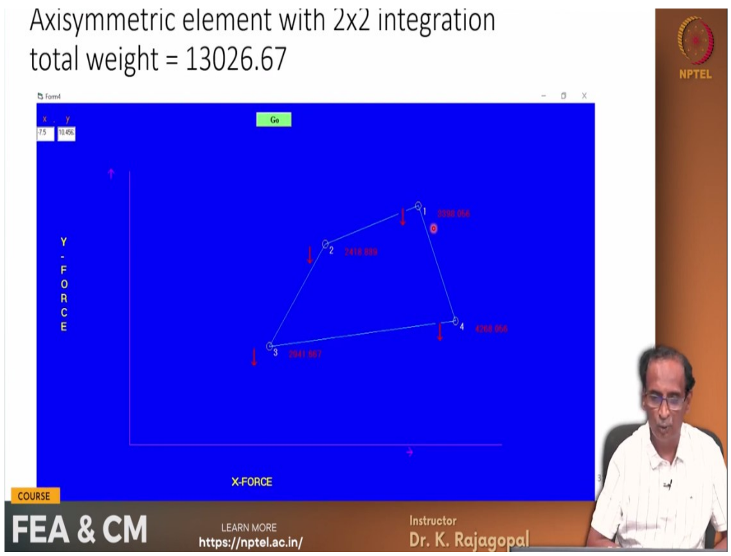

And with 2 by 2 integration we get different distribution at node 1, node 2, node 3, node 4.

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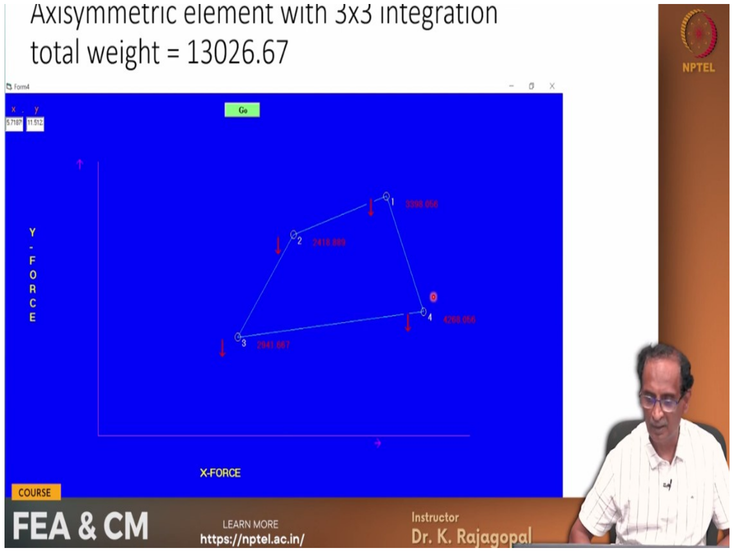

And even with 3 by 3 integration we get the same loads. See at node 1 with a 3 by 3 integration 3,398 and even with 2 point integration it is the same 3,398. So, that means that with 2 point integration itself you can get the accurate results and if you use the higher order of integration you are only going to spend extra time, but then you will not get any different result. I will show you this program a bit later.

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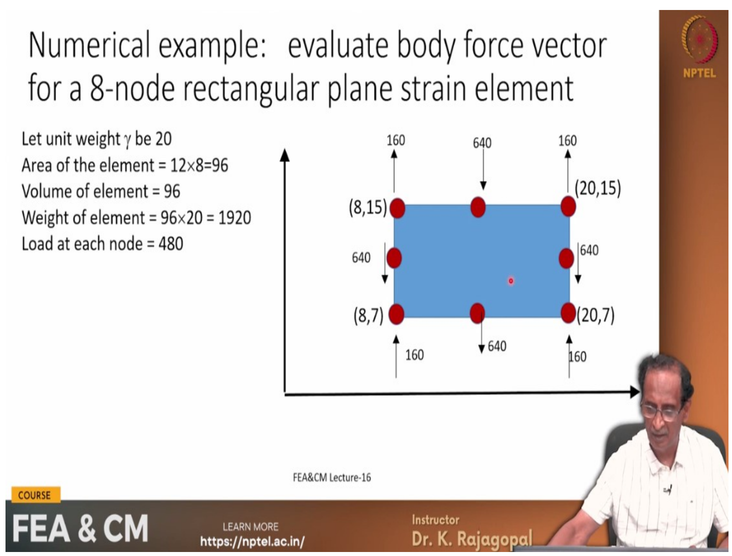
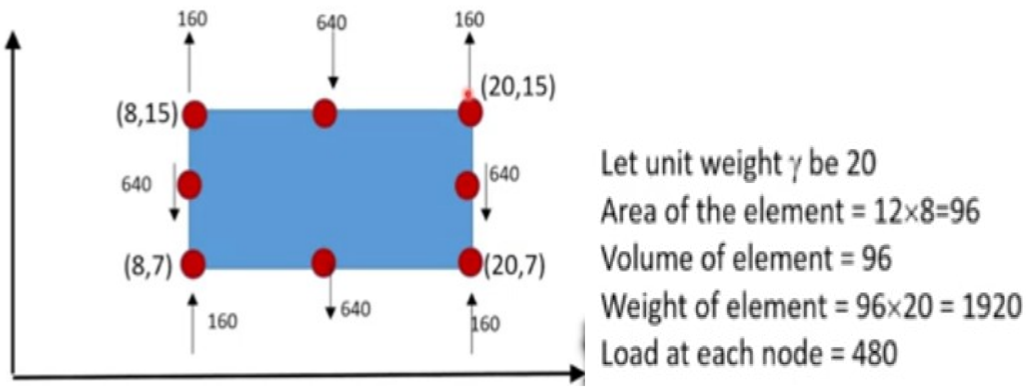


