FEM & Constitutive Modelling in Geomechanics
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Lecture - 17
Isoparametric Elements Part-II

Hello students, welcome back. So in the previous class, I had introduced you to the concept of isoparametric elements, which are defined from -1 to +1.

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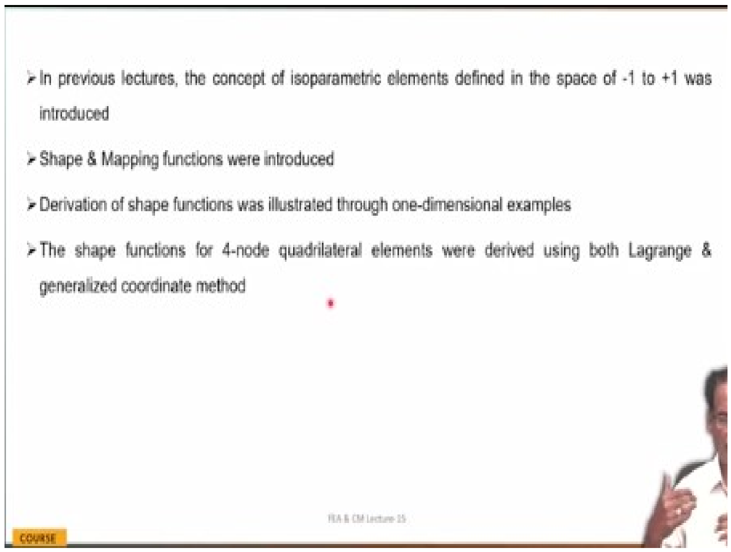

And we had the derived some shape functions and let us continue this concept for higher order elements.

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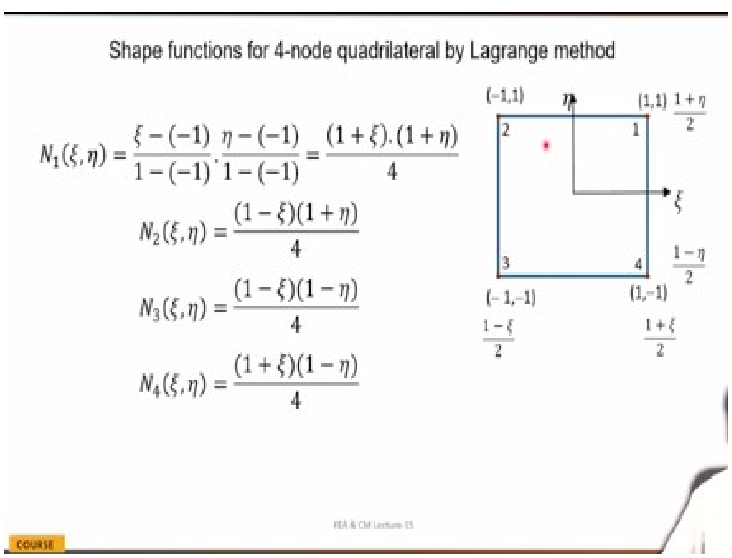

We have seen the derivation for 4-node quadrilateral element. We have derived the shape functions using both the Lagrange's method and then the generalized coordinate method.

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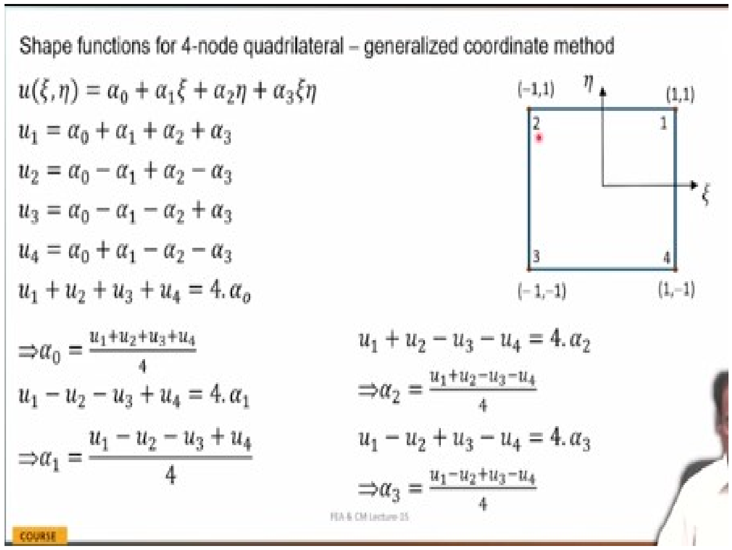

And we saw that we got the same functions.

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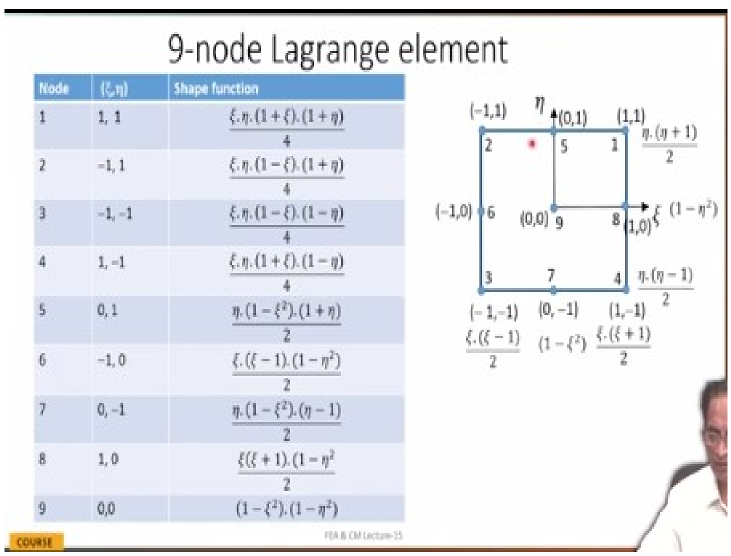
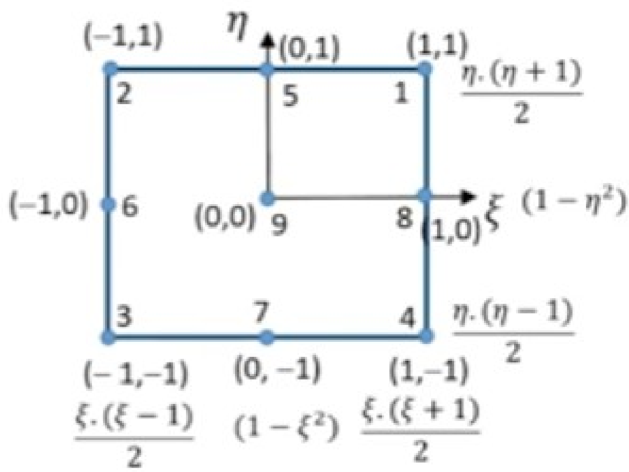


Node	(ξ,η)	Shape function
1	1, 1	$\frac{\xi.\eta.(1+\xi).(1+\eta)}{4}$
2	-1, 1	$\frac{\xi.\eta.(1-\xi).(1+\eta)}{4}$
3	-1, -1	$\frac{\xi.\eta.(1-\xi).(1-\eta)}{4}$
4	1, -1	$\frac{\xi.\eta.(1+\xi).(1-\eta)}{4}$
5	0, 1	$\frac{\eta.(1-\xi^2).(1+\eta)}{2}$
6	-1, 0	$\frac{\xi.(\xi-1).(1-\eta^2)}{2}$
7	0, -1	$\frac{\eta.(1-\xi^2).(\eta-1)}{2}$
8	1, 0	$\frac{\xi((\xi+1).(1-\eta^2}{2}$
9	0,0	$(1-\xi^2).(1-\eta^2)$

Now let us go on to the much higher order element say a 9-node Lagrange element. This is a pure Lagrange element because we have the same number of nodes along each line, whether along the psi direction or along eta direction. And node 1 is in the positive quadrant. Node 2 is in the second quadrant. Node 3 is in the third. Node 4 is in the fourth quadrant and so on.

And by looking at our 3-node bar element, we can say that our shape function at this point is psi of +1 is psi times psi + 1 by 2. And the shape function at -1 is psi times psi - 1 by 2 at this point at the mid side point at psi of 0, 1 minus psi square. Then in eta direction eta times eta + 1 by 2, 1 minus eta square.



And this and the advantage with the Lagrange's method is we get the shape function separately in the two directions at psi and eta directions and then take a product to get our shape function. So the N 1 will be the product of this and this. So the N 1 is psi eta times 1 plus psi, 1 plus eta by 4. Whereas N 9 is just simply the product of this and this, 1 minus psi square, 1 minus eta square. And then similarly, all the other at all the other nodes, okay?

