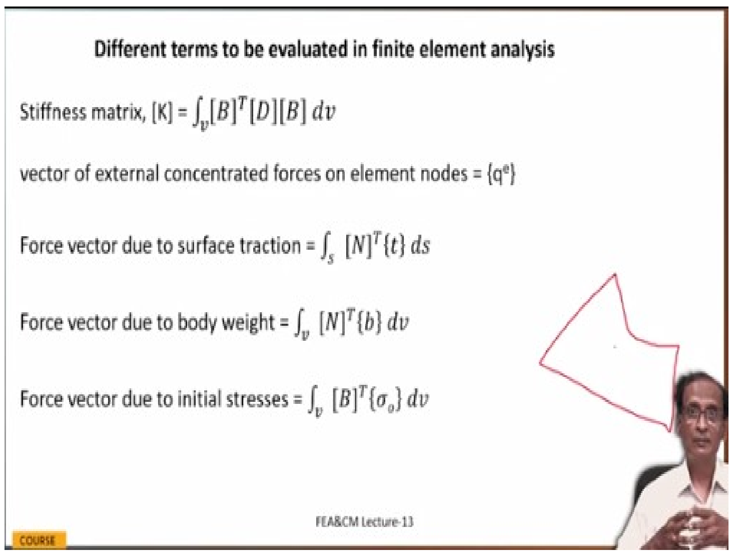
FEM & Constitutive Modelling in Geomechanics
Prof. K. Rajagopal
Department of Civil Engineering
Indian Institute of Technology-Madras

Lecture - 15
Numerical Integration Techniques

So very good morning to all of you. I hope you have read the previous lectures before listening to this lecture. Let us look at other aspects of computations. That is, let me start the laser pointer, okay. Let us look at the numerical integration techniques that we require.

(Refer Slide Time: 00:44)



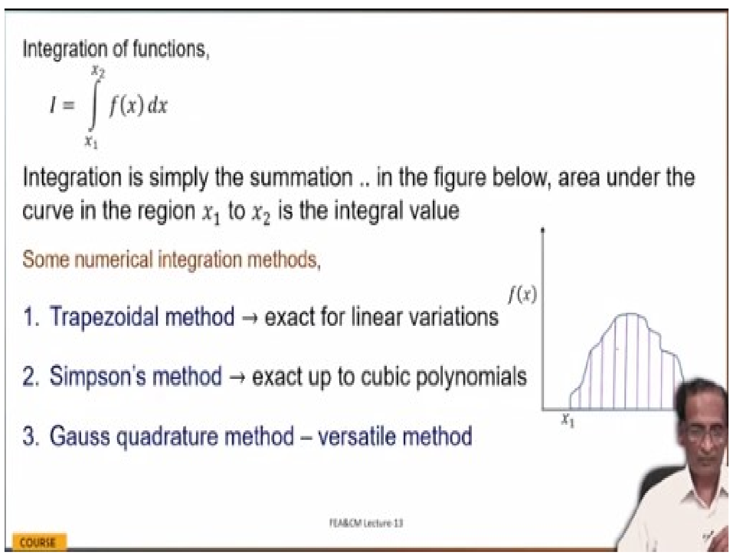
And just recall in the previous few classes, we had developed the equilibrium equations for a continuum. And we had seen number of integrals. See when starting from the stiffness matrix, integral B transpose DB integrated over the volume and then the force vector due to surface traction integrated over the surface. Then the force vector due to body weight and the force vector due to initial stresses.

We have number of these integrations and then we have also seen that the shape functions are expressed in terms of polynomial series. And then our B matrix is nothing but the matrix of the shape function derivatives. Then when we do this product B transpose DB or N transpose B or something, we will end up with polynomial of different orders.

Then how do we evaluate those integral quantities especially if the shape of the element is irregular. If it is a pure rectangle or something, it is okay we have the limits, but then say you have an irregular shape say something like this. So I have an element something like this, how do we do the integration? Because the x and y they are dependent on each other, and it is not possible.

So we need to look for other methods and especially those methods that are directly suitable for computer implementation, because if it is analytical integration, we know how to use, how to use our brains and do the calculations. But then when it comes to computer implementation, it will only do if you are able to program. And whatever method that we are going to develop, it should be easily programmable and that is what we are going to do that in this lecture.

(Refer Slide Time: 03:04)



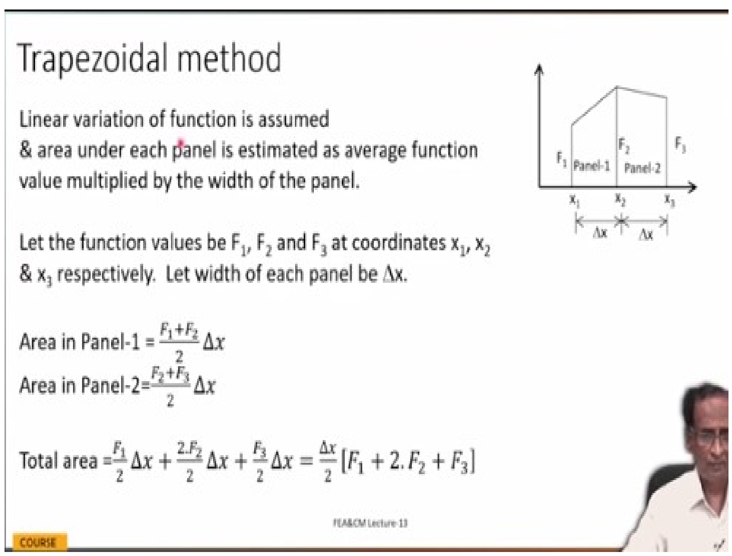
See as I said we have this integrations. Let me just see the, okay? So we have these integration quantities and the simplest one is let us say we have limits x 1 and x 2 and we have some function f and we want to integrate. And integration is just simply the summation.

Because actually, if you plot this function over x 1 and x 2 we can manually calculate the area of this function within each of these panels and then add them up to get the integral. And so there are different methods starting from trapezoidal method that is the simplest one. And in civil engineering, trapezoidal method is the one that we use for all the surveying calculations for estimating of volumes, at work volumes and so on.

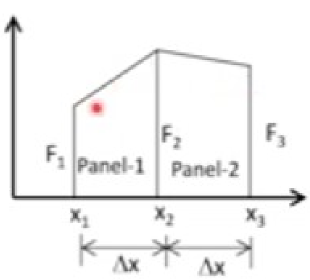
Then the trapezoidal method is good for linear variations like if you have a straight line curve, a straight line, then we can integrate. And Simpson's method is a bit more involved and it is good up to exactly integrating up to cubic polynomials. Then we are going to develop one method called as Gauss quadrature or Gauss-Seidel quadrature method.

And it is more versatile, more easy to implement and also given a polynomial order, we know how much, what is the order of Gauss quadrature that we require, okay? So we will see the theory behind, and then how to implement them also we will see.

(Refer Slide Time: 04:58)



See the trapezoidal method is actually suitable for linear functions, okay? And then the area of each panel, like for example say you have a limit of x 1 to x 3.



And here, I have shown you two panels. Let us say we divide this into two panels, panel 1 and panel 2. Then we can separately evaluate the area in each panel. Like this area of panel 1 is the average value of F 1 and F 2, multiplied by delta x.

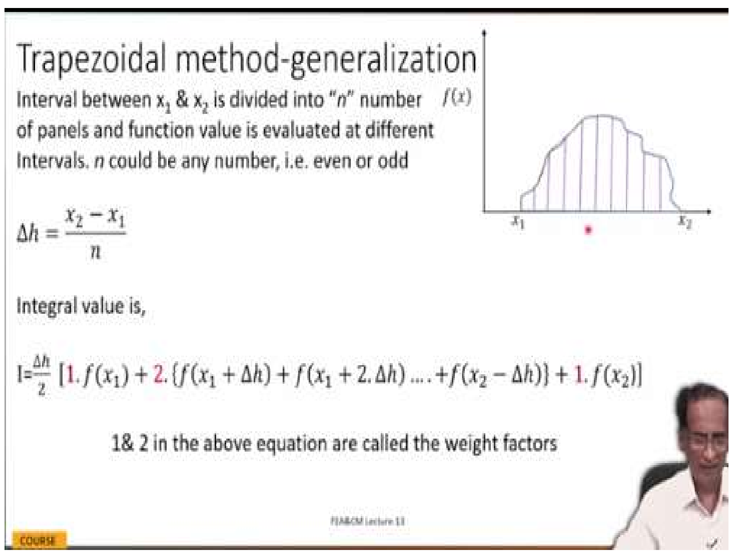
$$\text{Area in Panel-1} = \frac{F_1+F_2}{2}\Delta x$$

$$\text{Area in Panel-2} = \frac{F_2+F_3}{2}\Delta x$$

$$\text{Total area} = \frac{F_1}{2}\Delta x + \frac{2.F_2}{2}\Delta x + \frac{F_3}{2}\Delta x = \frac{\Delta x}{2}[F_1 + 2.F_2 + F_3]$$

And the area in the panel 2 is the average value of F 2 and F 3 multiplied by delta x. And so the total area comes out like delta x by 2, multiplied by F 1 plus F 3 plus 2F 2, right?