And now let us look at the 8 node quadrilateral element and this is 8 nodes 1, 2, 3, 4, 5, 6, 7, 8. So, when we have higher order elements we always numbers the corner nodes first because the corner nodes are the minimum required for defining any quadrilateral elements and then inside we can define the mid size nodes like 5, 6, 7, 8 are only some of them and see this N transpose b calculation is done for 4 node element and our gravity force is acting down.



But then you see at these 4 corner nodes you see the force going up whereas at the 4 this mid size nodes the force is going down 640 and of course the total of all the 8 nodes adds up to the total weight of the element 1,920, but if you get this type of result we should not be thinking that it is wrong because we call it as a consistent distribution of loads N transpose b integrated over the full volume.

And if you are not convinced whether this is correct or not you can go for higher order of integration set whether you get any different result and actually this distribution is consistent with the way the stiffness of the element is distributed between the different nodes.

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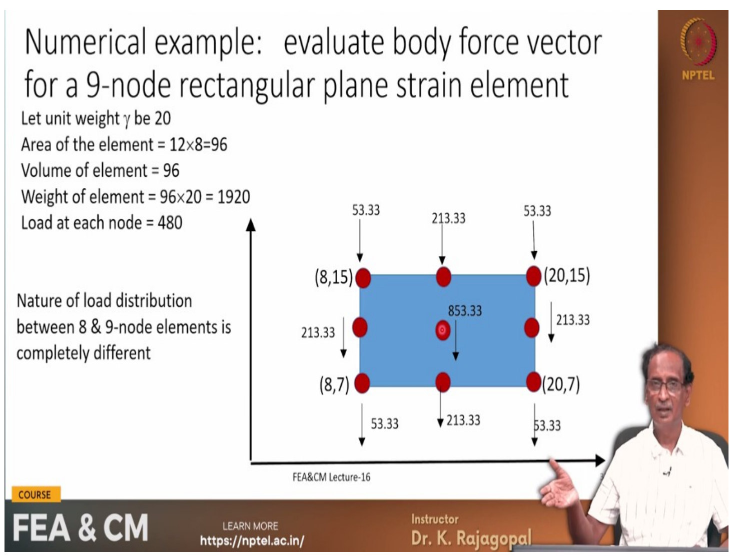
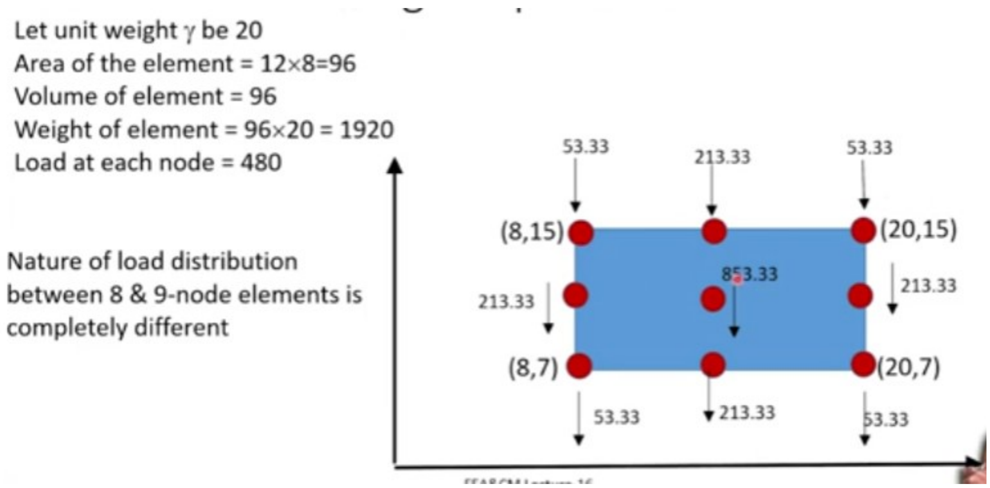


And here is the distribution for a 9 node quadrilateral. It is actually previously we had seen for 8 nodes now we have for the 9 nodes and when we have the 9th node you see all the loads are pointing downwards previously with 8 node element we had seen at the corner node the load is acting up and you see the difference between the load of the corner node and then the mid side node it is 213.33 by 53.33 that is 4 times.