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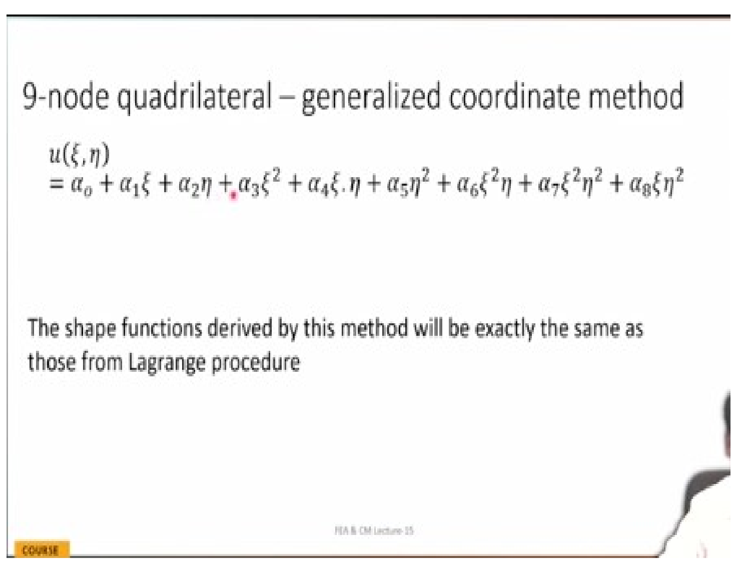

And we can get the same shape functions even by using the generalized coordinate method. The polynomial for the 9-node quadrilateral is alpha naught plus alpha 1 psi plus alpha 2 eta plus alpha 3 psi square plus alpha 4 psi eta and so on, okay? And we can derive the shape functions and we will see that the shape function that we get from the generalized coordinate method are exactly the same as what we get from the Lagrange's method.

$$u(\xi,\eta) = \alpha_o + \alpha_1\xi + \alpha_2\eta + \alpha_3\xi^2 + \alpha_4\xi.\eta + \alpha_5\eta^2 + \alpha_6\xi^2\eta + \alpha_7\xi^2\eta^2 + \alpha_8\xi\eta^2$$

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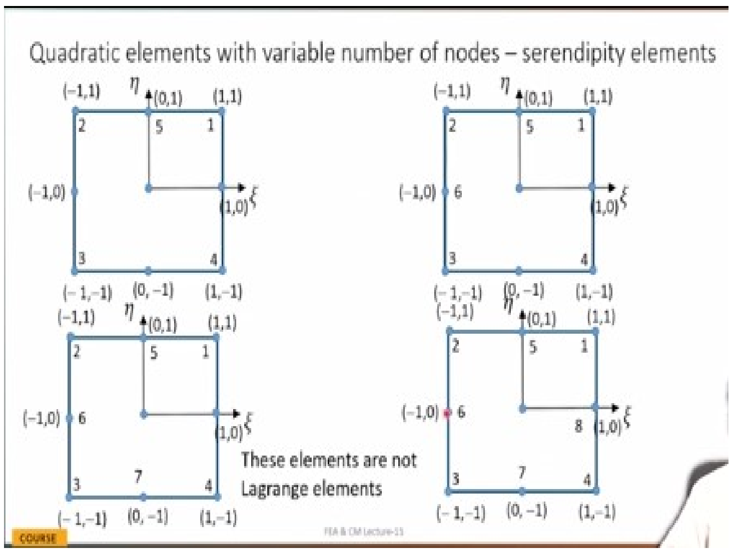

And the advantage with the serendipity method of correction is we can incorporate the elements with which are not exactly Lagrange elements like with any variable number of nodes. Say for example, we can think of a quadratic element with 5 nodes. Say let us say we have a fifth node here or we can have a 6-node elements 1, 2, 3, 4, 5, 6 or 7, 5, 6, 7 or 8-node element 5, 6, 7, 8.

And these are not Lagrange elements, these are called as serendipity elements. And for deriving the shape functions for this, we need a separate procedure. And so let us see what that is. So in fact, this 8-node element is very popular in finite element analysis because it is very powerful and we get a good accurate result in not only in the elastic analysis, but also in the elastic plastic analysis.

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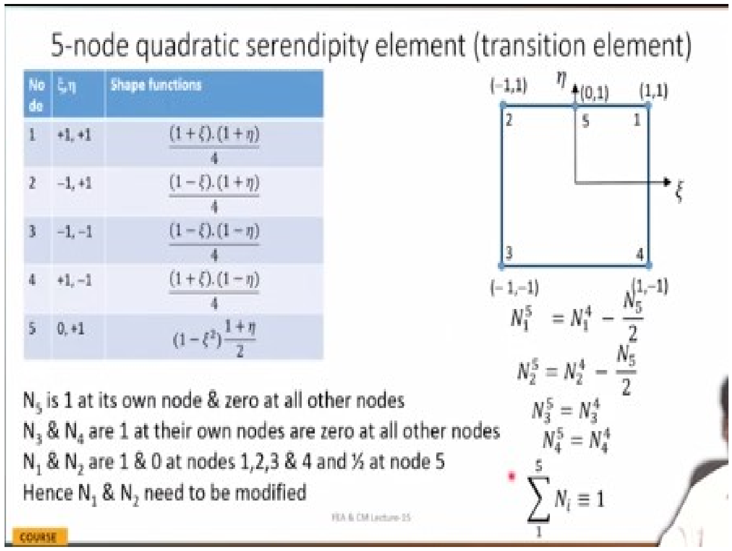
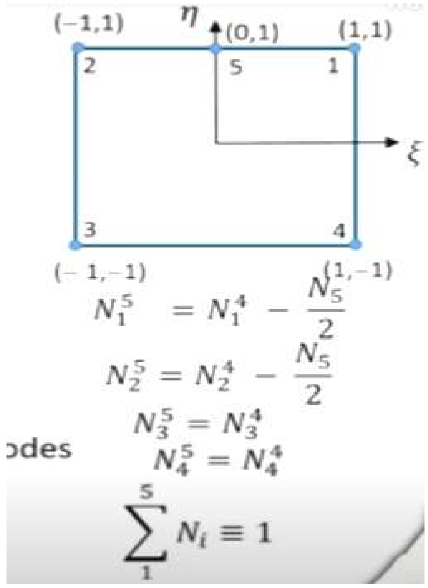


So for the 5-node element, we have the, we can start with the shape functions for the 4-node element 1, 2, 3, 4 and then introduce the fifth node and do the correction, okay? And so we can actually directly write our, these shape functions based on the Lagrange's procedure like this, okay? And if you see, at node 1, the shape function for 4-node element is 1 plus psi, 1 plus eta by 4.

Node	ξ,η	Shape functions
1	+1, +1	$\frac{(1+\xi).(1+\eta)}{4}$
2	−1, +1	$\frac{(1-\xi).(1+\eta)}{4}$
3	−1, −1	$\frac{(1-\xi).(1-\eta)}{4}$
4	+1, −1	$\frac{(1+\xi).(1-\eta)}{4}$
5	0, +1	$(1-\xi^2)\frac{1+\eta}{2}$

And at node 2, 1 minus psi times 1 plus eta by 4. And at node 3, 1 minus psi times 1 minus eta by 4 and node 4, 1 plus psi 1 minus eta by 4. Okay now, we have introduced the fifth node at this point. And that is 1 minus psi square because this is the middle node along the psi direction when you have three nodes. And then in the eta direction, this is at eta of +1 and there are only two nodes along the eta direction.



So this could be written as 1 plus eta by 2. But if you introduce this node, the problem is if you substitute eta of 0 and psi of 1 we have a problem. We do not get 0, because our object is the shape functions should have value of 1 at their own node and 0 at all the other nodes. So when you had a 4-node element, that was easy like, if you substitute psi of the eta 1, you get 1.

But if you substitute psi of -1 or eta of -1 you will get 0 and so on, okay? But when you introduce this node at node 5, then this is not 0 but it is half. So to make N 1 and N 2 as 0 at this point, I can say N 1 at fifth node is N 1 for the 4-node element minus N 5 by 2. And N 2 for 5-node element is N 2 for the 4-node element minus N 5 by 2.

And these two shape functions 3 and 4 need not be corrected because, if we evaluate these two along this line where eta is +1, they will automatically become 0. Like for example, if you substitute eta of 1, N 3 is 0 and N 4 is 0, okay? So just by correcting these two shape functions, we can get the shape functions for the 5-node serendipity element.

And then if you sum up all the shape functions, you get one, that is the corrected shape functions N 1 and N 2 for the 5-node element plus N 5 plus N 3 and N 4 you will get 1. That I have not done here, but you can do it yourself, okay?

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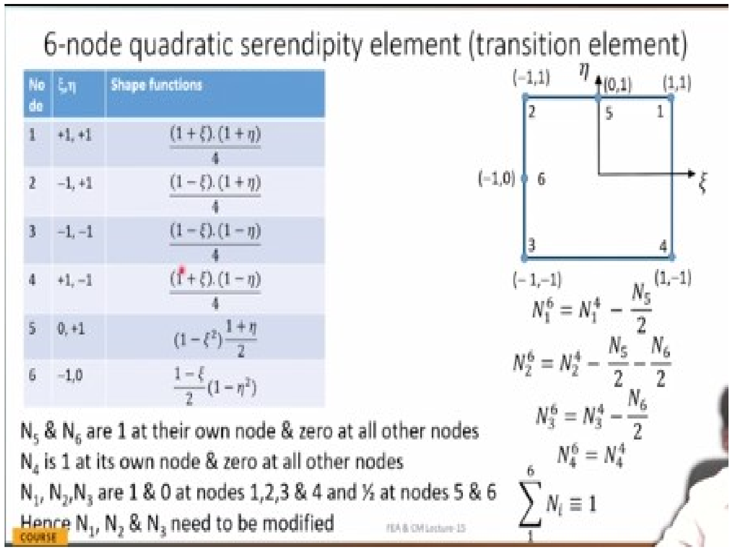
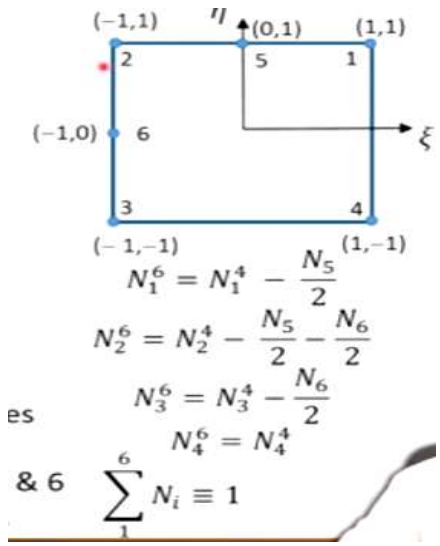


And now let us introduce the sixth node, okay? Let us start with the 4-node element. These are the basic shape functions for a 4-node quadrilateral and N 5 is 1 minus psi square times 1 plus eta by 2. And N 6 is 1 minus psi by 2 times 1 minus eta square because this is at psi of -1. And on the right hand side, the other one is at psi of +1 okay? So in eta direction, there are three nodes.