(Refer Slide Time: 05:58)



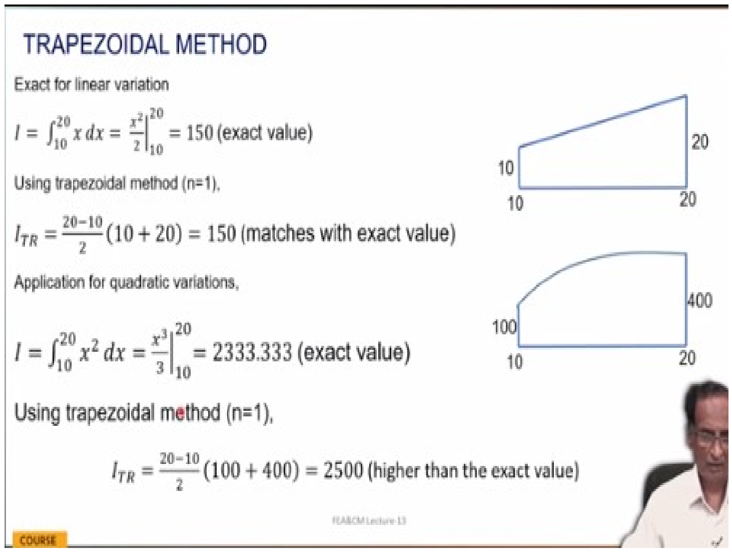
And in general, we can write like this. We can divide this length x 2 - x 1 into say n number of intervals. Or we can write a generic formula I is delta h by 2 or the function value at x 1, and then the function value at x 2, plus two times the sum total of all the intermediate function values, x 1 to say x 2 minus delta h okay? So these values that are highlighted in red, 1, 2 and 1 these are called as weight factors.

Integral value is,

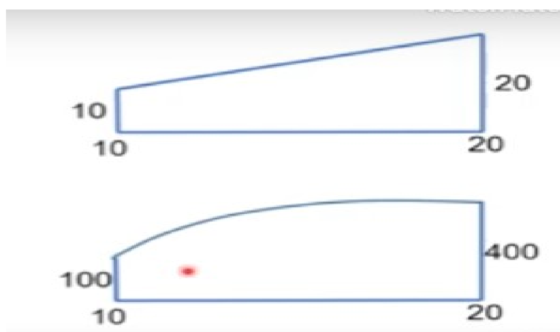
$$I=\frac{\Delta h}{2}\,[1.f(x_1) + 2.\{f(x_1+\Delta h) + f(x_1+2.\Delta h) \ldots.+f(x_2-\Delta h)\} + 1.f(x_2)]$$

They are multiplying the function values with some weight factors 1 and 2 okay? And it is actually it is a very simple method, the trapezoidal method. And if we have a higher order curve, or higher order polynomial, we need to divide this into more number of panels, okay?

(Refer Slide Time: 07:06)



So let us look at an application.



Let us say that you want to integrate a function x from 10 to 20. And the exact value is x square by 2 and the limits are 10 and 20. So this value is 150. And this being a linear function, it is varying like this. So we can just divide this entire length into one single panel, and then our integral value by trapezoidal method, is just simply delta x by 2 average value times 10 plus 20.

TRAPEZOIDAL METHOD

Exact for linear variation

$$I = \int_{10}^{20} x\,dx = \left.\frac{x^2}{2}\right|_{10}^{20} = 150 \text{ (exact value)}$$

Using trapezoidal method (n=1),

$$I_{TR} = \frac{20-10}{2}(10+20) = 150 \text{ (matches with exact value)}$$

Application for quadratic variations,

$$I = \int_{10}^{20} x^2\,dx = \left.\frac{x^3}{3}\right|_{10}^{20} = 2333.333 \text{ (exact value)}$$

10 plus 20 is divided by 2 is the average value, and delta x is this value. And that is 150. That is exactly matching with the result. That is because we have a linear polynomial. And now let us look at higher order polynomial, like let us take x square, integral of x square dx is integrated from 10 to 20 is 2333.333. That is the exact value. And let us try with one single panel, okay?

Using trapezoidal method (n=1),

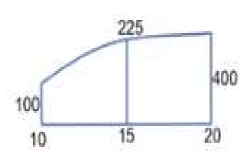
$$I_{TR} = \frac{20-10}{2}(100+400) = 2500 \text{ (higher than the exact value)}$$

So this I is at 10, the function value is 10 square, that is 100. And at 20, the function value is 400. And this comes out as 2500. And what we should notice is, this value is more than the exact value.

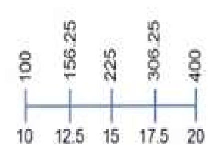
(Refer Slide Time: 08:59)

Considering two panels (n=2), $\Delta h=10/2=5$

$$I_{TR} = \frac{5}{2}(100 + 2 \times 225 + 400) = 2375$$



Considering four panels, $\Delta h=10/4=2.5$

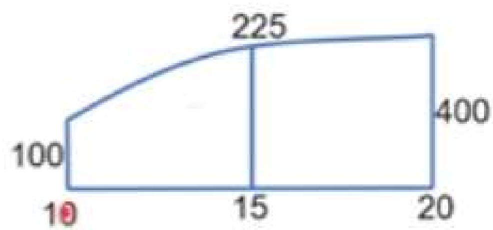


$$I_{TR} = \frac{2.5}{2}[100 + 2(156.25 + 225 + 306.25) + 400] = 2343.75$$

As the number of panels is increased, the numerical value approaches the exact va

how many panels are required to find the exact integral value?

So now, let us try with by dividing this into two panels, and see what happens to the accuracy of the solution.

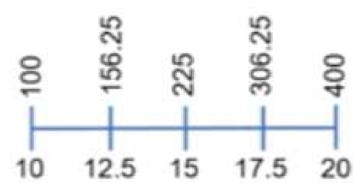


Considering two panels (n=2), $\Delta h=10/2=5$

$$I_{TR} = \frac{5}{2}(100 + 2 \times 225 + 400) = 2375$$

Considering four panels, $\Delta h=10/4=2.5$

So at 10, the function value is 100 x square. At 15 it is 225. At 20 it is 400. So this the integral values. Now our delta x is only 5. 5 by 2 100 plus two times 225 plus 100. That is 2375. And if you look at the exact value is 2333 okay? So we are coming closer to the exact result.



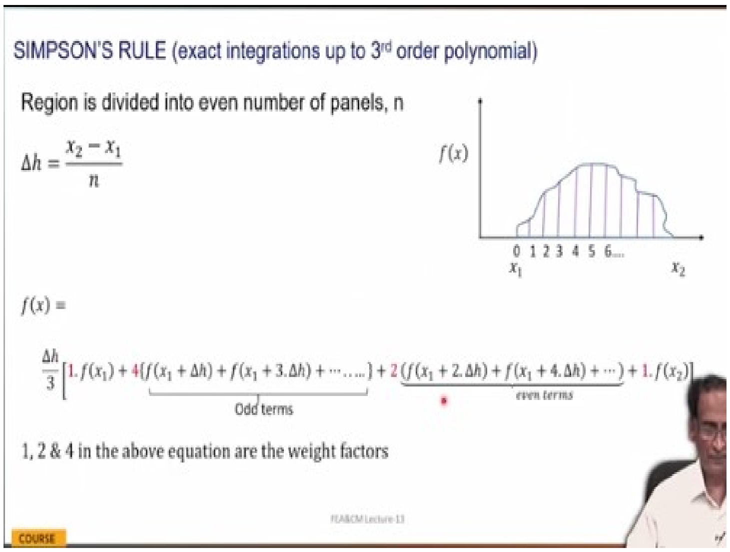
$$I_{TR} = \frac{2.5}{2}[100 + 2(156.25 + 225 + 306.25) + 400] = 2343.75$$

And now let us try with four panels. Instead of two panels let us take four and the x values are at x of 10, 12.5, 15, 17.5 and 20 okay because our delta h is 2.5 and the function values at each of these x values are like this. So our integral value is 2.5 by 2 that delta h by 2 multiplied by the function values at x 1 and x 2 and then two times all the intermediate function values.