And then 4 times of this load is distributed to the center point the center of the element and compared to the corner points this is 16 times. Actually we can see based on this the center point has more weight in terms of the stiffness like these degrees of freedom at this node, at this 9th node will have much higher stiffness contribution and correspondingly we also get higher load when we distribution this N transpose b dv and till now we have seen this N transpose b calculations.

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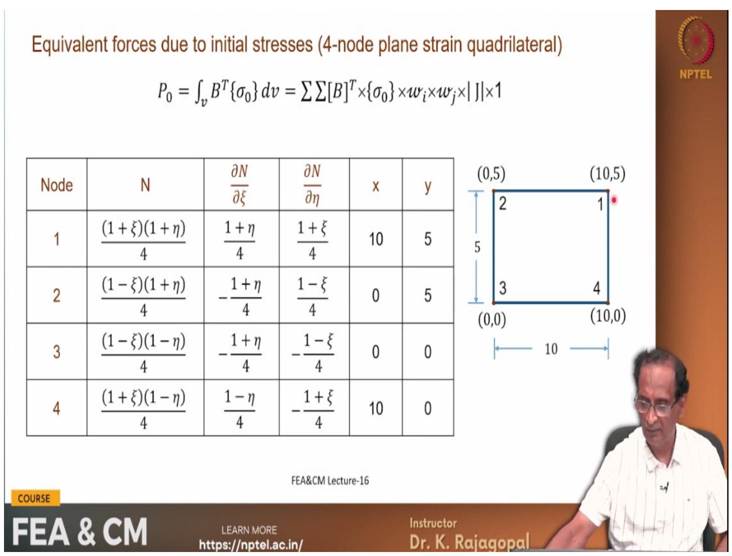


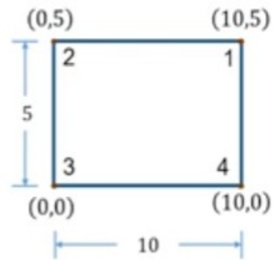
Node	N	$\frac{\partial N}{\partial \xi}$	$\frac{\partial N}{\partial \eta}$	x	y
1	$\frac{(1+\xi)(1+\eta)}{4}$	$\frac{1+\eta}{4}$	$\frac{1+\xi}{4}$	10	5
2	$\frac{(1-\xi)(1+\eta)}{4}$	$-\frac{1+\eta}{4}$	$\frac{1-\xi}{4}$	0	5
3	$\frac{(1-\xi)(1-\eta)}{4}$	$-\frac{1+\eta}{4}$	$-\frac{1-\xi}{4}$	0	0
4	$\frac{(1+\xi)(1-\eta)}{4}$	$\frac{1-\eta}{4}$	$-\frac{1+\xi}{4}$	10	0

And now let us do this b transpose sigma calculation. This is a very, very important calculation because in all our finite element analysis in geotechnical engineering we will be doing this not only for checking for the equilibrium of the system, but also for modeling construction and excavation and then the pre stressing and so on and let us do for rectangular element like this.

Equivalent forces due to initial stresses (4-node plane strain quadrilateral)

$$P_0 = \int_v B^T\{\sigma_0\}\,dv = \sum\sum[B]^T \times \{\sigma_0\} \times w_i \times w_j \times |J| \times 1$$

Node	N	$\frac{\partial N}{\partial \xi}$	$\frac{\partial N}{\partial \eta}$	x	y
1	$\frac{(1+\xi)(1+\eta)}{4}$	$\frac{1+\eta}{4}$	$\frac{1+\xi}{4}$	10	5
2	$\frac{(1-\xi)(1+\eta)}{4}$	$-\frac{1+\eta}{4}$	$\frac{1-\xi}{4}$	0	5
3	$\frac{(1-\xi)(1-\eta)}{4}$	$-\frac{1+\eta}{4}$	$-\frac{1-\xi}{4}$	0	0
4	$\frac{(1+\xi)(1-\eta)}{4}$	$\frac{1-\eta}{4}$	$-\frac{1+\xi}{4}$	10	0



And the coordinates the length is 10 and the height is 5 and these are the coordinates and then the 4 node 1, 2, 3, 4 and once again we have an anti symmetric sorry the numbering is in the anti clockwise direction 1, 2, 3, 4.