Node	ξ,η	Shape functions
1	+1, +1	$\frac{(1+\xi).(1+\eta)}{4}$
2	−1, +1	$\frac{(1-\xi).(1+\eta)}{4}$
3	−1, −1	$\frac{(1-\xi).(1-\eta)}{4}$
4	+1, −1	$\frac{(1+\xi).(1-\eta)}{4}$
5	0, +1	$(1-\xi^2)\frac{1+\eta}{2}$
6	−1,0	$\frac{1-\xi}{2}(1-\eta^2)$

So we can write like this. And if we evaluate N 1 and N 2, they are half here, and they are also half here. Then N 3 is also if you evaluate N 3 for the 4-node element here, you will get half. And N 3 along this line is 0. And N 4 is 0 if you evaluate along this line or along this line. So the N 4 does not require any correction. And N 1 requires a correction because of node 5. Whereas N 2 requires a correction for node 5 and node 6.



And N 3 requires a correction for node 6. So we can write N 1 for the 6-node element as N 1 for the 4-node element minus N 5 by 2. And N 2 for the 6-node element is N 2 4. That is N 2 for the 4-node element minus N 5 by 2 minus N 6 by 2. And N 3 for the 6-node element is N 3 for the 4-node element minus N 6 by 2. And N 4 for the 6-node element is exactly the same as N 4 for the 4-node element.

And the sum total of all the shape functions is equal to 1 because that we require for unique for the uniqueness in the mapping okay and that we will see later why we need the sum total of all the shape functions as 1, okay?

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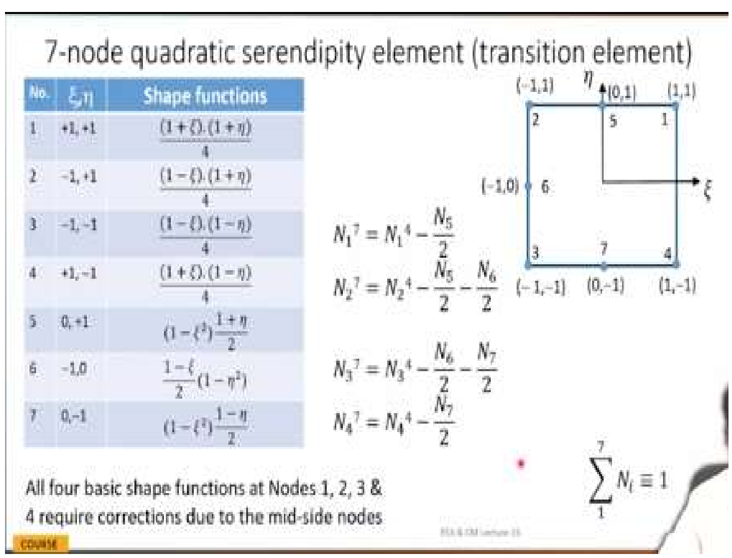
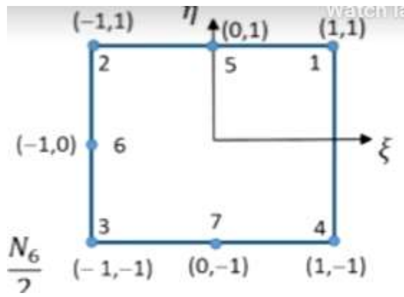


And we can introduce the seventh node. And now, our N 1 is corrected because of N 5. And the N 2 is corrected because of node 5 and 6. And N 3 is corrected because of nodes 6 and 7.



And N 4 is corrected only because of 7, okay? And that correction is, see the correction for N 1 is N 1 7 is N 1 4 minus N 5 by 2. That is N 1 for 7-node quadrilateral element is N 1 for the 4-node element minus N 5 by 2.

N 2 for the 7-node element is N 2 minus for the 4-node element minus N 5 by 2 minus N 6 by 2. Similarly, we can do for all the other nodes. And we can do this just by observation, we do not really need to derive them. And I forgot to ask you one question. See for the 5-node element we got these shape functions, but if you assume a polynomial and derive using the generalized coordinate method, you will see that you will not get the same shape functions.

Because earlier we got the same shape functions whether we used Lagrange method or the generalized coordinate method because we were applying the procedure for a purely Lagrange element, but these elements, these transition elements are not Lagrange elements. These are something else. So we have to use only the method that we are using, the serendipity method.

And basically all these elements with variable number of nodes these are called as transition elements.

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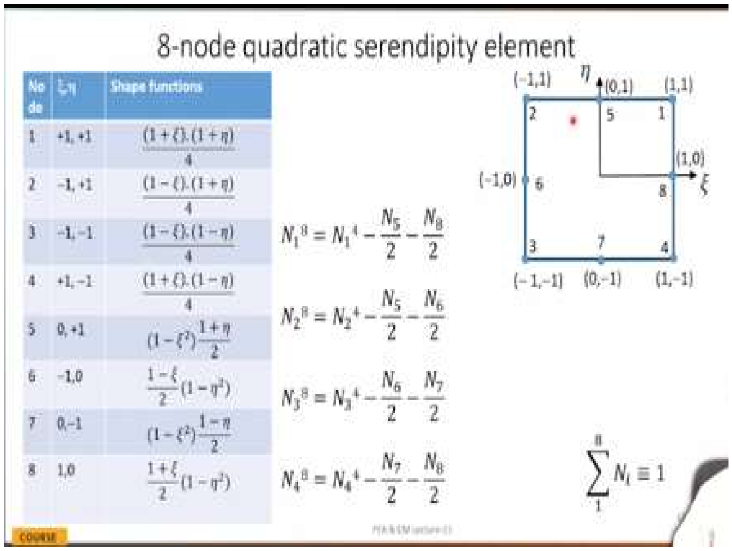
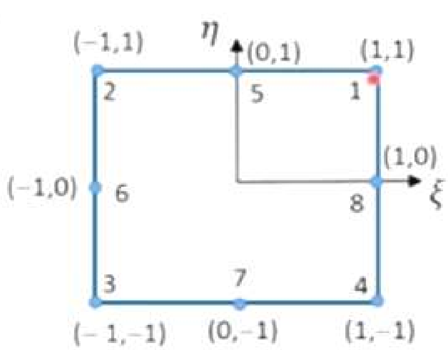


7 and then 8-node quadrilateral is actually it is **is** a very powerful element wherein all the 4 nodes N 1, N 2, N 3, N 4 they get corrected. Say N 4 for the 8-node element is N 4 for the 4-node element minus N 7 by 2 minus N 8 by 2 and so on, okay? N 1 8 is N 1 for the 4-node element minus and N 5 by 2 minus N 8 by 2 and so on, okay. So these directly by observation we can write.



No de	ξ,η	Shape functions
1	+1, +1	$\frac{(1+\xi).(1+\eta)}{4}$
2	−1, +1	$\frac{(1-\xi).(1+\eta)}{4}$
3	−1, −1	$\frac{(1-\xi).(1-\eta)}{4}$
4	+1, −1	$\frac{(1+\xi).(1-\eta)}{4}$
5	0, +1	$(1-\xi^2)\frac{1+\eta}{2}$
6	−1,0	$\frac{1-\xi}{2}(1-\eta^2)$
7	0,−1	$(1-\xi^2)\frac{1-\eta}{2}$
8	1,0	$\frac{1+\xi}{2}(1-\eta^2)$

$$N_1{}^8 = N_1{}^4 - \frac{N_5}{2} - \frac{N_8}{2}$$

$$N_2{}^8 = N_2{}^4 - \frac{N_5}{2} - \frac{N_6}{2}$$

$$N_3{}^8 = N_3{}^4 - \frac{N_6}{2} - \frac{N_7}{2}$$

$$N_4{}^8 = N_4{}^4 - \frac{N_7}{2} - \frac{N_8}{2}$$

And the main advantage here is we can easily program them. And these elements are formulated in terms of – 1 to + 1. So if it was x 1 to x 2, we do not know how much correction we need to apply, because we cannot do this type of simple calculations.