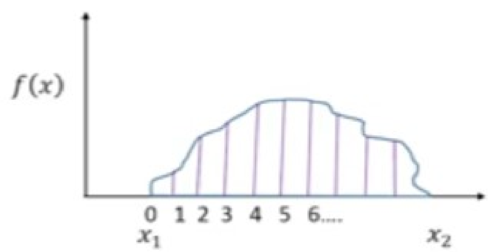
And this comes out as 2343.75. So in general we notice that, as we increase the number of panels, we are approaching towards the limit, to the solution that is 2333.33. But then, if I ask you how many panels you need to take so that you get the exact value? That we do not know. We have to go on trying out. And the number of panels also may require, may depend on the order of polynomial.

So in this case, we have a polynomial of x square. But let us say we have another polynomial of x to the power 5 or x to the power 6. We do not know how many panels we need.

(Refer Slide Time: 11:16)



So now let us look at some other method that is the Simpson's rule. And this method is exact up to a third order polynomial.



And this method involves in dividing the entire region into certain number of even number of panels; 2 panels, 4, 6, 8, 10 and so on. And our n is even. Whereas in the trapezoidal method, there is no such restriction. You can have any number of panels like 1, 2, 3, 4 and so on.

Region is divided into even number of panels, n

$$\Delta h = \frac{x_2 - x_1}{n}$$

And our delta h is x 2 – x 1 by n. And these function locations function value, locations are numbered as 0, 1, 2, 3, 4 and so on. So 0 and the last one, the function values at x 1 and x 2. And the function values at 1, 3, 5 so on they are called as odd numbers, odd terms. And the function value is evaluated at 2, 4, 6 and so on. These are the even locations.

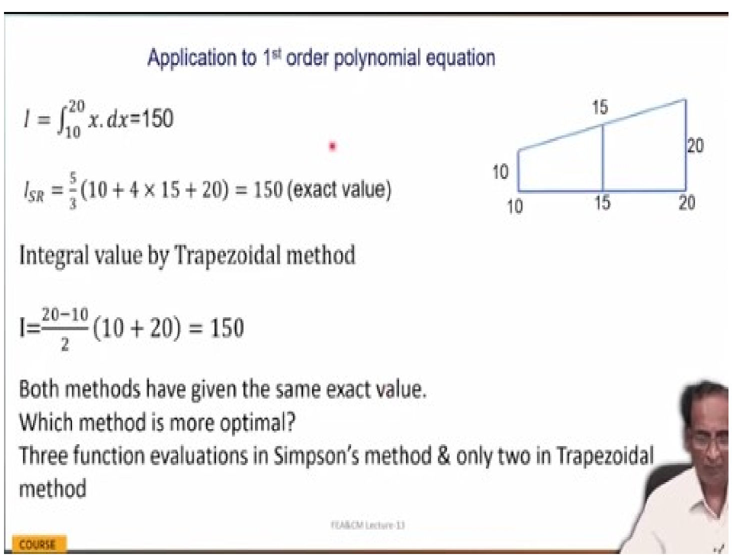
$$f(x) =$$

$$\frac{\Delta h}{3}\left[1.f(x_1) + 4\underbrace{\{f(x_1+\Delta h) + f(x_1+3.\Delta h) + \cdots\ldots\}}_{\text{Odd terms}} + 2\underbrace{(f(x_1+2.\Delta h) + f(x_1+4.\Delta h) + \cdots)}_{\textit{even terms}} + 1.f(x_2)\right]$$

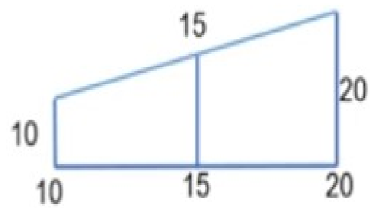
And as per the Simpson's rule, the integral value is delta h by 3 where delta h is x 2 – x 1 by n, one times the function value at x 1 and one times the function value at x 2, plus four times the sum total of all the function values evaluated at odd locations, 1, 3, 5, 7 and so on. So f of x 1 plus delta h x 1 plus 3 delta h and so on, plus two times the sum total of all the function values evaluated at even locations, x 1 plus 2 delta h, x 1 plus 4 delta h, and so on.

And so here, these factors 1, 4, and 2, these are called as weight factors. Previously, we had only two factors 1 and 2. But now in the Simpson's rule, we have three weight factors 1, 2 and 4.

(Refer Slide Time: 13:43)



And let us apply the Simpson's rule for different polynomials. Let us apply this for linear polynomial, the same one that we had seen earlier.



And we require minimum two panels because we have to divide the region of integration into even number of panels. So our x values are located at 10 that is x 1 and x 2 is at 20 and intermediate value is at 15.

$$I = \int_{10}^{20} x.dx = 150$$

$$I_{SR} = \frac{5}{3}(10 + 4 \times 15 + 20) = 150 \text{ (exact value)}$$

Integral value by Trapezoidal method

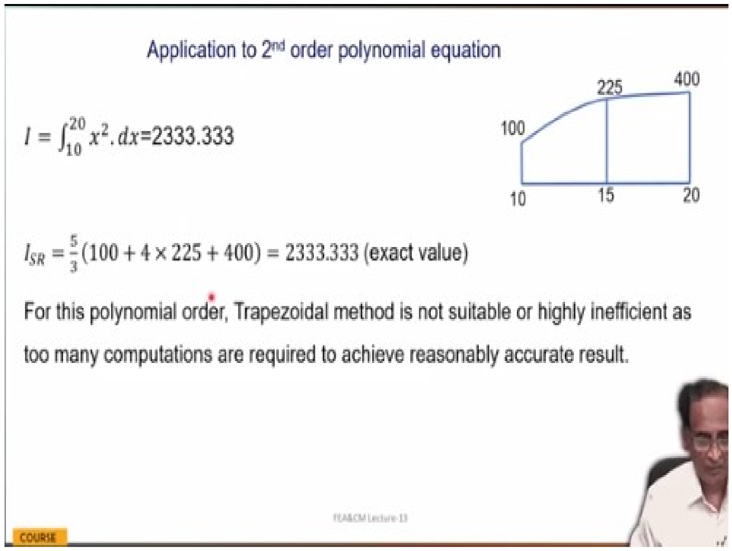
$$I = \frac{20-10}{2}(10 + 20) = 150$$

So our I the integral value is delta h by 3 that is 5 by 3 10 plus four times the function value at odd locations four times 15 plus the last one that is 20. That is exactly 150. That is the exact value. And the integral value by the trapezoidal method we already calculated 150. See both of them, they have given exactly the same value and both are correct. Both are matching with the exact result.

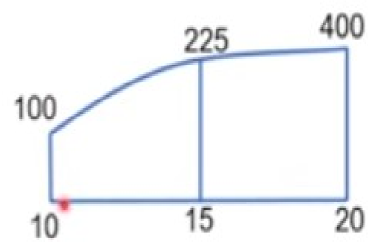
But which one is more optimal. So if you look at the trapezoidal method involved in only two function calculations 10 and 20. Whereas, the Simpson's method it required minimum three, 10, 15 and 20. So obviously the Simpson's rule is requiring more effort, even for a simple problem like this, the first order polynomial.

So when we decide on the choice of the numerical method, we also need to look at the computational effort that we need to spend on for getting the result.

(Refer Slide Time: 15:42)



So now let us look at the second order polynomial integral of x square, integrated from 10 to 20.