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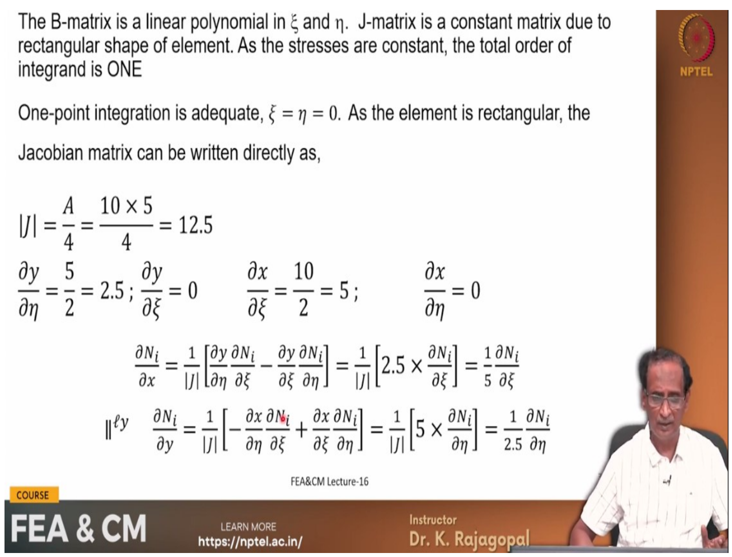

And the B matrix is a linear polynomial in Xi and eta and the J matrix is a constant matrix due to the rectangular shape because we have seen with rectangular shape earlier that dou x by dou Xi is constant and that is equal to length divided by 2 dou x by dou eta is 0 and dou y by dou eta is 0 and dou y and dou eta is half of the height. So, our determinant of the Jacobian matrix J is A by 4 that is 12.5 and dou y dou eta is a 5 by 2 and dou x by dou x is 10 by 2.

One-point integration is adequate, $\xi = \eta = 0$. As the element is rectangular, the Jacobian matrix can be written directly as,

$$|J| = \frac{A}{4} = \frac{10 \times 5}{4} = 12.5$$

$$\frac{\partial y}{\partial \eta} = \frac{5}{2} = 2.5\,; \quad \frac{\partial y}{\partial \xi} = 0 \qquad \frac{\partial x}{\partial \xi} = \frac{10}{2} = 5\,; \qquad \frac{\partial x}{\partial \eta} = 0$$

$$\frac{\partial N_i}{\partial x} = \frac{1}{|J|}\left[\frac{\partial y}{\partial \eta}\frac{\partial N_i}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial N_i}{\partial \eta}\right] = \frac{1}{|J|}\left[2.5 \times \frac{\partial N_i}{\partial \xi}\right] = \frac{1}{5}\frac{\partial N_i}{\partial \xi}$$

$$\frac{\partial N_i}{\partial y} = \frac{1}{|J|}\left[-\frac{\partial x}{\partial \eta}\frac{\partial N_i}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial N_i}{\partial \eta}\right] = \frac{1}{|J|}\left[5 \times \frac{\partial N_i}{\partial \eta}\right] = \frac{1}{2.5}\frac{\partial N_i}{\partial \eta}$$

And we can calculate dou N by dou x and dou N by dou y through our equation that we derived earlier and if you assume constant stress state we can just simply use 1 point integration because our B matrix is a linear polynomial.

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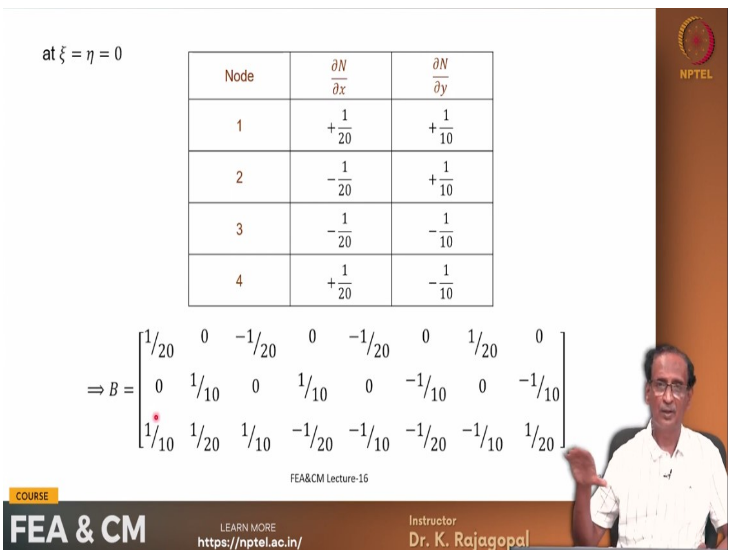

And at Xi and eta of 0 that is for 1 point integration our dou N by dou x at different points is like this and the dou N by dou y at different points is like this. This we have three rows because we have three strains and then 8 columns because 8 degrees of freedom.