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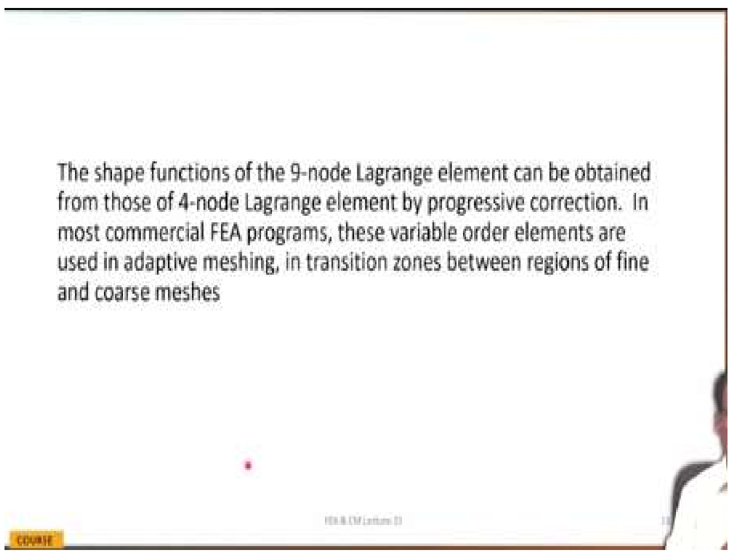

So these are the two elements, the 4-node quadrilateral and 9-node quadrilateral, they are pure Lagrange elements. And the others with 5 nodes, 7 nodes, 6 nodes, 8 nodes, these are all called as transition elements. And they are used in adaptive meshing. Sometimes or some programs they give you an option for adaptive meshing so that you get more closer mesh, where you have say some very high strain variation or some point is of interest for you.

And we can use this variable number of node elements for transition from one type of element to the other type of element.

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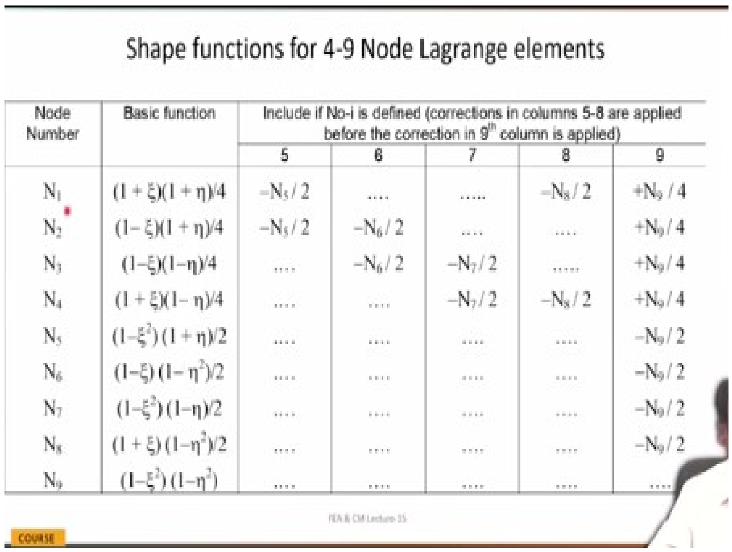

Shape functions for 4-9 Node Lagrange elements

Node Number	Basic function	Include if No-i is defined (corrections in columns 5-8 are applied before the correction in 9th column is applied)				
		5	6	7	8	9
N_1	$(1+\xi)(1+\eta)/4$	$-N_5/2$	....	.....	$-N_8/2$	$+N_9/4$
N_2	$(1-\xi)(1+\eta)/4$	$-N_5/2$	$-N_6/2$	....	....	$+N_9/4$
N_3	$(1-\xi)(1-\eta)/4$	....	$-N_6/2$	$-N_7/2$	.....	$+N_9/4$
N_4	$(1+\xi)(1-\eta)/4$	....	....	$-N_7/2$	$-N_8/2$	$+N_9/4$
N_5	$(1-\xi^2)(1+\eta)/2$	....	....	....	....	$-N_9/2$
N_6	$(1-\xi)(1-\eta^2)/2$	....	....	....	....	$-N_9/2$
N_7	$(1-\xi^2)(1-\eta)/2$	....	....	....	....	$-N_9/2$
N_8	$(1+\xi)(1-\eta^2)/2$	....	....	....	....	$-N_9/2$
N_9	$(1-\xi^2)(1-\eta^2)$	....	....	....	....	....

And so this table summarizes your different shape functions. Say N 1, N 2, N 3, N 4 these are the basic elements and N 5, N 6, N 7, N 8 and these columns, say if you introduce N 5, you correct N 1 and then N 2, okay? And if you introduce node 6, you correct N 2 and N 3. And with node 7 we correct N 3 and N 4 and so on. Like with, if you introduce eight node, we correct 1 and 4, okay?

And then with ninth node, we have correction for all the other 8 nodes. These are the corrections that we need to apply. And this table tells you what to do. And it is very easy to program or to change the number of nodes from 4 to 9. And this is what is done in most finite element programs. I will show you one program like later on how this is done to change the number of nodes from 4 to 9 and adapt any number of nodes that we have okay?

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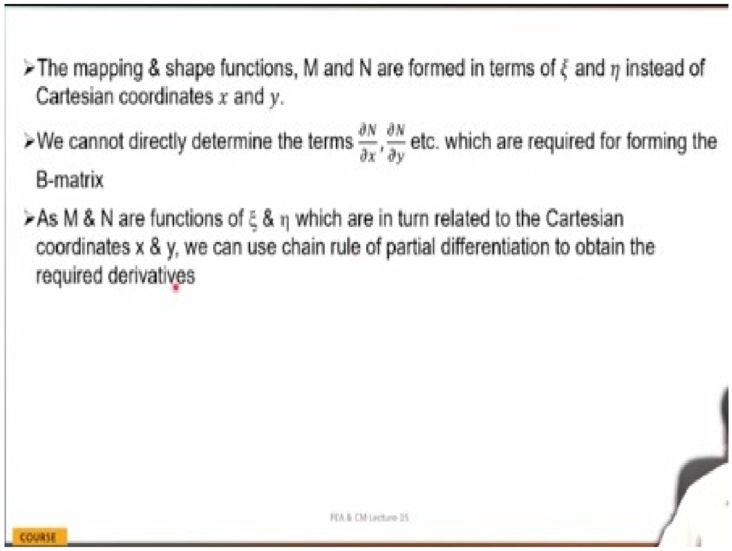

So now we have developed our mapping and shape functions in terms of psi and eta instead of the Cartesian coordinates x and y. And so we cannot directly determine our Cartesian derivatives of N, doh N by doh x and doh N by doh y. And unless you derive them, we will not be able to form the B-matrix, okay?

The B-matrix is required for forming our stiffness matrix or for calculating the equivalent loads due to your initial stresses and so on, okay? And as M and N both the mapping and shape functions, they are written in terms of psi and eta, which are in turn related to Cartesian coordinates x and y right? There is a relation certain relation between psi eta and x y okay?

So we can use the chain rule of partial differentiation to obtain the required derivatives, okay?

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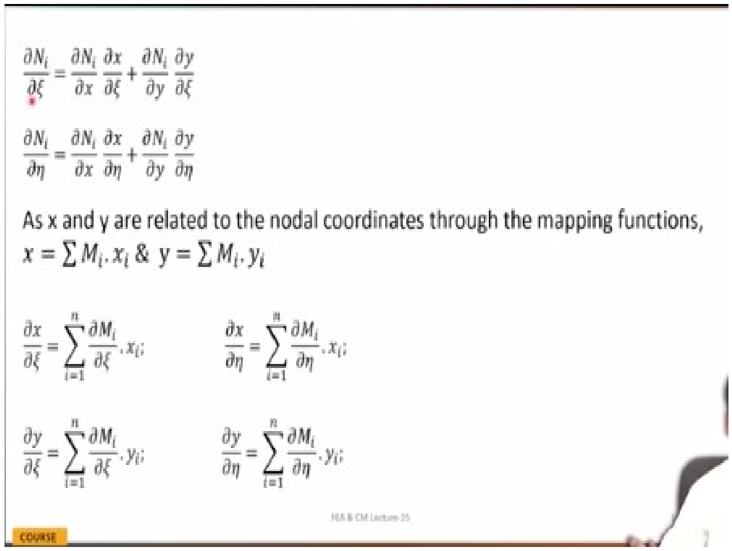

And what we do is, we can write our doh N by doh psi because our psi is related to both x and y. So I can write doh N by doh psi as doh N by doh x times doh x by doh psi plus doh N by doh y times doh y by doh psi. Because this is a chain rule of differentiation, because psi is dependent on both x and y. Similarly, the doh N by doh eta I can write as doh N by doh x times doh x by doh eta plus doh N by doh Y, doh Y by doh eta.

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta}$$

And here if you look at these two equations doh N by doh psi we can directly evaluate because our shape functions are formulated in terms of psi and eta. And our doh x by doh psi and doh y by doh psi also we can evaluate because our mapping functions are formulated in terms of psi and eta and our x is sigma of M i x i, okay. So our x is sigma of M i x i and y is sigma M i y i.

