FEM & Constitutive Modelling in Geomechanics
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Lecture - 16
Isoparametric Elements Part-I

So hello students, let us continue from our previous lecture on the numerical integration. We have seen that our Gauss quadrature method is developed in the space of -1 to +1. We have the sampling point locations and then the corresponding weight factors.

And now we have a new class of finite elements which are called as isoparametric elements that give us an advantage that whatever may be the type of element we can do the integration and we can control the accuracy of our integration, either we can get the exact value or slightly under predict our integral value.

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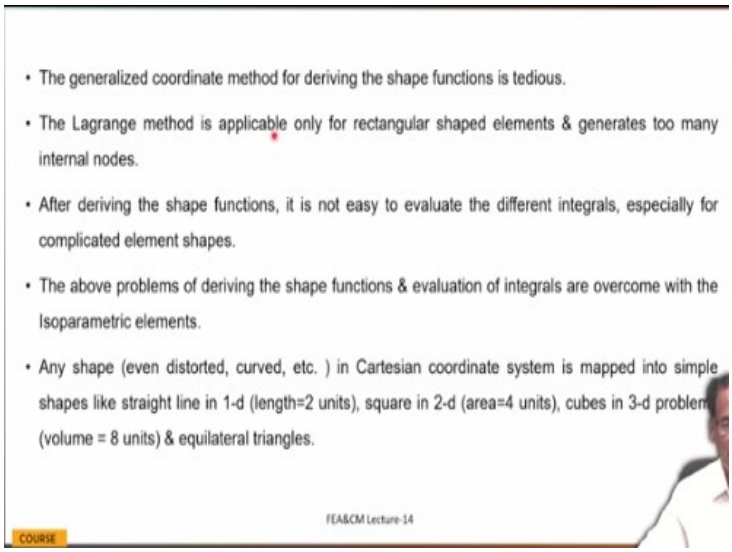

And so the need for this is felt because our generalized coordinate method for deriving the shape functions has resulted in some quantities with polynomials and we end up with lot of integrals, whether it is stiffness matrix or the load vector due to the self-weight or due to the initial stresses and so on. And so we need some method for evaluating them.

And when we have a irregular shaped elements, we cannot directly do the computations because we do not know exactly how to do the integrations. So we have one class of elements called as isoparametric elements that involve in converting any shape in the Cartesian coordinates to simple shape in the natural space. Say any shape is converted into, in the one dimensional problems, we convert them into a straight line.

Say you take a curve and we convert them to a straight line of length 2 units. And then if you have any quadrilateral shape, we convert that into a square in the two dimensional problems that has an area of 4 units. Then a three dimensional object we can convert into a cube in the 3-d problems and having volume of 8 units.

And then we can also convert them to or to triangles or some other shapes in the corresponding to the shape that we have in the Cartesian space.

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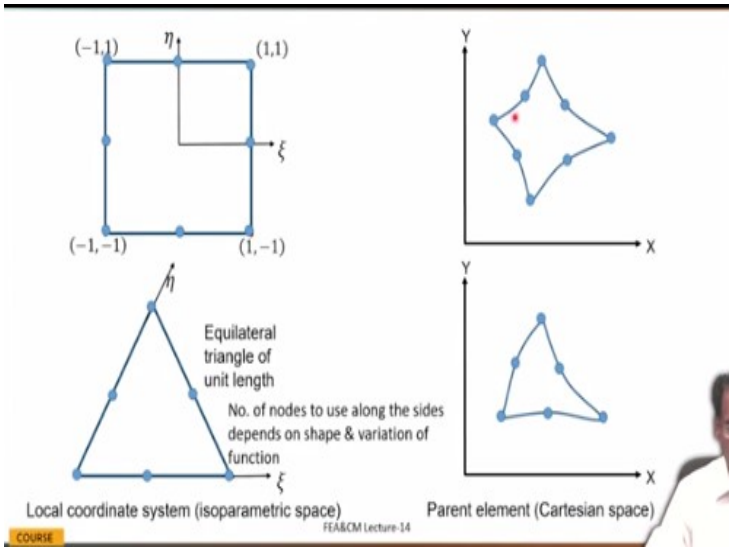

So here, so if you have a quadrilateral like this, an irregular quadrilateral, we can convert them into a regular shape of square having a length of 2 in each direction, okay. And say a triangular shape like this, we can convert to an equilateral triangle. And in the isoparametric space or in the one dimensional space, the length is 2. And in two dimensional area of 4, then in three dimensional space, volume of 8, okay?

And so actually programming becomes very simple because now all our elements, the coordinates are going to vary from -1 to +1 in the case of quadrilaterals. And in the

case of triangles, they vary from 0 to 1. The triangles we will deal with them later. But now let us only look at the quadrilateral elements.

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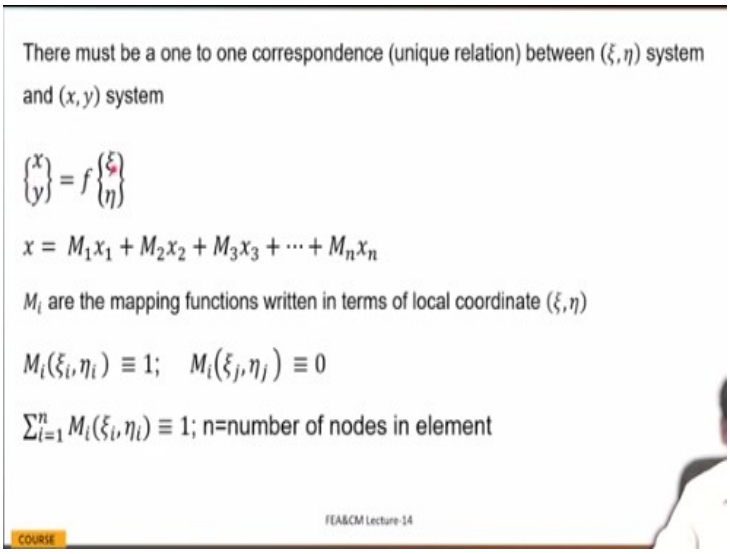

And we are going to develop some mapping functions to map between the natural space or the isoparametric space to the Cartesian space. So the natural space is in terms of psi and eta, whereas the Cartesian space is in terms of x and y, okay? This could be in a one dimensional or two dimensional or three dimensional. And we are mapping x and y to psi and eta through some functions.