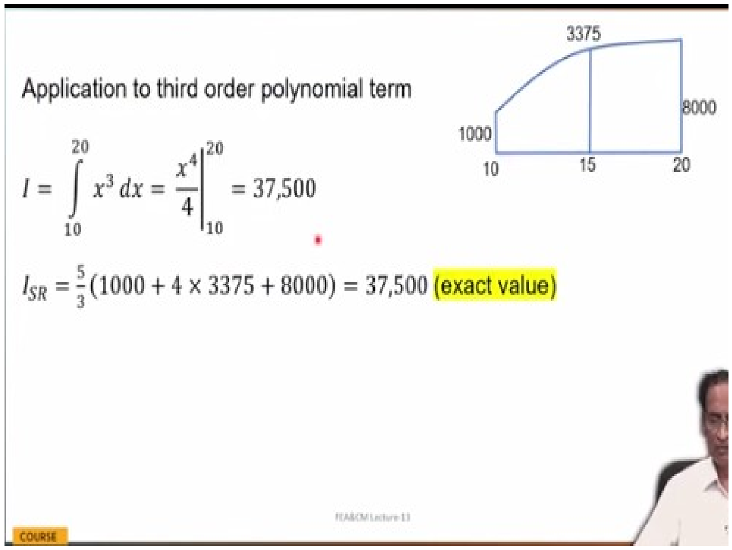
And the exact value is 2333.333. And it is divided into two panels. The x 1 is 10. And the function value is 100. And x 2 is 20. And the function value is 400 and intermediate values at 15. That is 225. And the integral as per the Simpson's rule is delta h by 3 that is 5 by 3 times 100 plus four times the function values at odd locations 225 plus 400.

$$I = \int_{10}^{20} x^2 . dx = 2333.333$$

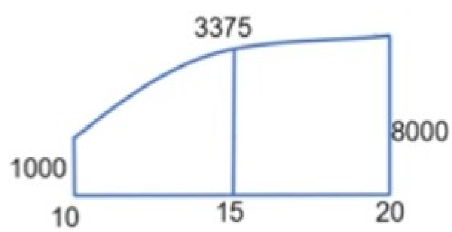
$$I_{SR} = \frac{5}{3}(100 + 4 \times 225 + 400) = 2333.333 \text{ (exact value)}$$

So that comes to 2333. And this is the exact value. And for this polynomial, the trapezoidal rule is not very accurate, and it is not suitable. Like we should not try applying method that does not work. Because we do not know exactly how many panels we need to take so that we get the exact result.

(Refer Slide Time: 16:57)



And let us apply this to a third order polynomial x cube dx, from the integrated from 10 to 20.

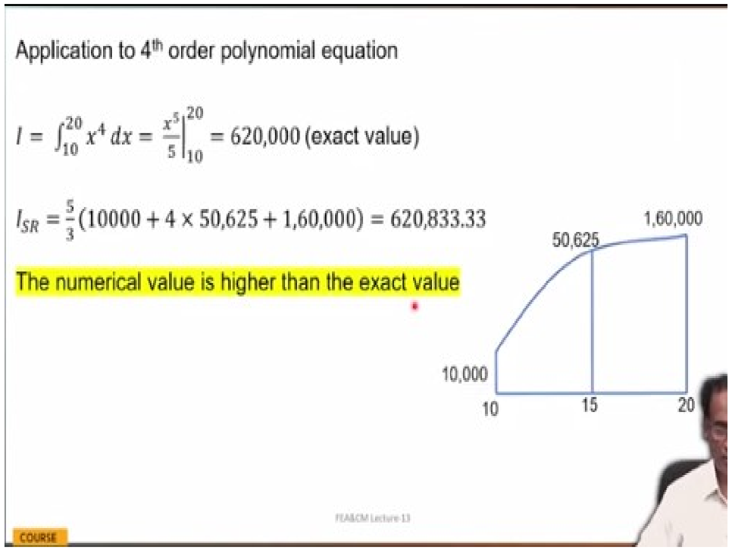


$$I = \int_{10}^{20} x^3\,dx = \left.\frac{x^4}{4}\right|_{10}^{20} = 37{,}500$$

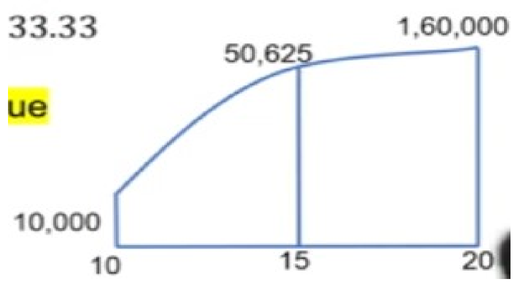
$$I_{SR} = \frac{5}{3}(1000 + 4 \times 3375 + 8000) = 37{,}500 \text{ (exact value)}$$

And the exact value is 37,500. So the function value at x 1 is 10 cube that is 1000. And at 20 it is 8000. And at 15, it is 3375. And the integral value as per the Simpson's rule is delta h by 3. Delta h is 5 by 3 multiplied by 1000, plus four times the function value at odd locations 3375 plus the last function value. And it is 37,500. It is exact.

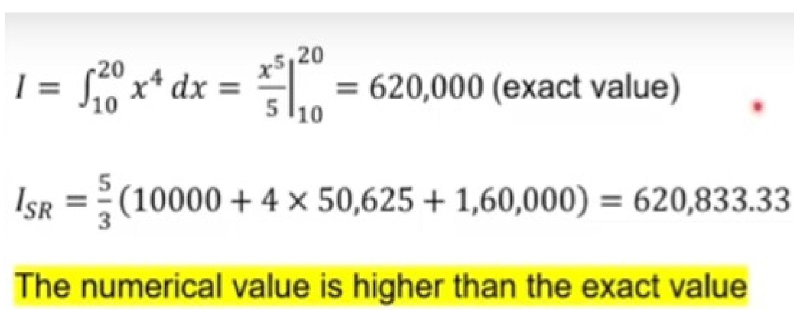
(Refer Slide Time: 17:52)



And now let us apply this to a higher order polynomial, like more than the third order polynomial, let us go to fourth order polynomial.

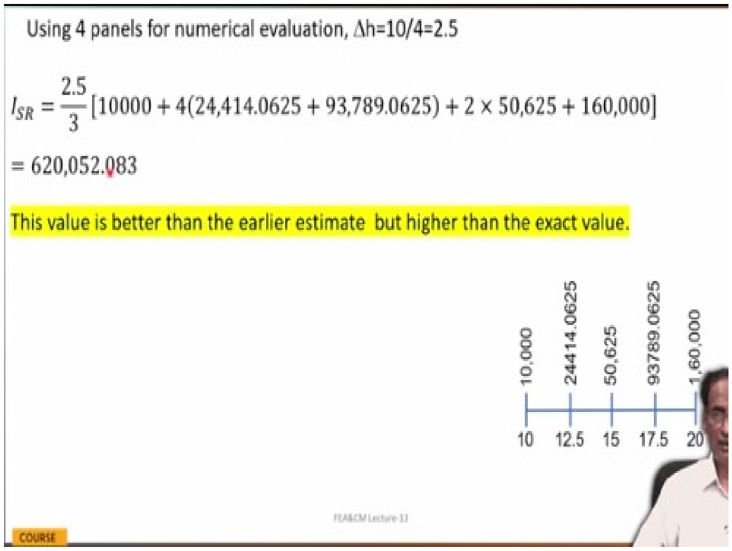


So the exact value here is 620,000. And if we apply the Simpson's rule with two panels, we get the value 620,833.33. And this value is not correct, and it is actually more than the exact value. See even in the trapezoidal method, our estimates are always more than the exact values.



And only for the case where we are able to handle that polynomial, we were able to get the exact result. But in all other cases, the values are higher than the exact value.

(Refer Slide Time: 18:50)



And now let us try with four panels. And with four panels, we get 620,052. And this value is better than the previous one, but then not exact, not an exact value. And once again, we do not know how many panels we need to take so that we can get exact value of 620,000. So that is another limitation of the Simpson's rule.

Using 4 panels for numerical evaluation, $\Delta h=10/4=2.5$

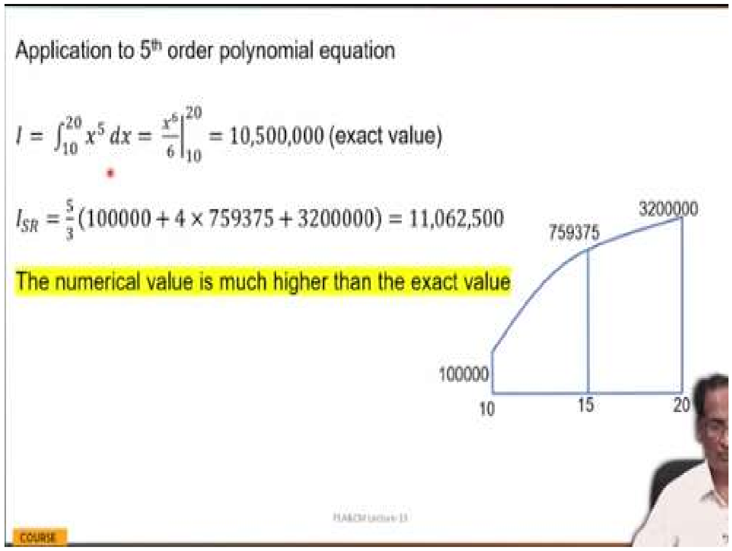
$$I_{SR} = \frac{2.5}{3}[10000 + 4(24,414.0625 + 93,789.0625) + 2 \times 50,625 + 160,000]$$

$$= 620,052.083$$

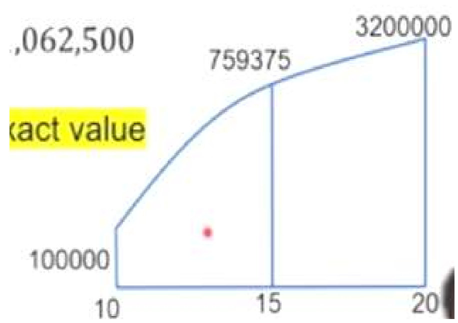
This value is better than the earlier estimate but higher than the exact value.