As x and y are related to the nodal coordinates through the mapping functions,
$x = \sum M_i . x_i$ & $y = \sum M_i . y_i$

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial M_i}{\partial \xi} . x_i; \qquad \frac{\partial x}{\partial \eta} = \sum_{i=1}^{n} \frac{\partial M_i}{\partial \eta} . x_i;$$

$$\frac{\partial y}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial M_i}{\partial \xi} . y_i; \qquad \frac{\partial y}{\partial \eta} = \sum_{i=1}^{n} \frac{\partial M_i}{\partial \eta} . y_i;$$

So we can write doh x by doh psi as doh M by doh psi x i and so on okay? So here we know the left hand side and then we know the doh x by doh psi, doh x by doh eta, doh y by doh psi, doh y by doh eta and so on okay?

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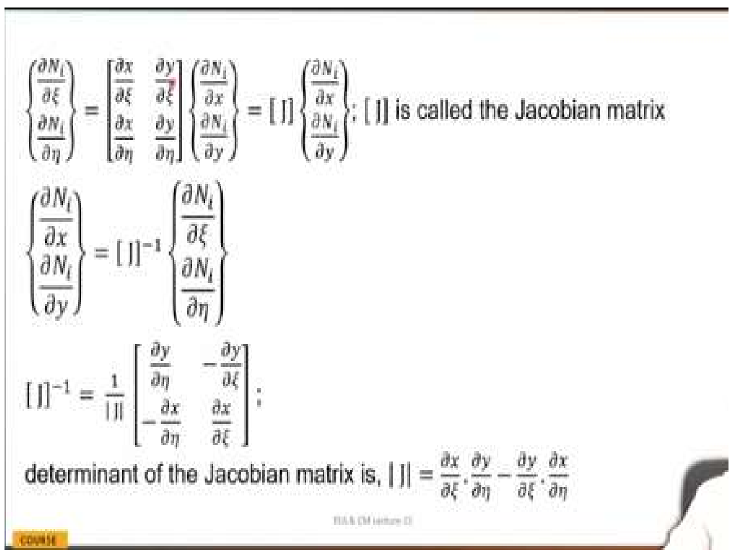

And so we can actually write this in matrix form like this. doh N by doh psi, doh N by doh eta and then doh x by doh psi, doh y by doh psi, doh x by doh eta, doh y by doh eta multiplied by doh N by doh x, doh N by doh y.

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}; \text{ [J] is called the Jacobian matrix}$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

$$[J]^{-1} = \frac{1}{|J|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix};$$

determinant of the Jacobian matrix is, $|J| = \frac{\partial x}{\partial \xi} . \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} . \frac{\partial x}{\partial \eta}$

And here we know the left hand side because our shape functions are written in terms of psi and eta. So we can determine this. Then this bracket is a very important quantity that we evaluate. This is called as Jacobian matrix that is consisting of the derivatives of x and y with respect to psi and eta. So by inverting this matrix, we can get doh N by doh x and doh N by doh y, right?

And our inverse of the Jacobian matrix is 1 by determinant of the Jacobian times doh y by doh eta minus doh y by doh psi minus doh x by doh eta doh x by doh psi okay. And the determinant of the Jacobian matrix is doh x by doh psi times doh y by doh eta minus doh x by doh eta times doh y by doh psi, okay? So this is also very important quantity, the Jacobian should be positive definite for unique mapping.

Because previously we said some things like our all the internal angles should be less than 180 degrees and then they should not be and then the intermediate point should be within the middle third and so on. And the more check is done by this Jacobean matrix that we derive, okay?

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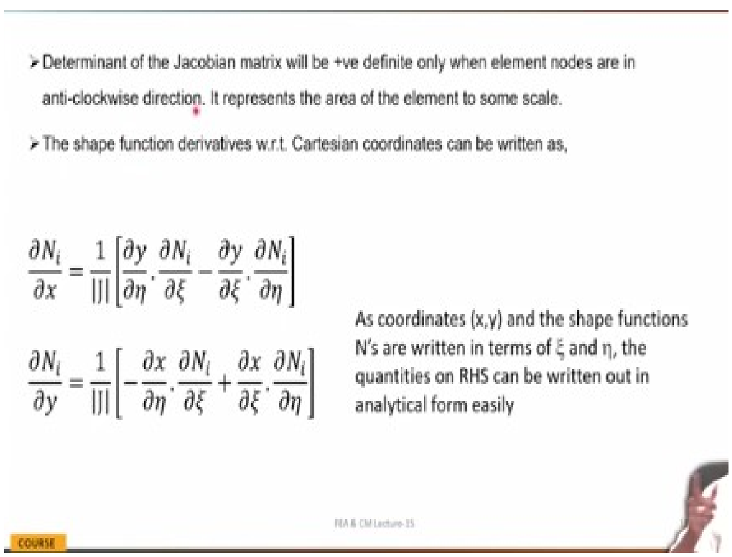

And the determinant of the Jacobian matrix will be positive definite only when element nodes are in anti-clockwise direction. We have to number all the nodes in the anti-clockwise direction and it has to be positive definite, because it represents the area of the element to some scale, okay? And then if it is positive definite, that means that we have unique mapping.

And the shape function derivatives with respect to Cartesian coordinates now, can be written like this, doh N by doh x, doh N by doh y in terms of all the other parameters, okay? And all the quantities on the right hand side, we can actually write out in analytical form when we are programming. It is very simple to program, okay?

$$\frac{\partial N_i}{\partial x} = \frac{1}{|J|}\left[\frac{\partial y}{\partial \eta}.\frac{\partial N_i}{\partial \xi} - \frac{\partial y}{\partial \xi}.\frac{\partial N_i}{\partial \eta}\right]$$

$$\frac{\partial N_i}{\partial y} = \frac{1}{|J|}\left[-\frac{\partial x}{\partial \eta}.\frac{\partial N_i}{\partial \xi} + \frac{\partial x}{\partial \xi}.\frac{\partial N_i}{\partial \eta}\right]$$

And now we got a relation to get your shape function derivatives with respect to Cartesian coordinates x and y. And now, we are dealing with integration in Cartesian space integral of x and y. But now, we have the, with mapping we are working in terms of -1 to +1.

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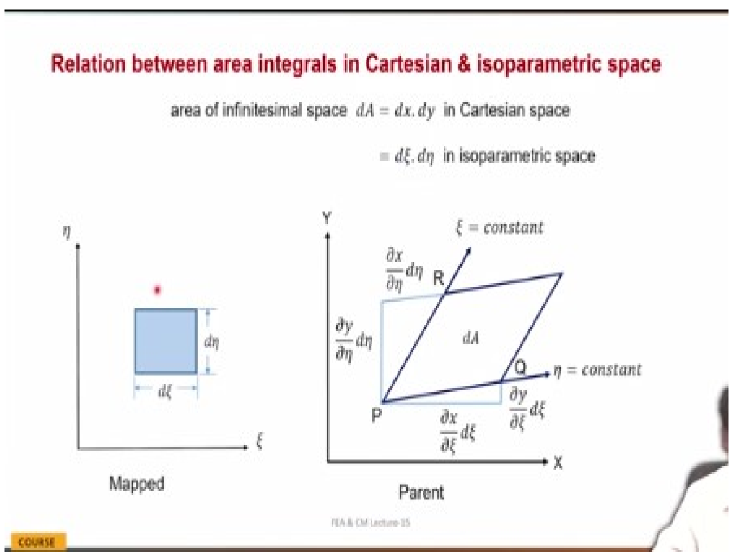
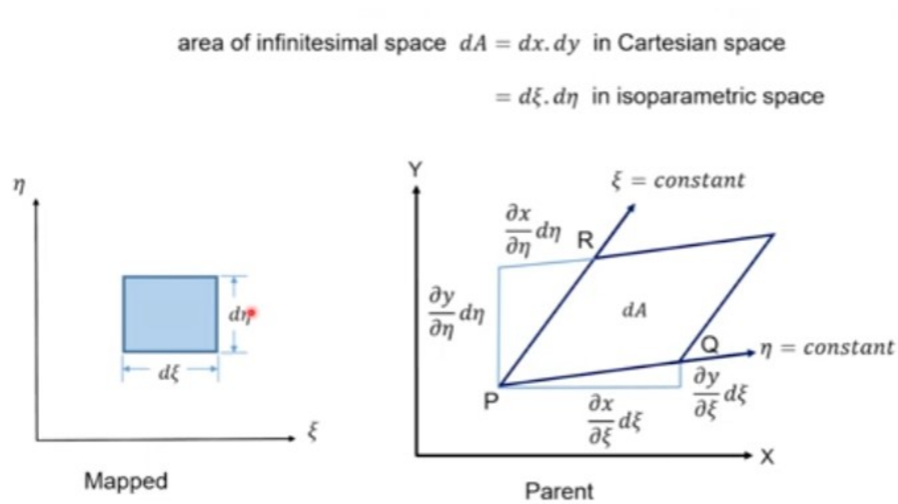


So how do we do that? And how do we establish the relation between the Cartesian space and then the isoparametric space, isoparametric or natural space, okay? So let us take an infinitesimal element dA of dx dy in the Cartesian space and then in the isoparametric space d psi d eta. And now, we want to establish a relation or the scale factor between dx dy and d psi d eta, right?



And how do we do that? So for that, we can go back to our vector calculus and let us imagine these two lines P Q and P R as two vectors. And along this line, your eta is constant right, because we are moving along psi. And then along this line your psi is constant, okay?