And let us develop some mapping functions M 1, M 2, M 3 and so on and which are written in terms of psi and eta so that we can map between Cartesian space and then the isoparametric space and our mapping functions, they also have some properties similar to our shape functions. The mapping function M i evaluated at its own node psi i and eta i is exactly equal to 1.

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = f \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

$$x = M_1x_1 + M_2x_2 + M_3x_3 + \cdots + M_nx_n$$

M_i are the mapping functions written in terms of local coordinate (ξ,η)

$$M_i(\xi_i,\eta_i) \equiv 1; \quad M_i(\xi_j,\eta_j) \equiv 0$$

$$\sum_{i=1}^{n} M_i(\xi_i,\eta_i) \equiv 1; \text{ n=number of nodes in element}$$

And mapping function evaluated at some other node is 0. And the sum total of all the mapping functions is equal to 1. This is required for unique mapping. Unique

mapping means for every point in the natural space, there should be only one point in the Cartesian space. For every psi and eta there should be only one x and y, we cannot have two points.

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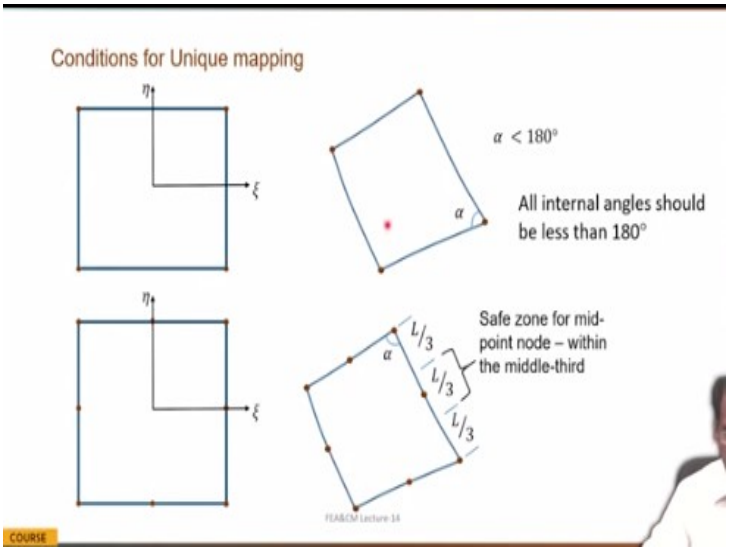

And what are the conditions for the unique mapping. So we need to have an element with all internal angles less than 180 degrees and then if we have any intermediate nodes along any line, they should not be too far away from the center like we have this middle third rule. Similarly, we have, we place these nodes in the within the middle third, okay?

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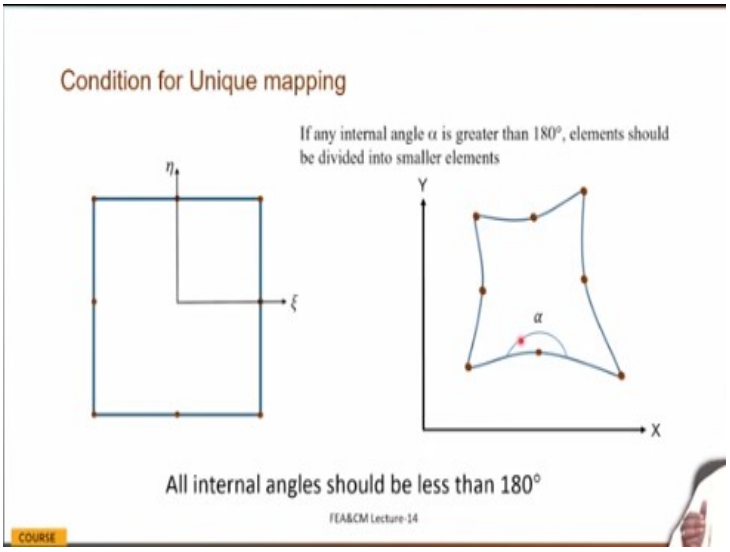

And then and any internal angle more than 180 degrees could lead to non-unique mapping. And so if you have a very complicated shape, then what we do is we divide

that shape into smaller number of elements, sorry larger number of elements so that at lower level at small element level we may have unique mapping. So all the internal angles should be less than 180 degrees.

So in this case, we can divide this into two elements. And then here also we may need a line. We can divide this into four elements, so that we get a unique mapping.

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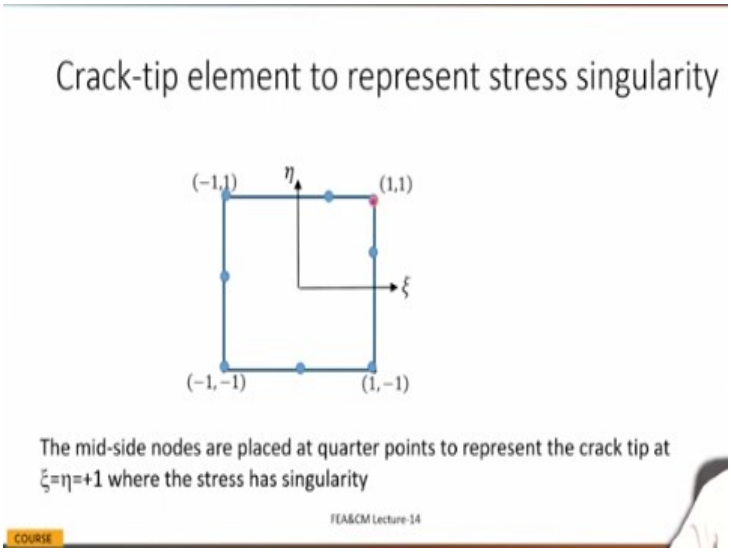

Or sometimes we can exploit the nature of these elements for special case like say one element called crack-tip element is developed by locating these intermediate points at one-fourth distance. These are called as quarter points. And by locating these two very close to this point, we can simulate the stress singularity at this point, okay?