at $\xi = \eta = 0$

Node	$\frac{\partial N}{\partial x}$	$\frac{\partial N}{\partial y}$
1	$+\frac{1}{20}$	$+\frac{1}{10}$
2	$-\frac{1}{20}$	$+\frac{1}{10}$
3	$-\frac{1}{20}$	$-\frac{1}{10}$
4	$+\frac{1}{20}$	$-\frac{1}{10}$

$$\Rightarrow B = \begin{bmatrix} 1/20 & 0 & -1/20 & 0 & -1/20 & 0 & 1/20 & 0 \\ 0 & 1/10 & 0 & 1/10 & 0 & -1/10 & 0 & -1/10 \\ 1/10 & 1/20 & 1/10 & -1/20 & -1/10 & -1/20 & -1/10 & 1/20 \end{bmatrix}$$

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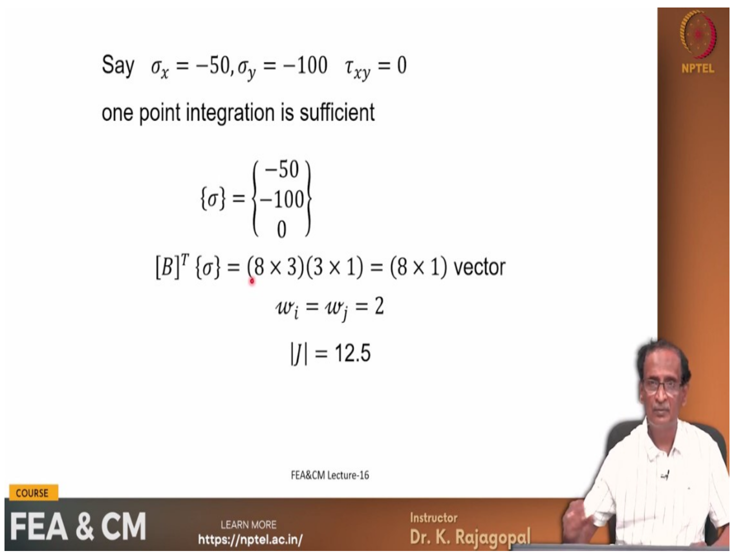

And let us say that our sigma x is -50 and sigma y is -100 and tau xy is 0 and everywhere and so our sigma in the vector form is – 50, - 100 and 0 and our B transpose sigma will have numbers of rows of 8 and columns 1 that we can easily see from here 8 by 3 multiplying 3 by 1 that is an 8 by 1 vector.

Say $\sigma_x = -50, \sigma_y = -100 \quad \tau_{xy} = 0$

one point integration is sufficient

$$\{\sigma\} = \begin{Bmatrix} -50 \\ -100 \\ 0 \end{Bmatrix}$$

$$[B]^T \{\sigma\} = (8 \times 3)(3 \times 1) = (8 \times 1) \text{ vector}$$

$$w_i = w_j = 2$$

$$|J| = 12.5$$

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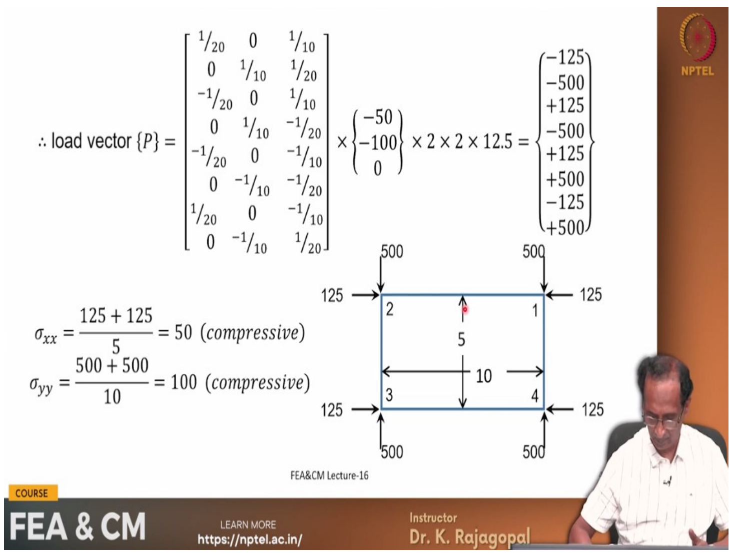
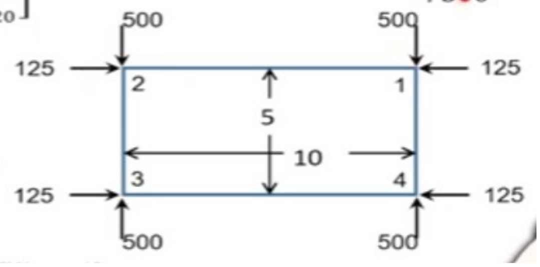