But at least the Simpson's rule is valid up to third order polynomial, whereas the trapezoidal rule is good only up to linear polynomials.

(Refer Slide Time: 19:37)



And let us apply this for slightly higher order polynomial x to the power 5.



And the exact integral value is 10,500,000. This is the exact value. And if you apply the Simpson's rule with the minimal number of panels, that is 2, we get 11,062,500 which is more than the exact value.

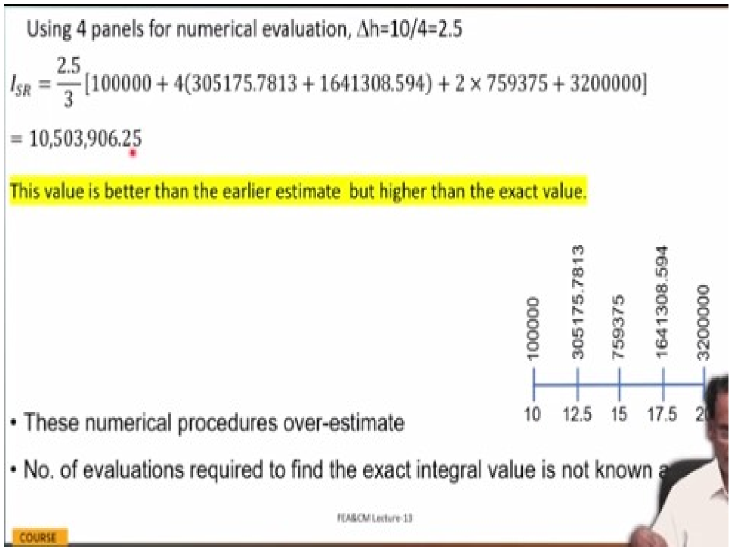
Using 4 panels for numerical evaluation, $\Delta h = 10/4 = 2.5$

$$I_{SR} = \frac{2.5}{3}[100000 + 4(305175.7813 + 1641308.594) + 2 \times 759375 + 3200000]$$

$$= 10{,}503{,}906.25$$

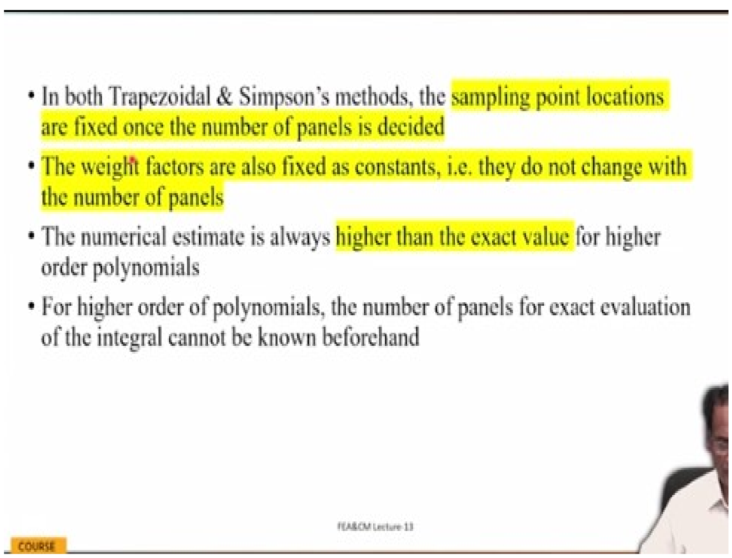
This value is better than the earlier estimate but higher than the exact value.

(Refer Slide Time: 20:06)



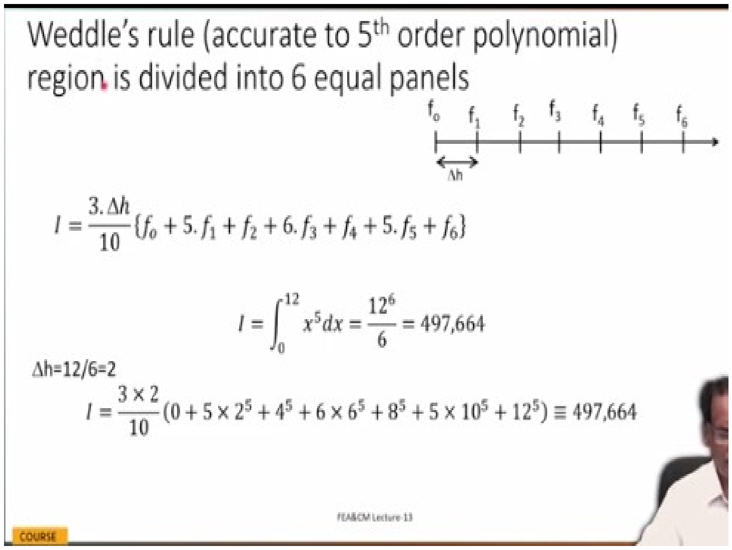
And if you divide this into four panels, you get better result. But it is more than the exact value. So both the trapezoidal method and then the Simpson's rule, they are exact up to the order of polynomial of 1 and 3. But beyond that, beyond their own capacity, they tend to overestimate the integral value.

(Refer Slide Time: 20:39)

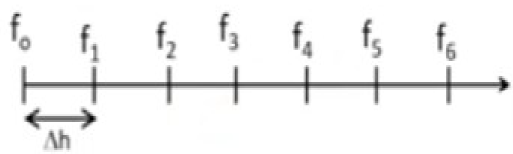


Okay that is one thing that we need to notice. And let us proceed further.

(Refer Slide Time: 20:46)



Let us look at one more similar integration rule, Weddle's rule. This is good up to fifth order polynomial. And we divide this into six even panels 1, 2, 3, 4, 5, 6 and evaluate the function values at f naught, f 1, f 2, f 3 and so on.



And the integral value is three times delta h by 10 times f naught + 5 f 1 + f 2 + 6 f 3 + f 4 + 5 f 5 + f 6. Actually here, this 1, 5, 6 and so on, these are all the weight factors.

$$I = \frac{3.\Delta h}{10}\{f_o + 5.f_1 + f_2 + 6.f_3 + f_4 + 5.f_5 + f_6\}$$

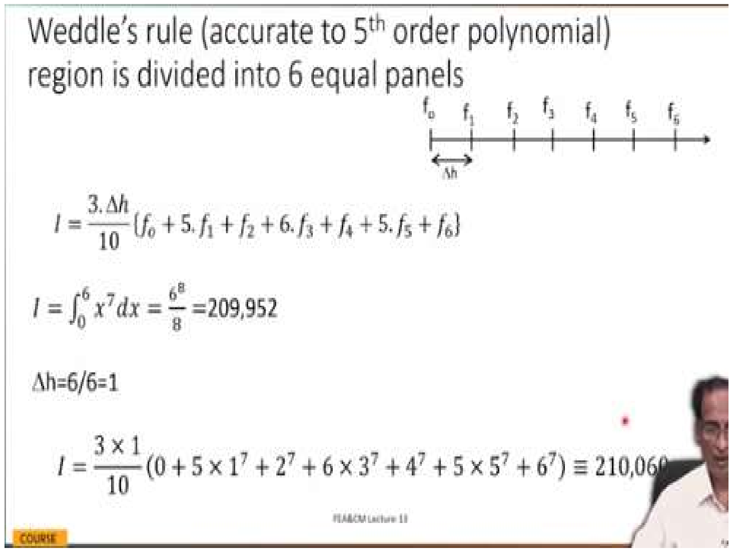
$$I = \int_0^{12} x^5 dx = \frac{12^6}{6} = 497{,}664$$