And so this particular one is specially used in the fracture mechanics problems to improve the solution accuracy. And we can see that if we have the stress at this point is very large, similar to our stress singularity.

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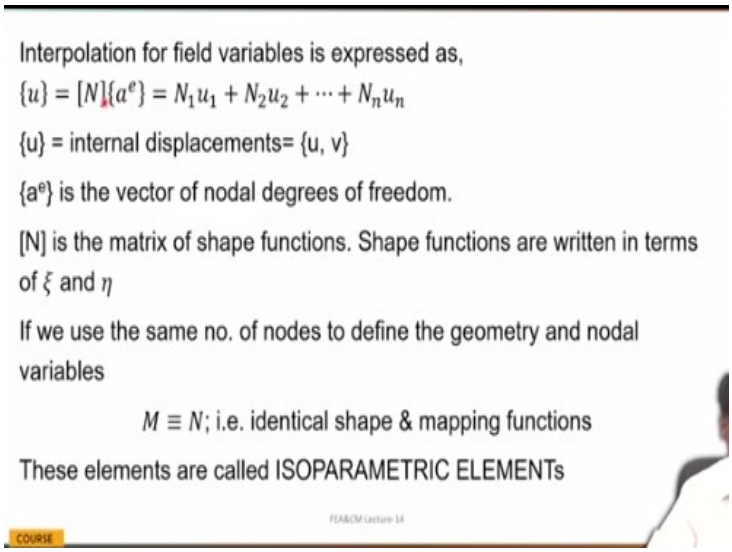

And now in these elements, we have mapping functions for Cartesian coordinates x and y. And for interpolation for the field variables, we have the shape functions, and our internal displacements u is N times a e, where N is the shape function matrix N 1 0, N 2 0, 0 N 1, 0 N 2 and so on, okay? And the N's are the shape functions, which are written in terms of psi and eta.

Interpolation for field variables is expressed as,
$\{u\} = [N]\{a^e\} = N_1u_1 + N_2u_2 + \cdots + N_nu_n$
{u} = internal displacements= {u, v}
{a^e} is the vector of nodal degrees of freedom.
[N] is the matrix of shape functions. Shape functions are written in terms of ξ and η

And if we use the same number of nodes for developing mapping and shape functions, we end up with the same functions for both mapping and then the shape functions. And M is exactly equal to N. And we call such elements as isoparametric elements. In most cases, we will be using only isoparametric type of elements where our mapping and shape functions are the same. But in some cases, we may have slight differences.

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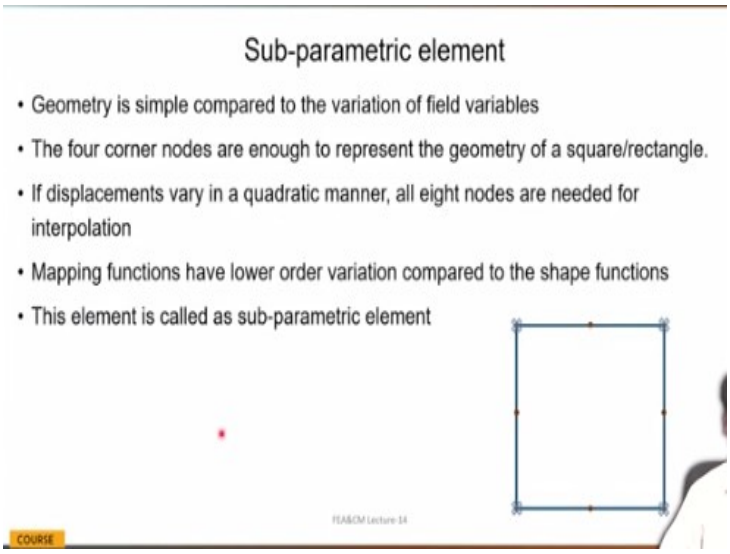

So we have a sub-parametric key element where our geometry is simple. Like let us say you have a square shape. We just need the four corner points for representing that shape. We do not need more than that. So we can use mapping functions of a lower order. But let us say how displacements vary in a quadratic manner. Then we need more number of points for describing the variation of the displacements.

So we may use in this element all the eight nodes for describing the variation of displacements. Whereas we use only four nodes for describing the geometry. And so the mapping functions have a lower order variation compared to the shape functions. And so this element is called as a sub-parametric element.

Actually we will come across the sub-parametric element later when we look at the infinite elements. But otherwise, we will be mostly dealing with only the isoparametric elements.

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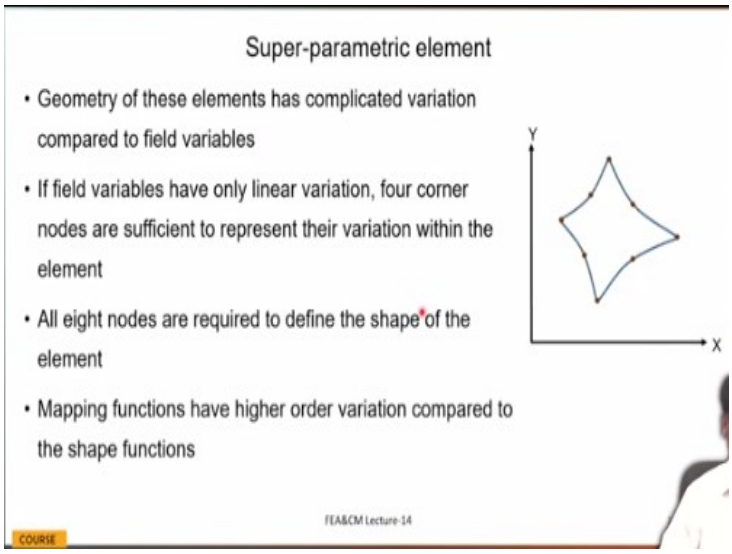

See on the other side, we have the super-parametric element. The shape is very complicated. So you need a higher order of mapping functions. And if your variables are having a lower order variation, we can just simply use these four coordinate points for our interpreting for interpolating the field variables. In this case, we call these as super-parametric elements.