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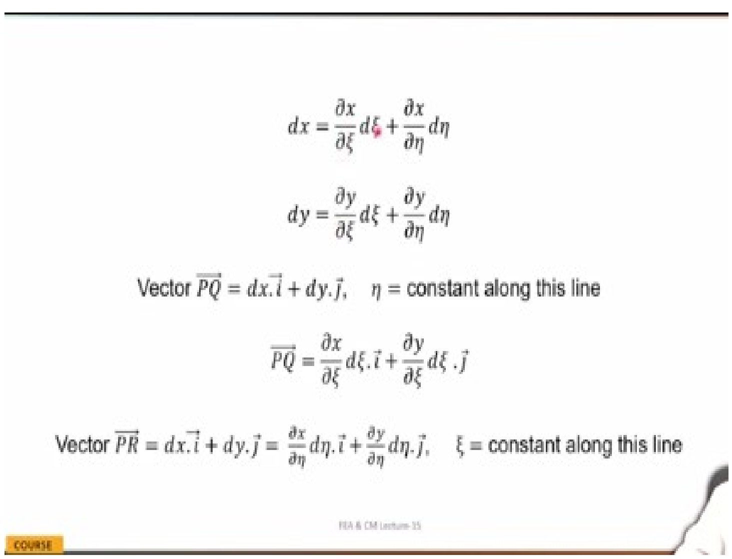

And in general, our vectors are written like a dx, we can write in terms of doh x by doh psi d psi plus doh x by doh eta d eta. Because our x is a function of both psi and eta, I can write it like this, doh x by doh psi d psi plus doh x by doh eta d eta. And dy also also can be written as doh y by doh psi d psi plus doh y by doh eta d eta because both x and y are related to psi and eta, okay?

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta$$

$$dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta$$

Vector $\overrightarrow{PQ} = dx.\vec{i} + dy.\vec{j}$, $\quad \eta$ = constant along this line

$$\overrightarrow{PQ} = \frac{\partial x}{\partial \xi} d\xi.\vec{i} + \frac{\partial y}{\partial \xi} d\xi\,.\vec{j}$$

Vector $\overrightarrow{PR} = dx.\vec{i} + dy.\vec{j} = \frac{\partial x}{\partial \eta} d\eta.\vec{i} + \frac{\partial y}{\partial \eta} d\eta.\vec{j}$, $\quad \xi$ = constant along this line

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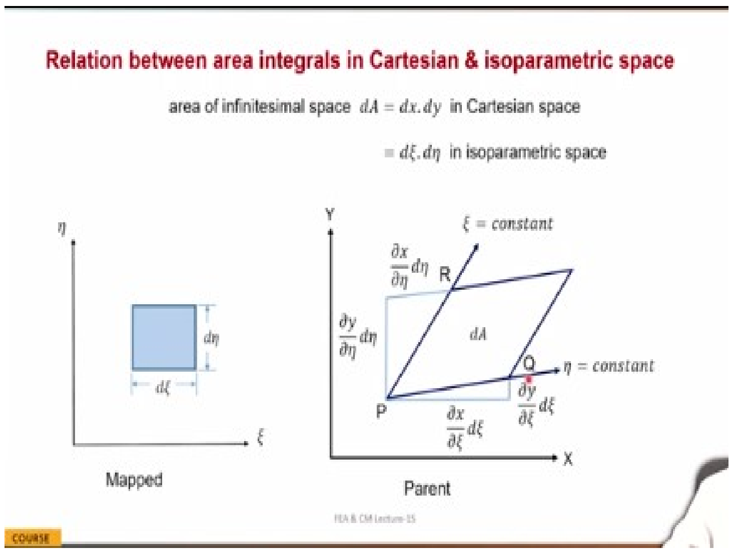

And so along this line, your eta is constant. That means d eta is 0. So we do not have the doh x by doh eta terms like and doh x by doh eta terms because d eta is 0. So we can write this length as doh x by doh psi times d psi. And this as doh y by doh psi d psi, okay? And the PQ is this is actually it is a vector doh x by doh psi d psi i plus doh y by doh psi d psi d psi in the j direction.

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$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta$$

$$dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta$$

Vector $\overrightarrow{PQ} = dx.\vec{i} + dy.\vec{j}$, η = constant along this line

$$\overrightarrow{PQ} = \frac{\partial x}{\partial \xi} d\xi.\vec{i} + \frac{\partial y}{\partial \xi} d\xi.\vec{j}$$

Vector $\overrightarrow{PR} = dx.\vec{i} + dy.\vec{j} = \frac{\partial x}{\partial \eta} d\eta.\vec{i} + \frac{\partial y}{\partial \eta} d\eta.\vec{j}$, ξ = constant along this line

FEA & CM Lecture-15

COURSE

So we can write the vector is in general written as dx i plus dy j. And along this PQ eta is constant. So I can write the PQ vector as doh x by doh psi times d psi i plus doh y by doh psi d psi j okay? And this is a vector and similarly along the vector PR your psi is constant. So d psi is 0. So we can write dx i plus dy j as doh x by doh eta d eta i plus doh y by doh eta d eta j okay?

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unit area in Cartesian space, dA = $\overrightarrow{PQ} \times \overrightarrow{PR}$

$$\frac{\partial x}{\partial \xi} d\xi \frac{\partial y}{\partial \eta} d\eta\, \vec{k} - \frac{\partial y}{\partial \xi} d\xi \frac{\partial x}{\partial \eta} d\eta\, \vec{k}$$

Note:

$\vec{i} \times \vec{j} = k \quad \vec{j} \times \vec{i} = -k$

$\vec{i} \times \vec{i} = 0 \quad \vec{j} \times \vec{j} = 0$

$$\therefore dA = dxdy = \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right) d\xi d\eta$$

$$\therefore dxdy = |J| d\xi d\eta$$

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} dxdy = \int_{-1}^{+1}\int_{-1}^{+1} |J| d\xi d\eta$$

FEA & CM Lecture-15

COURSE

And in the vector calculus the area is a cross product. So we have developed two vectors PQ and PR and if you take a cross product we will get the area, okay? So your, and then when we are doing the cross product i cross j is k, i cross i is 0, j cross j is 0 and j cross i is -k right? So if you do this cross product, we will get doh x by doh psi d psi doh y by data eta d eta k, positive k, minus this multiplied by k.

unit area in Cartesian space, dA = $\overrightarrow{PQ} \times \overrightarrow{PR}$

$$\frac{\partial x}{\partial \xi} d\xi \frac{\partial y}{\partial \eta} d\eta\, \vec{k} - \frac{\partial y}{\partial \xi} d\xi \frac{\partial x}{\partial \eta} d\eta\, \vec{k}$$

Note:

$$\vec{i} \times \vec{j} = k \qquad \vec{j} \times \vec{i} = -k$$

$$\vec{i} \times \vec{i} = 0 \qquad \vec{j} \times \vec{j} = 0$$

$$\therefore dA = dxdy = \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right) d\xi d\eta$$

$$\therefore dxdy = |J| d\xi d\eta$$

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} dxdy = \int_{-1}^{+1}\int_{-1}^{+1} |J| d\xi d\eta$$

So our, this is our dA that is dx dy and that is equal to, this is actually determinant of the Jacobian matrix multiplied by d psi d eta. So now, we can say dx dy is nothing but the determinant of the Jacobian matrix multiplied by d psi d eta. So we can write in general integral x 1 x 2 integral y 1 y 2 dx dy as integral of -1 to +1 integral -1 to +1 determinant of the Jacobian matrix j d psi d eta.

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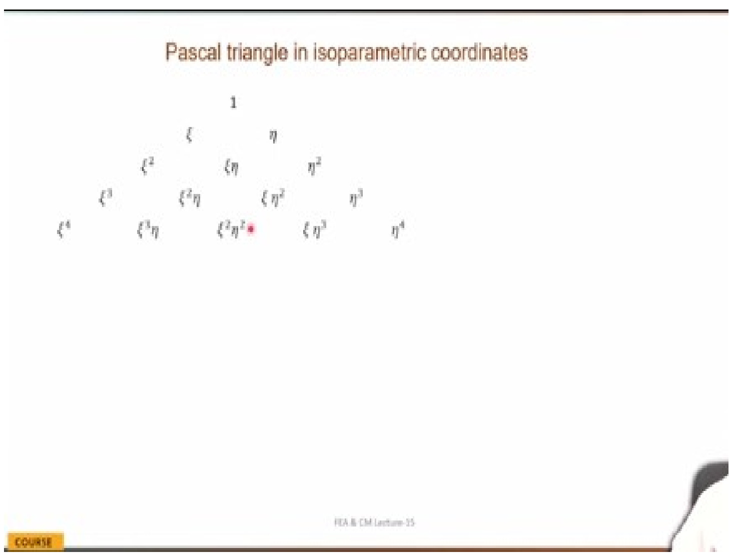

And so actually for the 9-node element this is what we have done. We took these 9 terms 1, psi eta, psi square, psi eta eta square, psi square eta psi eta square and psi square eta square. We have the 9 terms for the 9-node Lagrange element. So now, let us apply this and so if you want to do any calculations we can do this. And then how do we do this numerically.

So we can write in a summation form for doing the numerical calculation. So I can evaluate the psi and eta at some specific locations and then do the calculation. That is what we are going to do, okay?