Δh=12/6=2

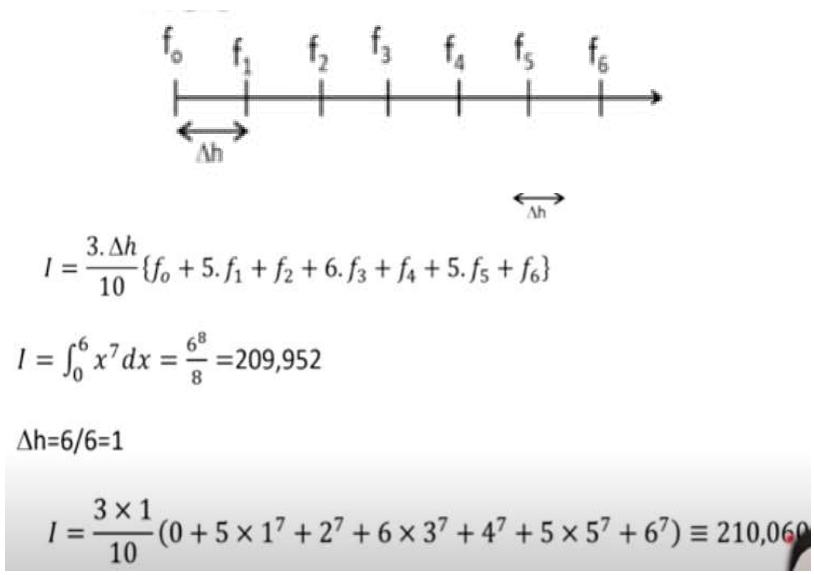
$$I = \frac{3\times 2}{10}(0 + 5\times 2^5 + 4^5 + 6\times 6^5 + 8^5 + 5\times 10^5 + 12^5) \equiv 497{,}664$$

And actually we are not interested in how this formula was derived. But we are only interested in how to apply this for our integrations. And let us take the fifth order polynomial and integrate from 0 to 12. The exact result is 497,664 and our delta h is 12 by 6 that is 2 and our integral value is three times delta h by 10 multiplied by all these quantities. And you get exact value of 497,664.

(Refer Slide Time: 22:19)



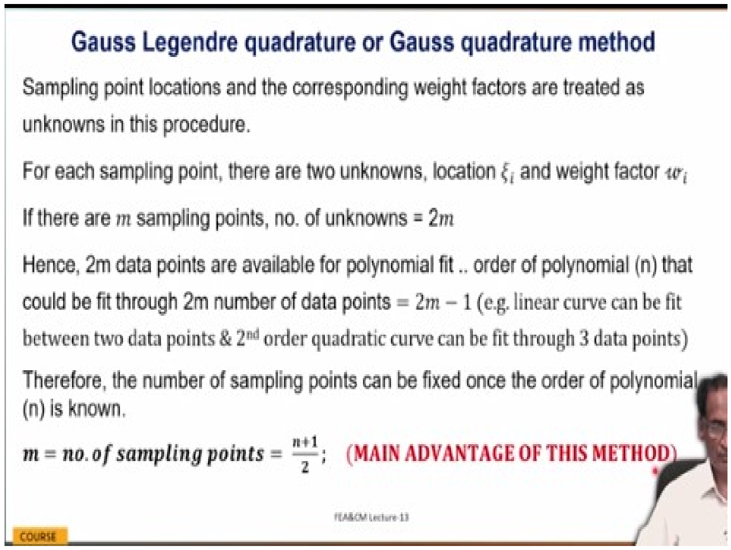
And let us apply this to a seventh order polynomial and the integral value in this particular case is 209,952.



And then the predicted value from the numerical analysis is 210,060 which is not correct. And also it is more than the exact value, right? So we see that all these other methods like the trapezoidal method, Simpson's rule, and then the Weddle's rule, they are good up to certain order of polynomial.

Beyond that they are not good. And in the case of trapezoidal and Simpson's rule, we can improve the accuracy by increasing the number of panels but we do not know exactly up to what number of panels we need to consider. So we need a better method for numerical integration.

(Refer Slide Time: 23:23)



And that we find in this Gauss Legendre quadrature or the Gauss quadrature method. And this is actually it is a more mathematical procedure. But in this course, I am not going into the mathematics but in a very simple manner we will see how to apply this for our calculations. In this procedure, both the sampling point locations and then the corresponding weight factors are treated as unknowns.

So in the previous case, once you fix the number of panels, we know these sampling point locations x naught, x naught plus delta h, x naught plus 2 delta h and so on. And then the weight factors are also fixed 1, 2, 1 in the case of trapezoidal method. 1, 4, 2, 1 in the case of the Simpson's rule. And similarly in the Weddle's rule, we have 1, 5 and 6 and so on.

And so whereas in the Gauss quadrature method, the sampling points and also their corresponding weight factors are treated as unknowns. And we will come out with different set of weight factors depending on the number of points that we consider. So for in this case, for each sampling point, there are two unknowns, the location and then the weight factor, the location psi and the weight factor w.

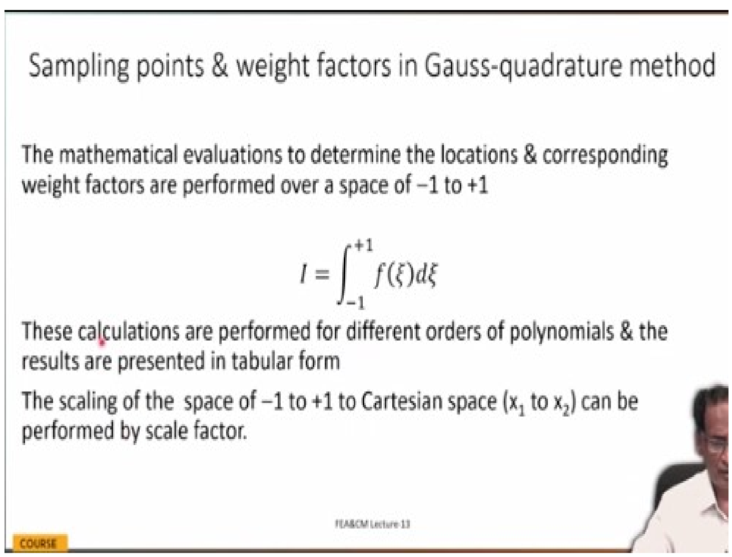
So if we have m number of sampling points, the number of unknowns is 2m okay? And if you have 2m data points for fitting a polynomial, the order of polynomial that could be fit through 2m number of data points is 2m - 1. So it is very simple. Like if we have two data points, we can fit a straight line and the straight line is a first order. And if you have three data points, we can fit a quadratic curve of second order.

And if we have four points, you can fit a cubic curve and so on. So actually this method is advantageous because if we know the order of polynomial, we can decide on the number of sampling points and we can equate n to 2m - 1 and then we get the number of sampling points as n + 1 by 2. So if you know the order of polynomial that you need to integrate, we can get the number of sampling points as n + 1 by 2.

And this is the main advantage of the Gauss quadrature method or Gauss Legendre quadrature. Because in advance, we know how many sampling points we need to consider like similar to number of panels that we need to consider. In the case of trapezoidal any number like odd or even does not matter. But in the case of Simpson's rule, we need only even number of panels.

Whereas in the Weddle's rule we need six panels. Whereas here in the Gauss quadrature method, it is more flexible. And we can decide on the number of data points based on the order of polynomial that we have.