And if you go through this full calculation we will get a force vector like this and we can plot them on the element like this. This – 125 at node 1 is in the negative x direction and – 500 is in the negative y direction at node 1 like this and at node 2 it is positive 125 and then negative 500 at node 3 both x and y are positive at node 4 x is negative force and y is a positive force. So, here we see that the equilibrium is satisfied like there is a whatever force that you have downwards is also the same force acting up and then from the sides also we have so.

$$\therefore \text{load vector } \{P\} = \begin{bmatrix} 1/20 & 0 & 1/10 \\ 0 & 1/10 & 1/20 \\ -1/20 & 0 & 1/10 \\ 0 & 1/10 & -1/20 \\ -1/20 & 0 & -1/10 \\ 0 & -1/10 & -1/20 \\ 1/20 & 0 & -1/10 \\ 0 & -1/10 & 1/20 \end{bmatrix} \times \begin{Bmatrix} -50 \\ -100 \\ 0 \end{Bmatrix} \times 2 \times 2 \times 12.5 = \begin{Bmatrix} -125 \\ -500 \\ +125 \\ -500 \\ +125 \\ +500 \\ -125 \\ +500 \end{Bmatrix}$$

$$\sigma_{xx} = \frac{125 + 125}{5} = 50 \ (compressive)$$

$$\sigma_{yy} = \frac{500 + 500}{10} = 100 \ (compressive)$$



That means that at the element level we have perfect equilibrium and then this sigma x is the net lateral force divided by the height multiplied by 1 so that is 125 + 125 by 5 that is 50 compressive then sigma yy is 500 plus 500 divided by 10 that is 100 compressive that is what we had specified and then there is no shear stress because along any surface there is no shear force 125 and – 125 so it is they compensate each other.

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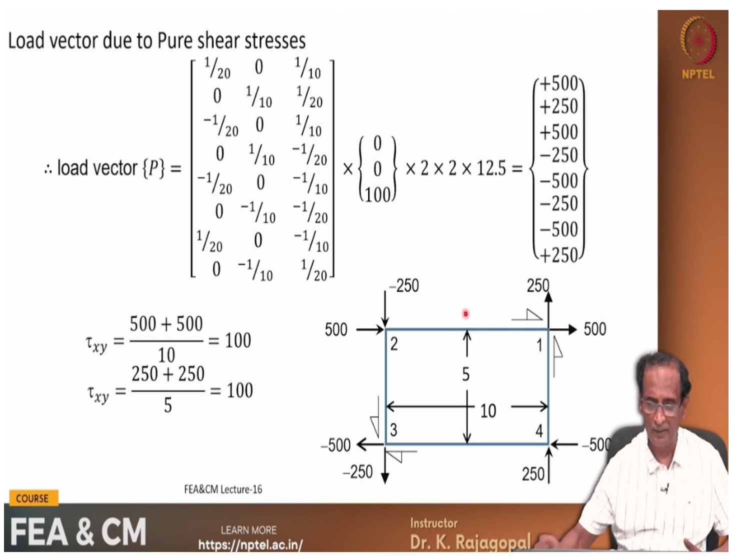
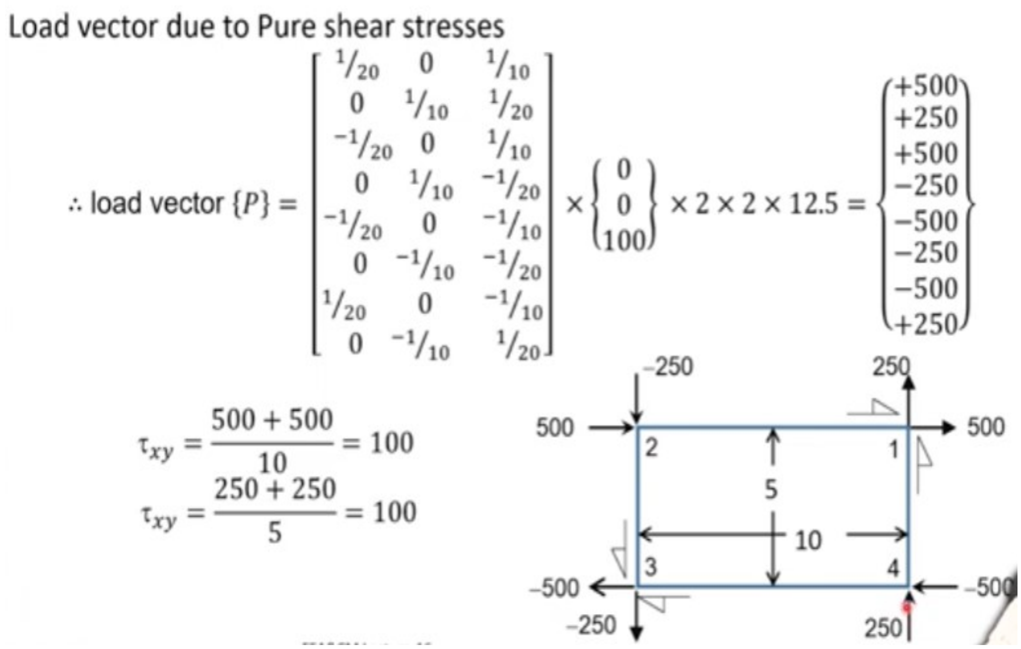