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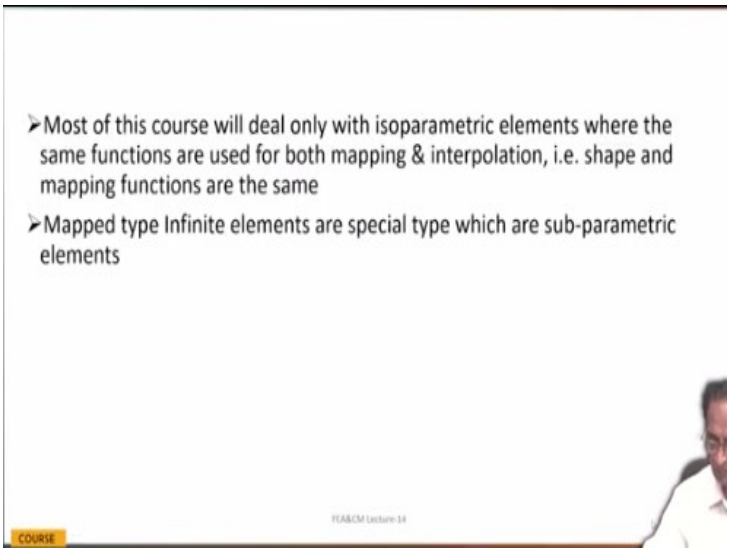

And in our course we will be, sorry I think this should be sub-parametric elements. Our most of the course will be dealing with only the isoparametric elements where we have the same mapping and the shape functions.

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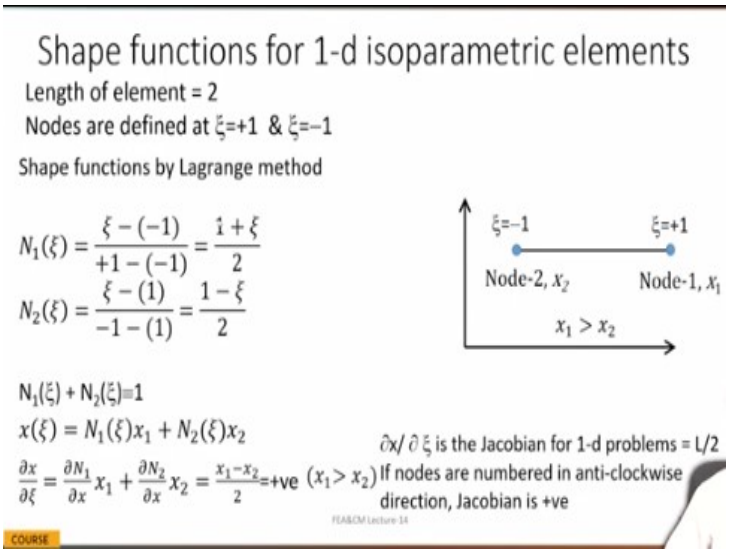
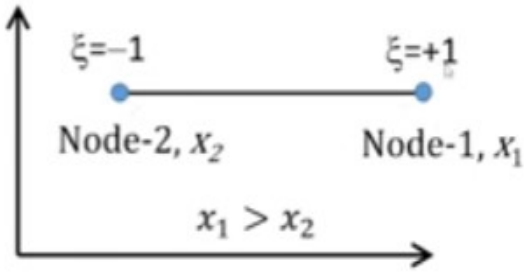


So let us try to derive the shape function, isoparametric shape functions. And let us start with only one dimensional element so that it is more easy for us to conceptualize. And let us consider two points, node 1 and node 2 at psi of +1 node 1 and at psi of -1 node 2 x 1 and x 2 and x 1 is greater than x 2. And we can get the shape functions by different methods by Lagrange method or the generalized coordinate method.



Let us try to get our shape functions by the Lagrange method. And N 1 of psi is the psi minus of -1. The psi minus the location at the other point and that is -1 divided by the coordinate value at node 1 is 1 minus of the coordinate value at other node that is -1. So that is 1 plus psi by 2. And similarly N 2 of psi is psi -1 divided by -1 the coordinate at this point minus of the coordinate at the other point.

So that is 1 minus psi by 2. And these are the two shape functions that we get. N 1 is 1 plus psi by 2 and N 2 is 1 minus psi by 2. And the sum total of the two shape functions N 1 + N 2 is 1. And in general, we can write x as N 1 x 1 + N 2 x 2 because our element is defined only in terms of -1 to +1. But in the Cartesian coordinates, it is from x 1 to x 2. So we can write x as N 1 x 1 + N 2 x 2.

Length of element = 2

Nodes are defined at ξ=+1 & ξ=−1

Shape functions by Lagrange method

$$N_1(\xi) = \frac{\xi - (-1)}{+1 - (-1)} = \frac{1+\xi}{2}$$

$$N_2(\xi) = \frac{\xi - (1)}{-1 - (1)} = \frac{1-\xi}{2}$$

$N_1(\xi) + N_2(\xi) \equiv 1$

$$x(\xi) = N_1(\xi)x_1 + N_2(\xi)x_2$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial x}x_1 + \frac{\partial N_2}{\partial x}x_2 = \frac{x_1 - x_2}{2} = +ve \quad (x_1 > x_2)$$

And our doh x by doh psi is doh N 1 by doh x x 1 plus doh N 2 by doh x x 2 and that is x 1 – x 2 by 2. And this is positive only when x 1 is greater than x 2. Actually, this is similar to our node numbering in the anti-clockwise direction. In fact this is anti-clockwise numbering. And the doh x by doh psi represents the scale factor. Because in fact this is x 1 minus x 2 by 2.

This itself is the scale factor because in the natural space the length is 2. In the Cartesian space, the length is x 1 – x 2, okay. So this doh x by doh psi is called as the Jacobean for 1-d problem and that is equal to L by 2. And if the nodes are numbered in the anti-clockwise direction, the Jacobean value is positive.