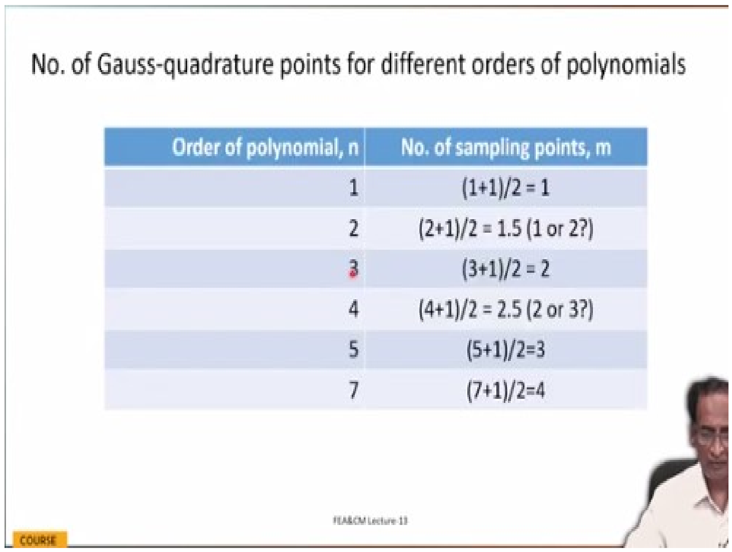
(Refer Slide Time: 27:00)



And so, in the mathematical derivation is done in the space of -1 to +1. See the sampling point locations psi are derived in the space of -1 to +1 so that we can scale them later. And we can scale the space of -1 to +1 to Cartesian space of x 1 to x 2 okay through the mapping factors or scale factors.

$$I = \int_{-1}^{+1} f(\xi)d\xi$$

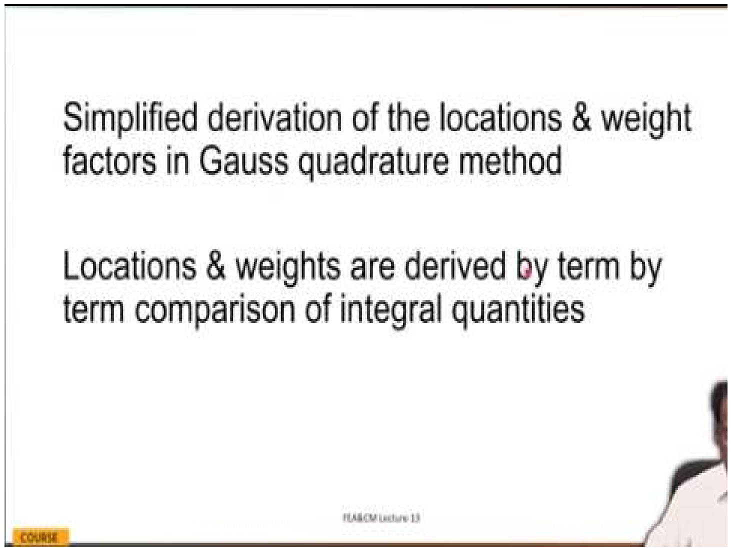
(Refer Slide Time: 27:35)



So if you have an order of polynomial of 1, the number of sampling points is 1 plus 1 by 2 that is 1. And if you have 3, 3 plus 1 by 2 is 2. And if you have 5, fifth order polynomial, you need 3 points. If you have seventh order polynomial you have 4. So if you have an even number of polynomials like 2 and 4, if you apply that formula 2 plus 1 by 2 that comes to 1.5.

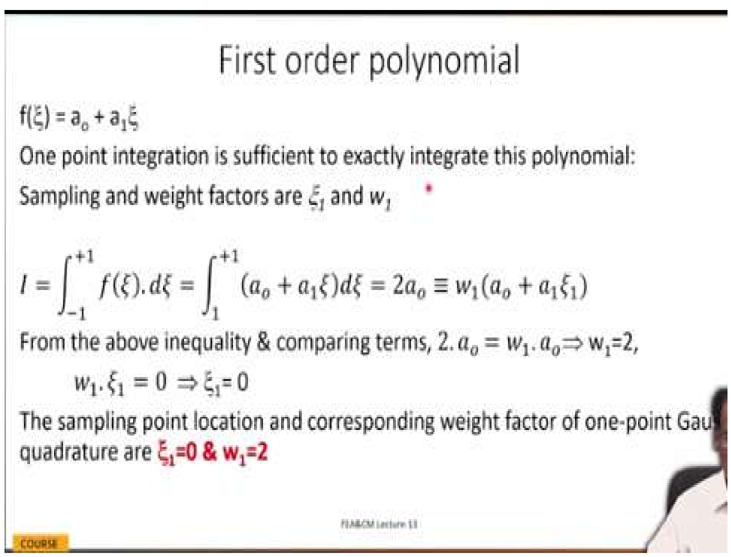
And obviously, we cannot have 1.5 number of sampling points. We need to decide either 1 or 2. Or when you have a fourth order polynomial, it comes out as 2.5. So you need to decide between 2 and 3, whether you want to use 2 or 3. So obviously, that we will decide later. I will not tell you the answer now how to decide whether 2 points or 3 points or 1 point or 2 points, okay? That we will discuss later.

(Refer Slide Time: 28:46)



And here I am going to derive the sampling point locations and their corresponding weight factors in a very simple manner by term by term comparison. We are not going to go through the Gauss Legendre mathematics, because we are not really interested in the mathematics behind this, but we need to know how to apply this procedure for our finite element calculations.

(Refer Slide Time: 29:15)



Let us look at the first order polynomial and this function could be like our a naught plus a 1 psi.

$$f(\xi) = a_o + a_1\xi$$

This is similar to our polynomial that we had assumed for deriving our shape functions or in the Rayleigh-Ritz procedure also we had a polynomial. And let us integrate this from -1 to +1. See if you integrate this quantity from -1 to +1, because psi is an odd function, if you integrate it becomes an even function.

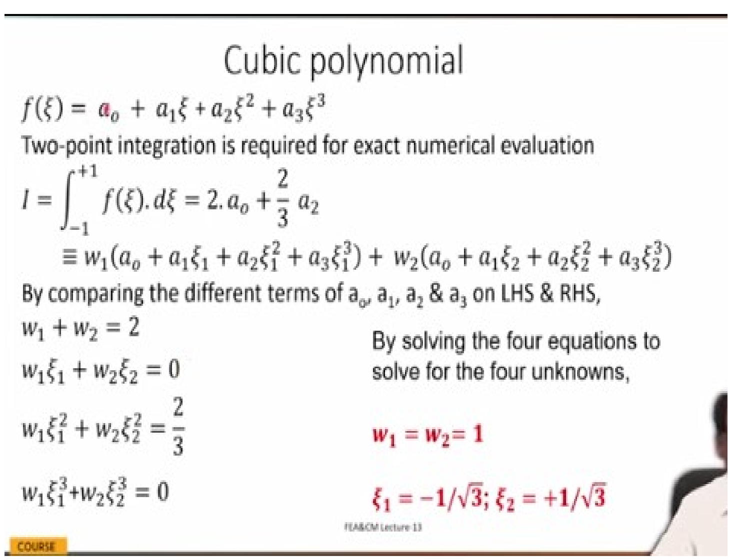
$$I = \int_{-1}^{+1} f(\xi).d\xi = \int_{1}^{+1} (a_o + a_1\xi)d\xi = 2a_o \equiv w_1(a_o + a_1\xi_1)$$

And if you substitute the limits of -1 and +1, that gets cancelled out. So you are left with only 2a naught, a naught psi and psi varying from +1 to -1. Sorry it should be -1, okay? And so if we have one sampling point, the sampling point location is psi 1 and then the weight factor is w 1. And we can evaluate this function as w 1 multiplied by a naught plus a 1 psi a 1 psi 1 okay?

And this is 2a naught is the exact value, exact integral value and we are making it exactly equal to w 1 times a naught plus a 1 psi 1. And by comparing the left hand side to the right hand side we can determine our weight factor w 1 and then the location psi 1 okay. So if you compare the term by term, 2a naught is equal to w 1 a naught. So this gives w 1 is 2. And then w 1 times psi 1 is 0.

So that means that the sampling point location one is 0. So the sampling point location and then the corresponding weight factor for one point Gauss quadrature is 0 and 2. Psi 1 is 0 and w 1 is 2. So it is very simple. It is actually we take a polynomial and then integrate it and then equate that to w times that function value, okay?