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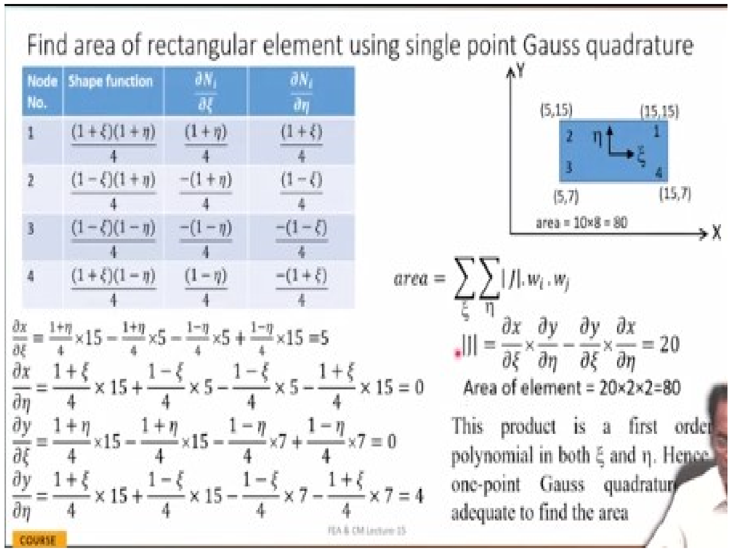
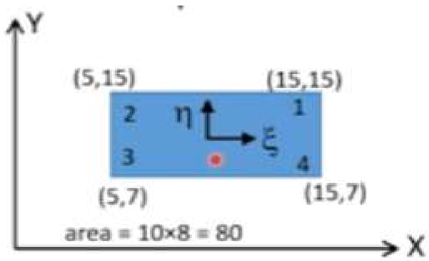


Node No.	Shape function	$\frac{\partial N_i}{\partial \xi}$	$\frac{\partial N_i}{\partial \eta}$
1	$\frac{(1+\xi)(1+\eta)}{4}$	$\frac{(1+\eta)}{4}$	$\frac{(1+\xi)}{4}$
2	$\frac{(1-\xi)(1+\eta)}{4}$	$\frac{-(1+\eta)}{4}$	$\frac{(1-\xi)}{4}$
3	$\frac{(1-\xi)(1-\eta)}{4}$	$\frac{-(1-\eta)}{4}$	$\frac{-(1-\xi)}{4}$
4	$\frac{(1+\xi)(1-\eta)}{4}$	$\frac{(1-\eta)}{4}$	$\frac{-(1+\xi)}{4}$

Let us say that we are given a 4-node quadrilateral or a rectangle with these coordinates and then we are interested in finding the area of this element. And the area of the element we can get as, okay? So this is the area of the element, integral dx dy integrated from x 1 to x 2 y 1 to y 2 is determinant of the Jacobian matrix j d psi d eta integrated from -1 to +1.



$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} dxdy = \int_{-1}^{+1}\int_{-1}^{+1} |J| \, d\xi \, d\eta$$

And this is in a numerical form we can write it in the summation form like this. We can write the area as the summation over psi summation over eta determinant of the Jacobian matrix w i w j where the weight factors in the psi direction and eta direction. So our shape functions at these four nodes are like this 1 plus psi, 1 plus eta by 4, 1 minus psi, 1 plus eta by 4 and so on.

Doh N by doh psi, doh N by doh eta. These are just simple derivatives with respect to psi and eta. And now we can calculate doh x by doh psi as doh N by doh psi times x i right? And we have four x values; x, x 1 is 15, x 2 is 5, x 3 is 5, and x 4 is 15, okay? So doh N 1 by doh psi times x 1 plus doh N 2 by doh psi x 2 plus doh N 3 by doh psi x 3 plus doh N 4 by doh psi times x 4

And these are the different derivatives and these are the coordinates. And if you do the calculation, it is a constant value, 5. It is not a function of psi or eta. Then similarly, doh x by doh eta if you calculate, it is coming out as 0. So why does it, why did we get this type of answer? doh x by doh psi is an absolute constant. And doh x by doh eta is also an absolute constant but it is 0, why?

$$\frac{\partial x}{\partial \xi} = \frac{1+\eta}{4}\times 15 - \frac{1+\eta}{4}\times 5 - \frac{1-\eta}{4}\times 5 + \frac{1-\eta}{4}\times 15 = 5$$
$$\frac{\partial x}{\partial \eta} = \frac{1+\xi}{4}\times 15 + \frac{1-\xi}{4}\times 5 - \frac{1-\xi}{4}\times 5 - \frac{1+\xi}{4}\times 15 = 0$$
$$\frac{\partial y}{\partial \xi} = \frac{1+\eta}{4}\times 15 - \frac{1+\eta}{4}\times 15 - \frac{1-\eta}{4}\times 7 + \frac{1-\eta}{4}\times 7 = 0$$
$$\frac{\partial y}{\partial \eta} = \frac{1+\xi}{4}\times 15 + \frac{1-\xi}{4}\times 15 - \frac{1-\xi}{4}\times 7 - \frac{1+\xi}{4}\times 7 = 4$$

So actually when you look at this, the shape of the element rectangle, your psi and eta are coinciding with x and y. So psi is coinciding with x coordinate and eta is coinciding with y coordinate. So as you are moving along X axis, your eta is remaining constant. Similarly, if you are moving along Y your psi is remain constant, okay? Or as you are moving along psi, your Y is constant.

And as you are moving along X your eta is constant. So our doh x by doh eta, the variation of x along eta direction is 0. That is what we can see from this figure itself. And then similarly, doh y by doh psi we get it as equal to 0. That is the variation of y along psi direction. That is constant because this element itself is a rectangular shaped element. And doh y by doh eta that is the variation of y with respect to eta that is coming out as 4.

So this doh x by doh psi is 5 and doh y by doh eta is 4. And this 5 is coming from the half-length. See the length along the x axis is 15 - 5, that is 10. And 10 divided by 2 is 5 okay? And the length along the y axis is 15 - 7, that is 8, divided by 2 is 4. So now, the area of this element can be written as double summation over psi and eta mod j multiplied by w i w j where w i w j are the scale factors.

And we can just simply use one point integration for doing this calculation because the polynomial is anyway constant. Like this, we do not see any polynomial terms in the doh x by doh psi or doh x by doh eta and so on; 5, 0, 0, 4 okay? So our determinant of the Jacobian matrix is 20. And the area of element is j multiplied by w 1. That is 2 times w 1 that is 2 that is 80.

$$area = \sum_{\xi}\sum_{\eta} |J|.w_i.w_j$$

$$|J| = \frac{\partial x}{\partial \xi}\times\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\times\frac{\partial x}{\partial \eta} = 20$$

5 = 0 Area of element = 20×2×2=80

So that is exactly equal to the area of this element, 20 times sorry 10 times 8, okay? And with just one point integration, we can get the area of this element because, in fact the polynomial order is 0. There is no variation, okay?

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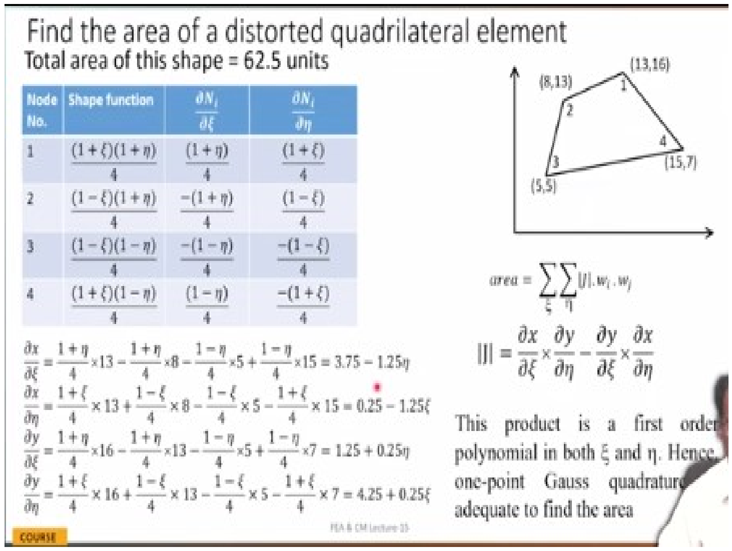
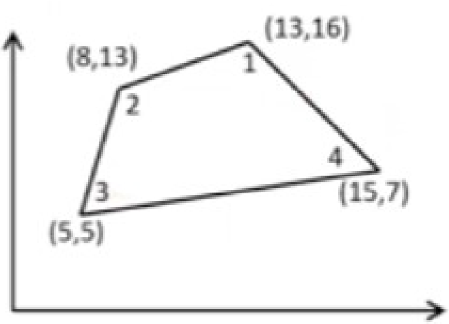


So now, let us take a distorted shape. Let us take the same 4 nodes 1, 2, 3, 4 and assign some properties so that we have a distorted shape like this.