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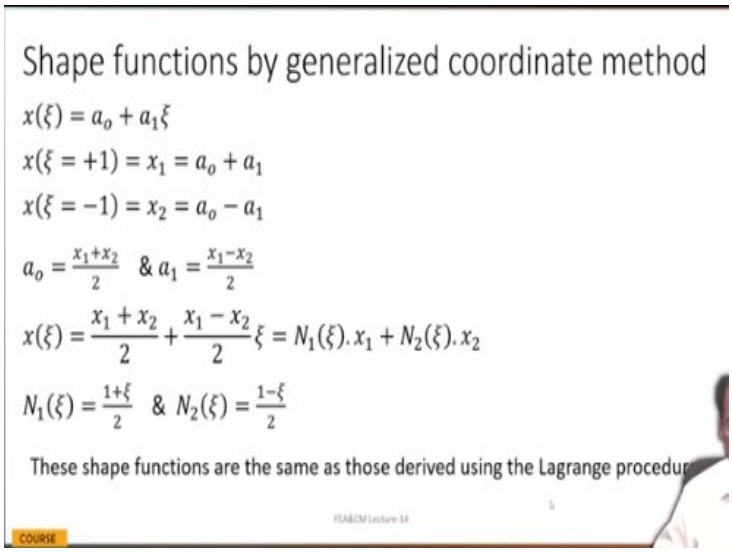

And let us derive the same shape function by the generalized coordinate method. We can assume a polynomial. And because we have two nodes, we need two terms in the polynomial. And we can only have first order polynomial, because we have only two points. So x of psi is a naught plus a 1 psi. So at node 1 psi is +1, x 1 is a naught plus a 1. At node 2 psi is -1 that is x 2 that is a naught minus a 1.

So by solving this, a naught is x 1 + x 2 by 2. And a 1 is x 1 – x 2 by 2. So once again, we see that our N 1 is 1 plus psi by 2. And N 2 is 1 minus psi by 2. These are the same as what we got from the Lagrange's method.

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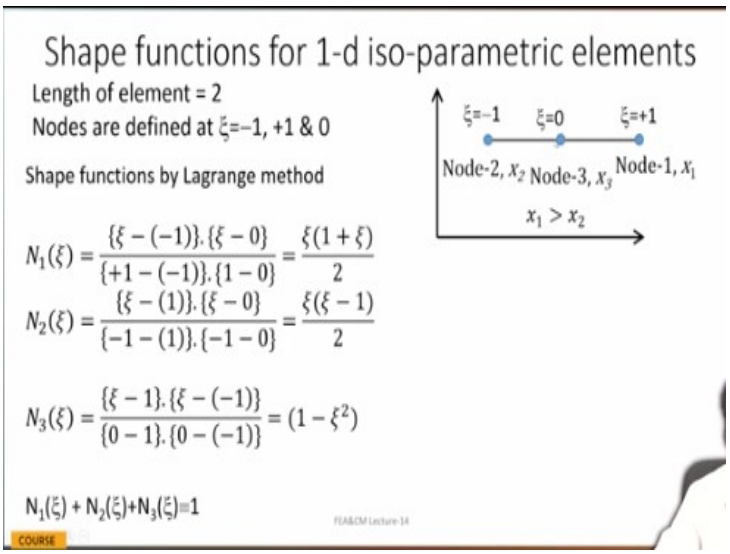

And now let us introduce a third node. N 1 is at psi of +1, N 2 is at psi of -1 and the third node is introduced at psi of 0. Actually, the concept here is we need minimum two nodes for defining any line. And if you have a curved line, then we require intermediate node. And that intermediate node is introduced at psi of zero. And once again, this length is 2 and the nodes are at psi of -1, +1 and 0.

Shape functions by generalized coordinate method

$$x(\xi) = a_o + a_1\xi$$

$$x(\xi = +1) = x_1 = a_o + a_1$$

$$x(\xi = -1) = x_2 = a_o - a_1$$

$$a_o = \frac{x_1+x_2}{2} \quad \& \; a_1 = \frac{x_1-x_2}{2}$$

$$x(\xi) = \frac{x_1 + x_2}{2} + \frac{x_1 - x_2}{2}\xi = N_1(\xi).x_1 + N_2(\xi).x_2$$

$$N_1(\xi) = \frac{1+\xi}{2} \quad \& \; N_2(\xi) = \frac{1-\xi}{2}$$

And we can get the shape functions by the Lagrange's method. See the shape function at 1 is psi minus of -1 times psi minus 0 divided by 1 minus of -1 and divided by 1 – 0, okay. So this comes to psi of psi times 1 plus psi by 2. Similarly, N 2 will be psi times psi minus 1 by 2. And N 3 is psi minus of -1 multiplied by psi -1 okay and divided by 0 of minus of -1 and 0 of **+1** minus of +1.

And this comes to 1 minus psi square. So these are the three shape functions that we have for the three node bar element. Psi times 1 plus psi by 2 and psi times psi minus 1 by 2, and 1 minus psi square. And if we add up all these three shape functions, N 1 + N 2 + N 3 that is exactly equal to 1.