And let us consider the pure shear stress. Let us apply a shear stress of 100 and if you do the calculation the force distribution is like this. So, at node 1 500, 250, node 2 500, 250 going down and node 3 it is – 500 and – 250 node 4 – 500 and + 250 and the shear stress along any surface say along this horizontal surface we have 500 + 500 that is 1,000 divided by 10 times 1 that is 100 and then on the vertical phase 250 + 250 by 5 times 1 that is 100.



So, that is the applied shear stress and we get the same shear stress even by manual calculation and so actually this is our sign convention. So, if any shear force develops a clockwise moment around an interior point we call that as a positive shear force and this is the positive shear stress, positive shear force and positive shear stress.

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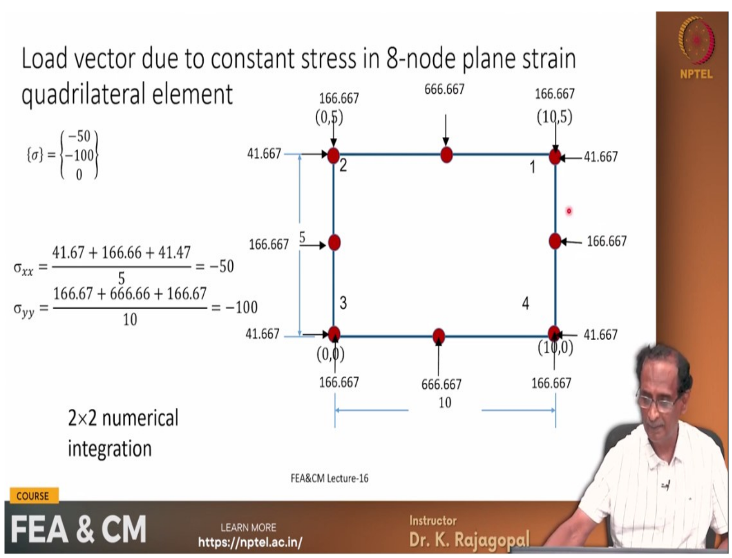
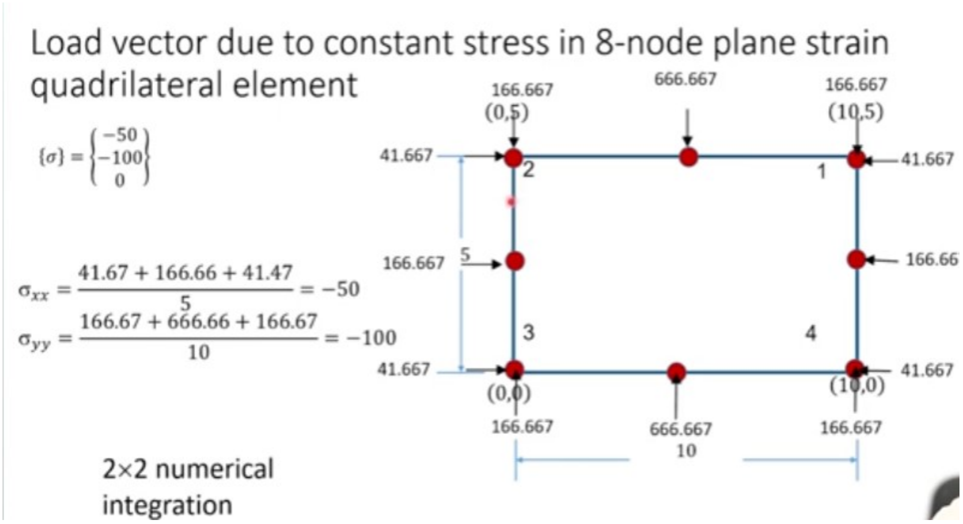


And so the load vector for the same stress state on a 8 node quadrilateral is like this. It is actually it is a bit more complicated because the mid side node we have 4 times of the loaded the corner nodes whether it is in the vertical direction or the lateral direction 166.66 multiplied by 4 is this 41.66 times 4 is this one 166.