Now we see that doh x by doh psi, doh x by doh eta they are not 0, they are varying as a function of eta and psi. Similarly, doh y by doh psi doh y by doh eta they are functions of psi because they are varying. And so and actually here, how many integration points do we require?

$$\frac{\partial x}{\partial \xi} = \frac{1+\eta}{4}\times 13 - \frac{1+\eta}{4}\times 8 - \frac{1-\eta}{4}\times 5 + \frac{1-\eta}{4}\times 15 = 3.75 - 1.25\eta$$
$$\frac{\partial x}{\partial \eta} = \frac{1+\xi}{4}\times 13 + \frac{1-\xi}{4}\times 8 - \frac{1-\xi}{4}\times 5 - \frac{1+\xi}{4}\times 15 = 0.25 - 1.25\xi$$
$$\frac{\partial y}{\partial \xi} = \frac{1+\eta}{4}\times 16 - \frac{1+\eta}{4}\times 13 - \frac{1-\eta}{4}\times 5 + \frac{1-\eta}{4}\times 7 = 1.25 + 0.25\eta$$
$$\frac{\partial y}{\partial \eta} = \frac{1+\xi}{4}\times 16 + \frac{1-\xi}{4}\times 13 - \frac{1-\xi}{4}\times 5 - \frac{1+\xi}{4}\times 7 = 4.25 + 0.25\xi$$

Say our doh x by doh psi is only eta, that is a first order polynomial. And doh y by doh eta is also a first order polynomial, because it is 4.25 plus 0.25 psi and so on, okay? So this product, the determinant of the Jacobian matrix is a first order polynomial. So that means that if we use one point integration, you should get the exact result.

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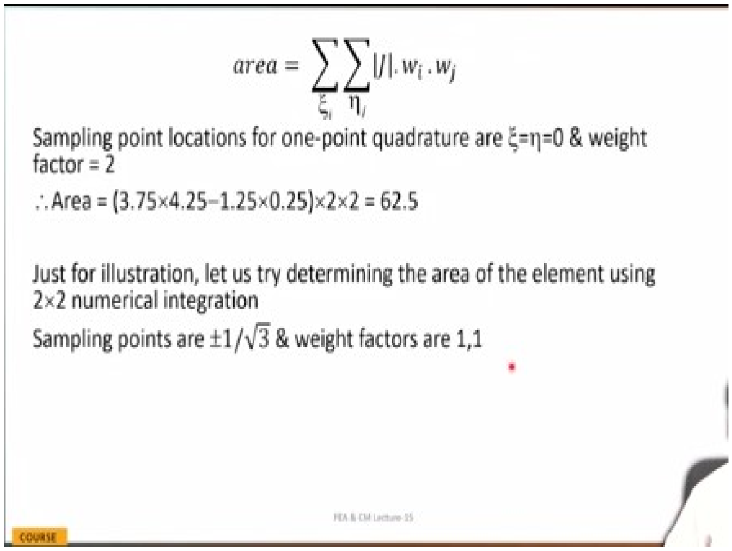
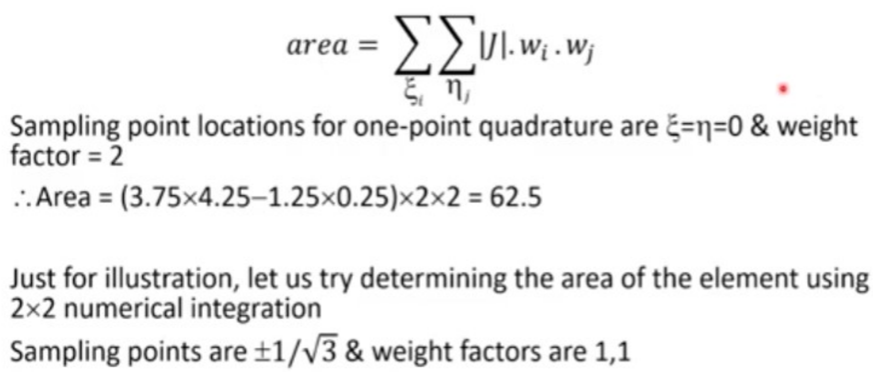


And so by one point integration, we get 62.5. And the area of this element, actually I have calculated separately by splitting this into two triangles. And then we have a formula for estimating the area of the triangle as the determinant of the coordinate matrix 1, x 1 1, y 1 1, x 2 y 2, 1, x 3 y 3. The determinant of that matrix will give you the area of the element, okay. And by doing that procedure, we got the area of this element as 62.5, okay?



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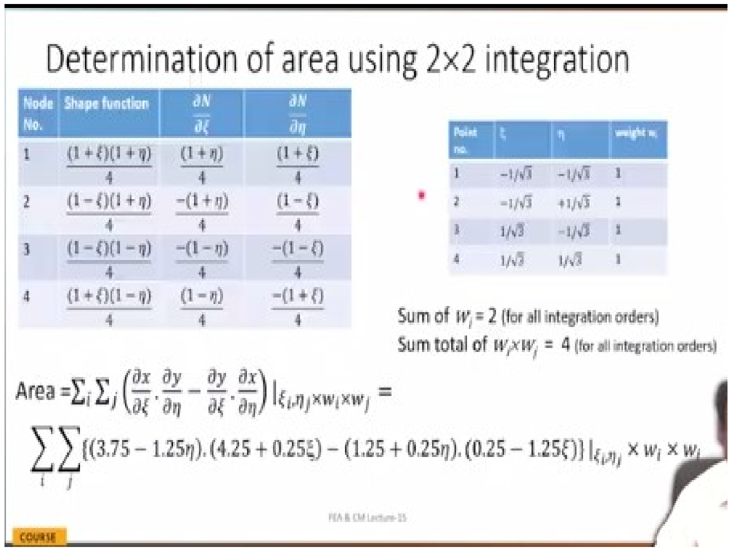

And let us see if we apply higher order of integration like 2 by 2 or 3 by 3, whether you get any different answer. But of course, it is a tedious process. With 2 by 2 integration, our sampling point locations are -1 by root 3, -1 by root 3, and the weight factor is 1. And -1 by root 3, +1 by root 3, and so on, okay? And so if you substitute the various values, you will actually get a very complicated equation.

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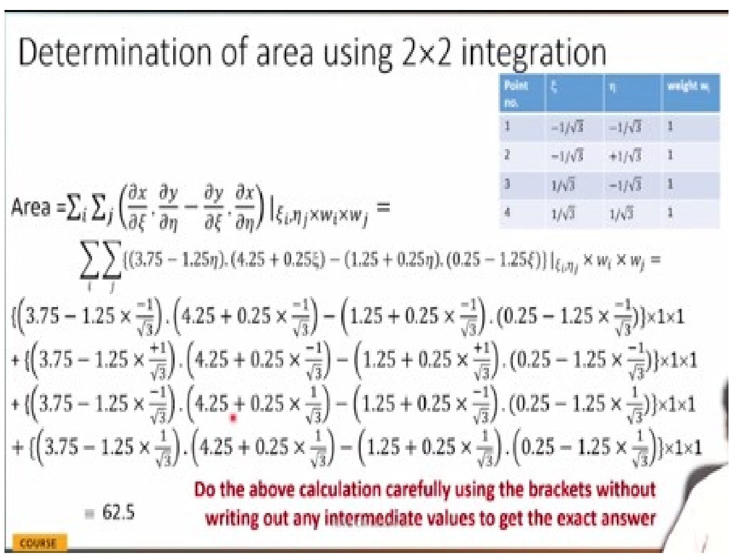

And then on top of that, we have to substitute psi and eta values at four points because it is a 2 by 2 integration. And we will get a long equation. And if we solve this long equation without rounding off any numbers or writing out intermediate values, we will see that this is 62.5, okay?

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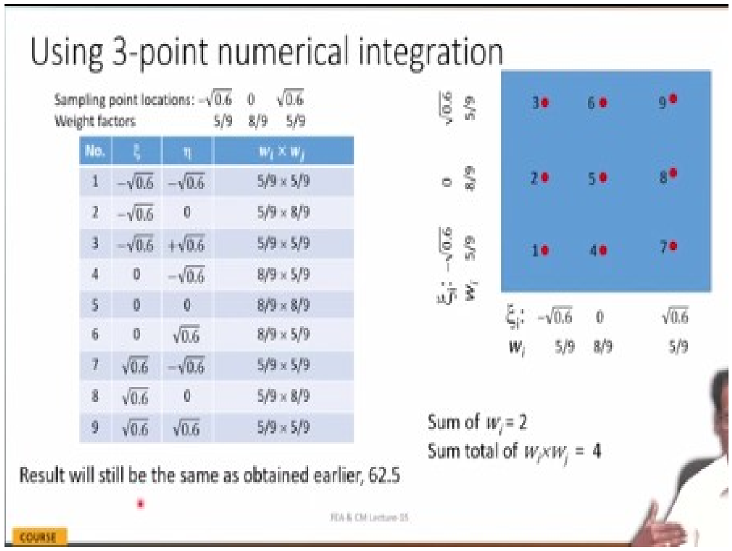

And we can apply even the 3-point integration, but then we will end up with the same value. And if you use a 4-point integration, we are evaluating all the quantities four times. And if you use a 3-point integration, we need to do 9 evaluations. See for something that could have done with just 1-point integration, if you use higher order of integration, your computational effort is more but the result is not going to change, okay?

So this is how we can do these computations. And in this class, we have got a relation between the Cartesian space and then the natural space. And then we found a way of determining the Cartesian derivatives or the shape functions doh N by doh x and doh N by doh y. And now we are ready for doing all the finite element calculations using our isoparametric elements.

Okay that we will do from next class. So if you have any questions, please send emails to this address profkrg@gmail.com. And before you come for the next class, please do try to understand all the previous lectures so that it becomes more easy, okay? So thank you very much.