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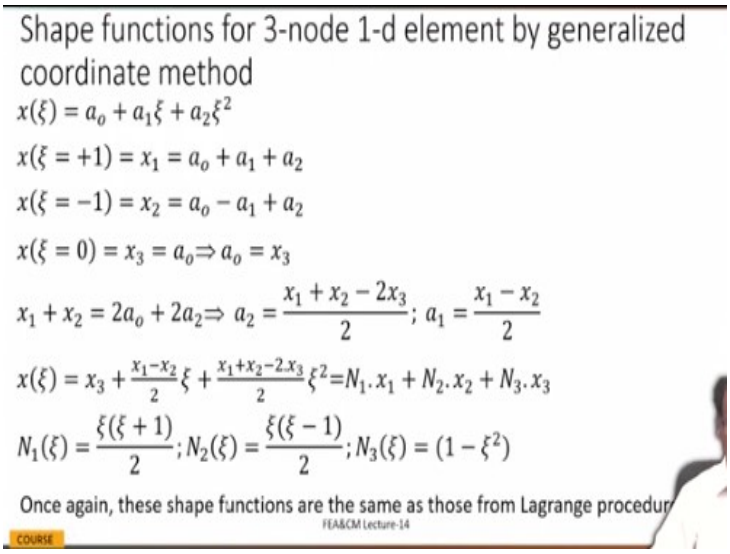

And let us derive the same shape functions by the generalized coordinate method x of psi is a naught plus a 1 psi plus a 2 psi square and at psi of +1 you have x 1. That is a naught + a 1 + a 2. Psi of -1 is x 2. a naught minus a 1 plus a 2. Psi of 0. It is just simply a naught that is x 3. So if you solve these three equations, once again we get that N 1 is psi times psi plus 1 by 2.

$$x(\xi) = a_o + a_1\xi + a_2\xi^2$$

$$x(\xi = +1) = x_1 = a_o + a_1 + a_2$$

$$x(\xi = -1) = x_2 = a_o - a_1 + a_2$$

$$x(\xi = 0) = x_3 = a_o \Rightarrow a_o = x_3$$

$$x_1 + x_2 = 2a_o + 2a_2 \Rightarrow a_2 = \frac{x_1 + x_2 - 2x_3}{2};\ a_1 = \frac{x_1 - x_2}{2}$$

$$x(\xi) = x_3 + \frac{x_1 - x_2}{2}\xi + \frac{x_1 + x_2 - 2x_3}{2}\xi^2 = N_1.x_1 + N_2.x_2 + N_3.x_3$$

$$N_1(\xi) = \frac{\xi(\xi + 1)}{2};\ N_2(\xi) = \frac{\xi(\xi - 1)}{2};\ N_3(\xi) = (1 - \xi^2)$$

N 2 is psi times psi minus 1 by 2. And N 3 is 1 minus psi square. See these are exactly the same shape function that we got from the Lagrange method. So we actually should notice that all these are Lagrange elements. And we have only one dimension. So we do not see the other dimension, but we can imagine this as a Lagrange element.

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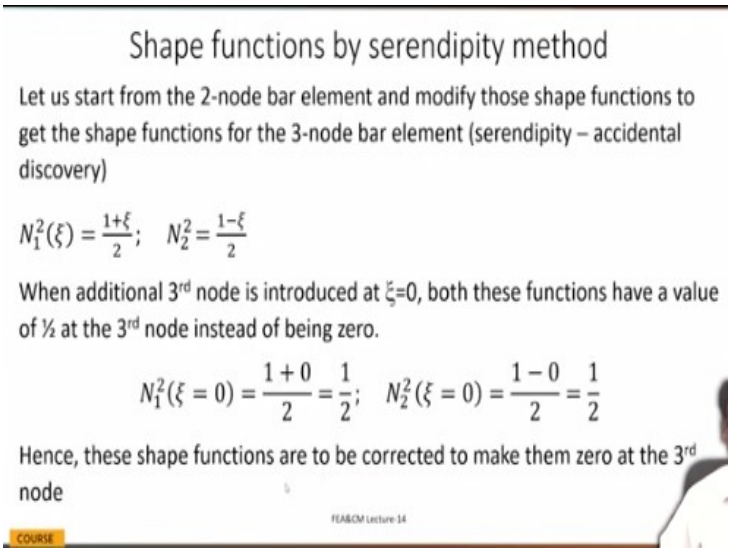

And now let us follow a slightly different method, which is called as a serendipity method. Actually, serendipity is a statistical term and it is actually accidental discovery like the literal meaning is serendipity discovery is accidental discovery. Like just by chance, we discovered this method or this procedure.

And this method of deriving the shape functions for higher order elements is, we start with the shape functions for a lower order element and then go on applying some corrections to get the shape functions for higher order elements, okay? Now let us

start with this 2-node bar element. Our N 1 was 1 plus psi by 2 and N 2 is 1 minus psi by 2. And here the subscript refers to node number 1 and 2.

$$N_1^2(\xi) = \frac{1+\xi}{2}; \quad N_2^2 = \frac{1-\xi}{2}$$

And the superscript refers to the number of nodes in the element. For 2-node bar element, 1 plus psi by 2 and 1 minus psi by 2. And now in this element, let us introduce a node at psi of 0. That is the node 3. And then we have a problem. Because these two shape functions, they do not become 0 at the node 3, they become half. So how do we make them 0 at node 3?

$$N_1^2(\xi = 0) = \frac{1+0}{2} = \frac{1}{2}; \quad N_2^2(\xi = 0) = \frac{1-0}{2} = \frac{1}{2}$$

And we notice that at node 3, N 3 is 1. So we can just simply write this, the shape function for a 3-node bar element as N 1. For 2-node bar element –N 3 by 2, okay?

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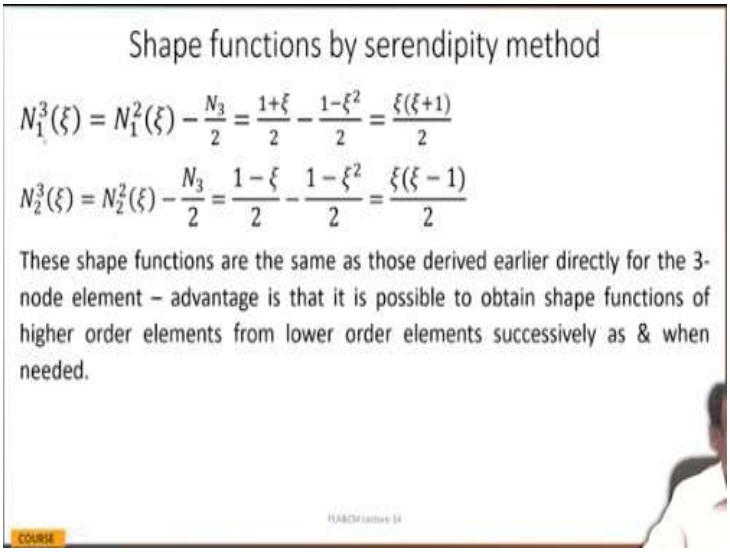