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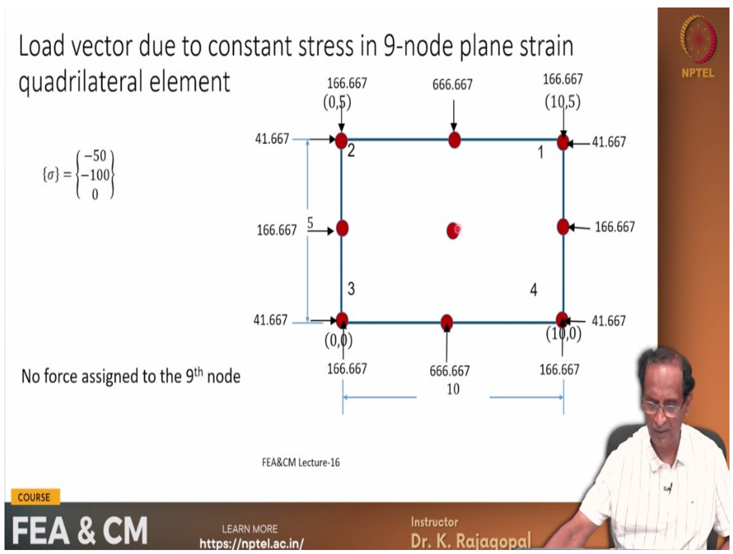
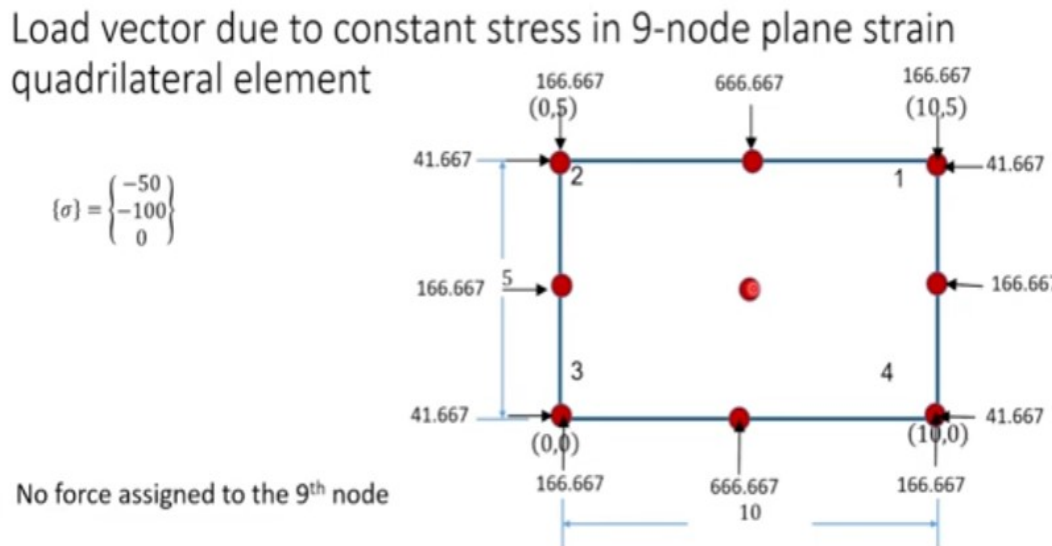


And if you look at the 9 node element interestingly there is no force here that is understandable because that is the point of center of gravity and for maintaining symmetry the forces at the center of gravity should be 0 because there are no deformations if that is the center of sorry that is in the symmetry plane and for doing these calculations I am going to show you one computer program very simple computer program that you can run because see for a 4 node quadrilateral we can do hand calculations, but not for 8 and 9 node because the terms is more.



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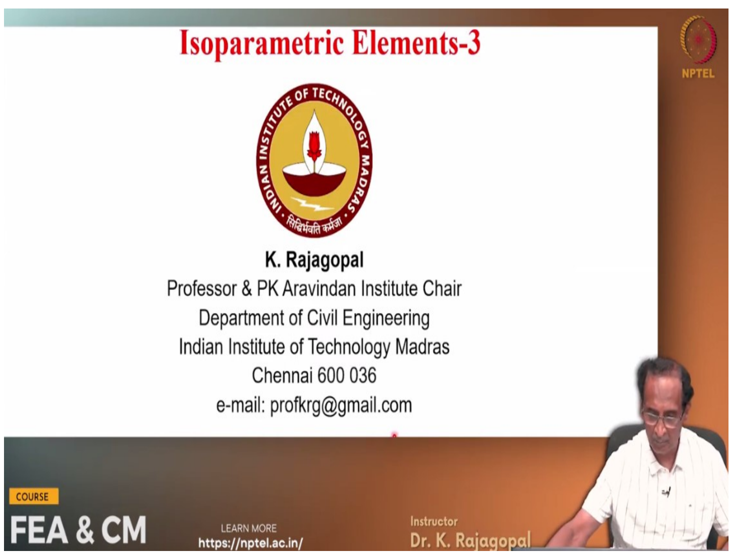

So, if you have any question please write an email to this address profkrg@gmail.com and I will reply back. So, thank you very much.