And so we can do the correction like this. N 1 for the 3-node bar element is N 1, for the 2-node bar element –N 3 by 2. That is 1 plus psi by 2 minus 1 minus psi square by 2. And that comes to psi times psi plus 1 by 2. Similarly, the N 2 for the 3-node bar element is N 2 for the 2-node bar element minus N 3 by 2, that is psi times psi minus 1 by 2.

$$N_1^3(\xi) = N_1^2(\xi) - \frac{N_3}{2} = \frac{1+\xi}{2} - \frac{1-\xi^2}{2} = \frac{\xi(\xi+1)}{2}$$

$$N_2^3(\xi) = N_2^2(\xi) - \frac{N_3}{2} = \frac{1-\xi}{2} - \frac{1-\xi^2}{2} = \frac{\xi(\xi-1)}{2}$$

So once again we see that these shape functions that we got by the serendipity method are exactly the same as what we got from the Lagrange method or from the generalized coordinate method. So actually the serendipity method gives us an advantage that we can use this method if the third node is present. If not, we will just work with the shape functions for the 2-node bar element.

So we can actually convert a lower order element into higher order element by doing this correction, okay. And as and when needed, we can correct them. And so in the program, we can have both 2-node bar elements and 3-node bar elements and then if necessary, 4 and 5 node bar elements. And we can successfully apply these corrections for getting the shape functions for higher order elements.

So we do not really need to invent a new method for getting the shape functions for the higher order elements.

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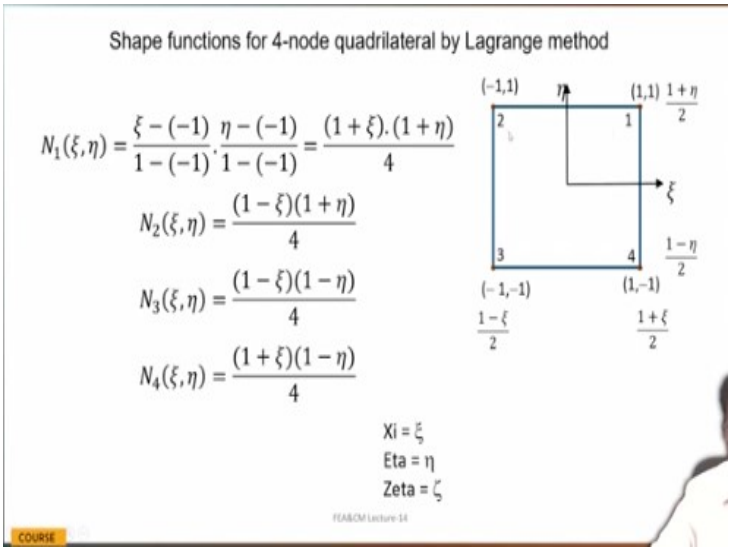
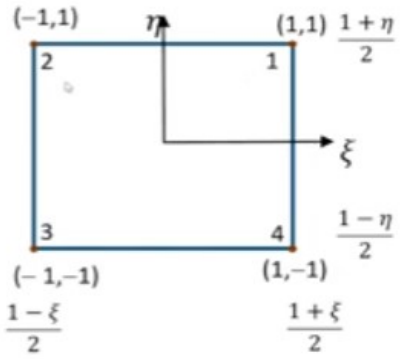


And the procedure that we have developed for one dimensional elements can also be extended to two dimensions.



Now let us consider a 4-node quadrilateral. And this is a typical Lagrange element, because we have the same number of nodes in each direction. And both along the horizontal direction and vertical direction. So let us define node 1 in quadrant 1 that is 1 and 1.

At node 2, in the second quadrant psi is -1 and eta is +1. The node 3 is in the negative quadrant -1 -1. Node 4 is in the fourth quadrant psi of +1 and eta of -1. So we can use the Lagrange procedure for getting the shape functions. So N 1, we can get it as a function in in terms of psi multiplied by a function in eta to get the N 1 of psi eta. So in the, so these are the shape functions that we already derived for the bar elements.

$$N_1(\xi,\eta) = \frac{\xi-(-1)}{1-(-1)}.\frac{\eta-(-1)}{1-(-1)} = \frac{(1+\xi).(1+\eta)}{4}$$

$$N_2(\xi,\eta) = \frac{(1-\xi)(1+\eta)}{4}$$

$$N_3(\xi,\eta) = \frac{(1-\xi)(1-\eta)}{4}$$

$$N_4(\xi,\eta) = \frac{(1+\xi)(1-\eta)}{4}$$

At node 1 that is psi of +1 our shape function is 1 plus psi by 2. At node 2 at psi of -1 we have 1 minus psi by 2. And by extending the same procedure for eta direction, we have the shape functions 1 plus eta by 2 at eta of +1. And 1 minus eta by 2 at eta of -1. So our N 1 of psi eta is just simply 1 plus psi times 1 plus eta by 4. And N 2 of psi eta is 1 minus psi times 1 plus eta by 4.

And N 3 is 1 minus psi times 1 minus eta by 4. And N 4 is 1 plus psi times 1 minus eta by 4, okay? So these are the shape functions that we can directly write based on our Lagrange's procedure.

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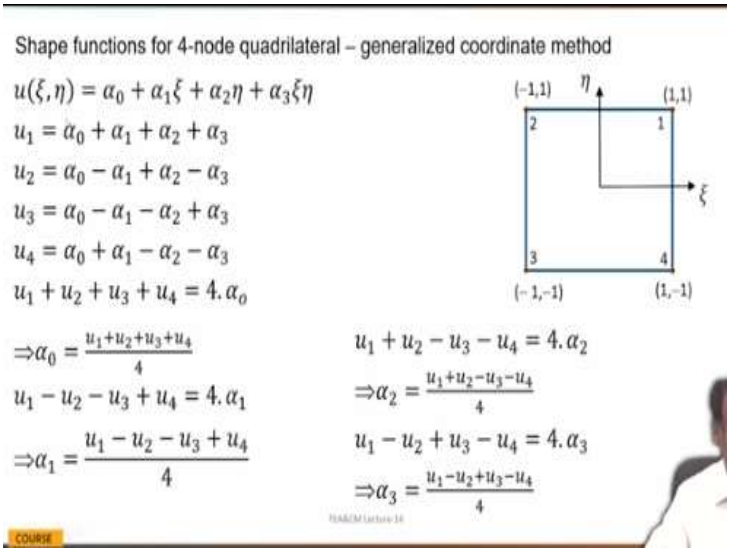
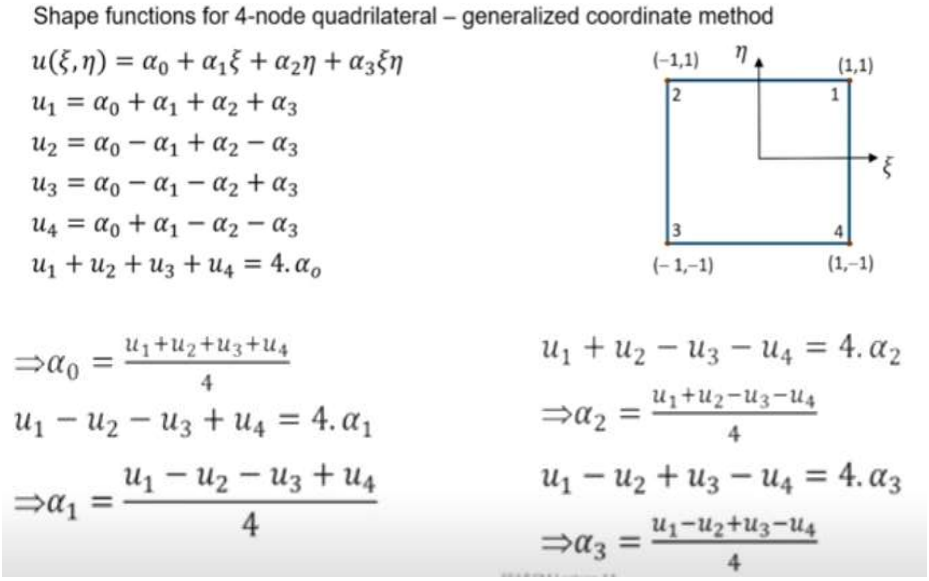


And we can also apply the generalized coordinate method. And we will see that we get the same values, okay? And in the generalized coordinate method or the displacement function u of psi eta, we can write as alpha naught plus alpha 1 psi plus alpha 2 eta plus alpha 3 psi eta, right. And by substituting psi and eta of 1 or -1, and we get the u 1, u 2, u 3, u 4.



At node 1 our psi is 1 and eta is 1. And at node 3 psi is -1 and eta is -1 and so on. We can substitute all these values and get four simultaneous equations so that we can get

the four constants alpha naught, alpha 1, alpha 2, alpha 3. These are the generalized coordinates.

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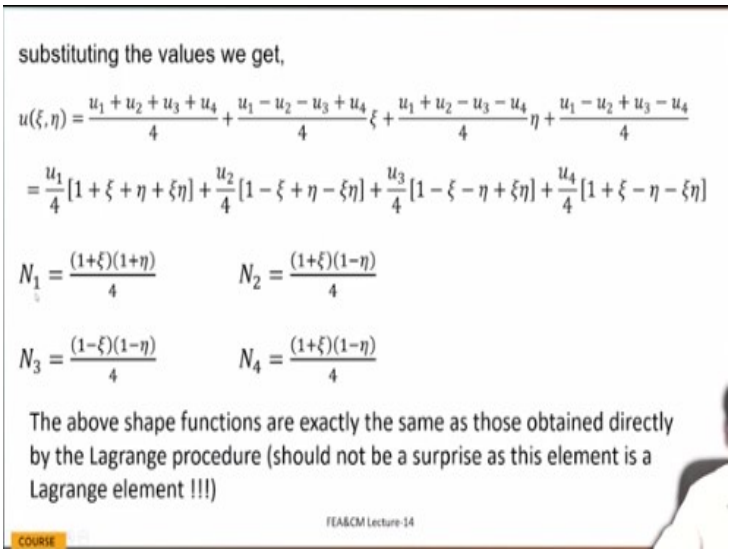

Then once you substitute and simplify, we get our N 1 as 1 plus psi times 1 plus eta by 4. N 2 is 1 plus, sorry I think it is I think I am making too many mistakes. So our N 2 is 1 minus psi times 1 plus eta by 4. And N 3 is 1 minus psi 1 minus eta by 4, and so on. And once again, we get the same shape functions as we got from the generalized coordinate method.

substituting the values we get,
$u(\xi,\eta) = \frac{u_1+u_2+u_3+u_4}{4} + \frac{u_1-u_2-u_3+u_4}{4}\xi + \frac{u_1+u_2-u_3-u_4}{4}\eta + \frac{u_1-u_2+u_3-u_4}{4}$
$= \frac{u_1}{4}[1+\xi+\eta+\xi\eta] + \frac{u_2}{4}[1-\xi+\eta-\xi\eta] + \frac{u_3}{4}[1-\xi-\eta+\xi\eta] + \frac{u_4}{4}[1+\xi-\eta-\xi\eta]$
$N_1 = \frac{(1+\xi)(1+\eta)}{4}$ $N_2 = \frac{(1+\xi)(1-\eta)}{4}$
$N_3 = \frac{(1-\xi)(1-\eta)}{4}$ $N_4 = \frac{(1+\xi)(1-\eta)}{4}$

But then it is not real surprise, because this is a pure Lagrange element. So whether you use a generalized coordinate method or Lagrange method we will get the same shape functions. So I think that is the end of my lecture today. And if you have any questions, please send an email to profkrg. So please do listen to all the lectures before you go for listening to the next lecture.

Because all of them are in a sequence. If you miss any sequence, then it is difficult for you to understand. So thank you very much. We will meet in the next class.