(Refer Slide Time: 32:07)



So now, let us take a cubic polynomial. The function is a naught + a 1 psi + a 2 psi square + a 3 psi cube. And for exactly integrating the third order polynomial we require two Gauss quadrature points. Why because 2 times 2 - 1 is 3 so that the 3 is the order of polynomial, okay? So here the integral value is integrated from -1 to +1 is 2a naught plus 2 by 3 a 2. And we need two sampling points psi 1 and psi 2.

$$f(\xi) = a_o + a_1\xi + a_2\xi^2 + a_3\xi^3$$

Two-point integration is required for exact numerical evaluation

$$I = \int_{-1}^{+1} f(\xi).d\xi = 2.a_o + \frac{2}{3}a_2$$

$$\equiv w_1(a_o + a_1\xi_1 + a_2\xi_1^2 + a_3\xi_1^3) + w_2(a_o + a_1\xi_2 + a_2\xi_2^2 + a_3\xi_2^3)$$

By comparing the different terms of a_o, a_1, a_2 & a_3 on LHS & RHS,

$$w_1 + w_2 = 2$$

$$w_1\xi_1 + w_2\xi_2 = 0$$

$$w_1\xi_1^2 + w_2\xi_2^2 = \frac{2}{3}$$

$$w_1\xi_1^3 + w_2\xi_2^3 = 0$$

By solving the four equations to solve for the four unknowns,

$$w_1 = w_2 = 1$$

$$\xi_1 = -1/\sqrt{3};\ \xi_2 = +1/\sqrt{3}$$

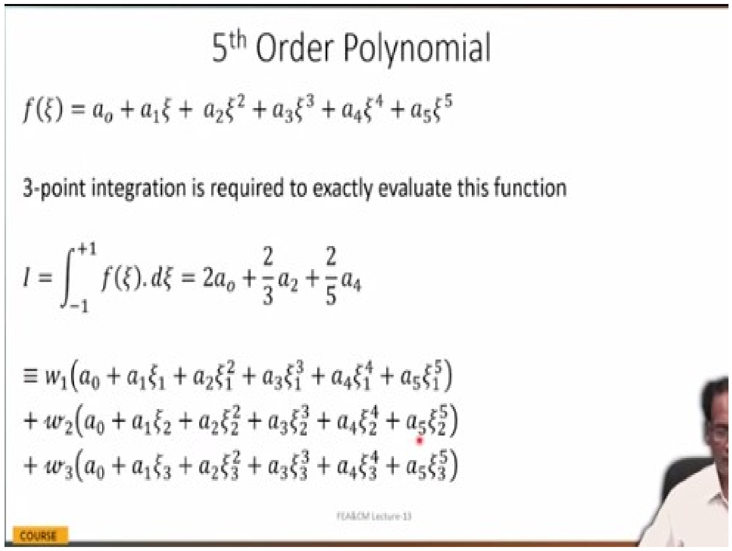
And then the corresponding weight factors w 1 and w 2. So let us equate this to this numerical value exactly w 1 times a naught plus a 1 psi 1 plus a 2 psi 1 square plus a 3 psi 1 cube plus w times w 2 times a naught plus a 1 psi 2 plus a 2 psi 2 square plus a 3 psi 2 cube. So now, we can set up four simultaneous equations, because we have four unknowns; psi 1, psi 2, w 1 and w 2.

And by comparing the term by term against a naught, a 1, a 2, and a 3 we get four simultaneous equations. So our w 1 + w 2 is equal to 2. w 1 plus two times a naught is 2 a naught. So we can write w 1 + w 2 is 2. And similarly, our a 2 term is associated with psi 1 square. So w 1 times psi 1 square plus w 2 times psi 2 square is two thirds. And there are no a 1 and a 3 terms.

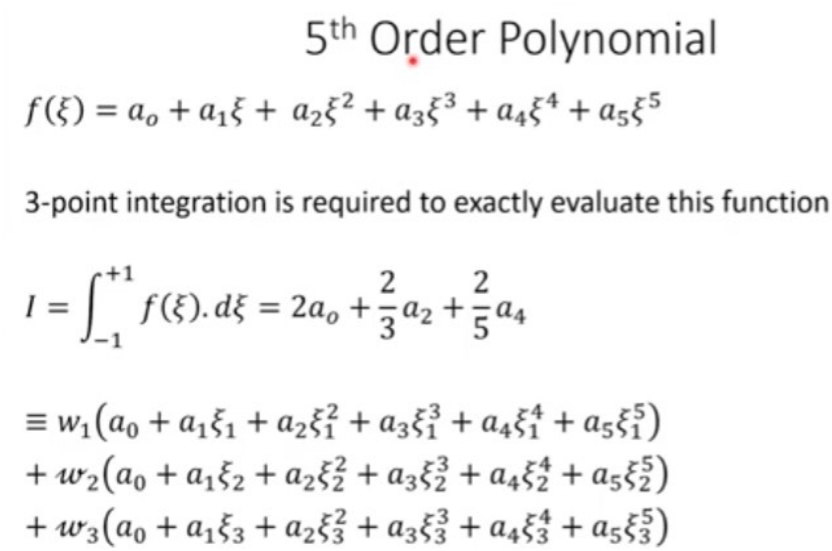
So w 1 times psi 1 plus two times psi 2 is 0. Then there is no a 3 term. So w 1 times psi 1 cube plus w 2 times psi 2 cube that is 0. And by solving these four simultaneous equations, we get w 1 and w 2 as 1 and psi 1 is -1 by root 3 and psi 2 is +1 by root 3. It is, we can if you are not able to solve by hand we can you use MATLAB or some other program.

We can give these four equations in symbolic form and then we can ask the program to solve it.

(Refer Slide Time: 35:10)



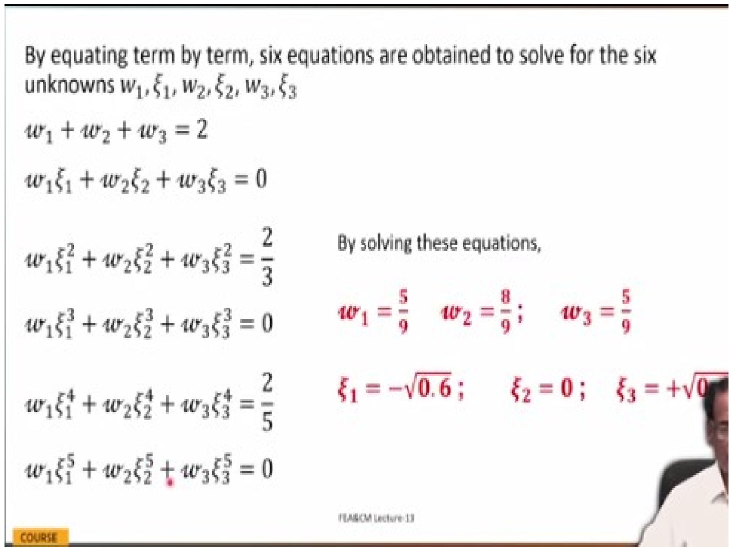
So the fifth order polynomial, actually I am not considering second order and fourth order, because we end up with odd number of points like 1.5, 2.5 and so on, which is not possible. And when you have a fifth order polynomial, we need three data points or three sampling points because n is equal to 2m - 1. So m is 5 + 1 by 2 that is 3.



And let us take a fifth order polynomial a naught + a 1 psi + a 2 psi square + a 3 psi cube + a 4 by psi 4 + a 5 psi to the power 5. And so if you integrate this fifth order polynomial, you get 2a naught + 2 by 3 a 2 + 2 by 5 a 4. And we require three sampling points psi 1, psi 2 and psi 3 and the three weight factor is w 1, w 2, w 3.

So this should be exactly equal to w 1 times a naught plus a 1 psi 1 plus a 2 psi 1 square plus a 3 psi 1 cube plus a 4 psi 1 4 plus a 5 times psi 1 to the power 5 plus w 2 times the function evaluated at psi 2 and w 3 multiplied by the function value at psi 3.

(Refer Slide Time: 36:47)



So now, we need six simultaneous equations because we have three sampling locations and then three weight factors psi 1, psi 2, psi 3, w 1, w 2, and w 3, okay? So by comparing the term by term a naught, a 1, a 2, a 3, a 4, a 5, we get six simultaneous equations. And by solving them, we can get w 1 is 5 by 9. w 2 is 8 by 9 and w 3 is 5 by 9. And psi 1 is minus of square root 0.6.

$$w_1 + w_2 + w_3 = 2$$

$$w_1\xi_1 + w_2\xi_2 + w_3\xi_3 = 0$$

$$w_1\xi_1^2 + w_2\xi_2^2 + w_3\xi_3^2 = \frac{2}{3}$$

$$w_1\xi_1^3 + w_2\xi_2^3 + w_3\xi_3^3 = 0$$

$$w_1\xi_1^4 + w_2\xi_2^4 + w_3\xi_3^4 = \frac{2}{5}$$

$$w_1\xi_1^5 + w_2\xi_2^5 + w_3\xi_3^5 = 0$$

By solving these equations,

$$w_1 = \frac{5}{9} \quad w_2 = \frac{8}{9}; \quad w_3 = \frac{5}{9}$$

$$\xi_1 = -\sqrt{0.6}; \quad \xi_2 = 0;$$

Psi 2 is 0 and psi 3 is plus square root of 0.6. So actually here I have done only up to a fifth order polynomial. But if you see any mathematics textbooks or finite element textbooks, they will give you a whole page.

(Refer Slide Time: 37:43)

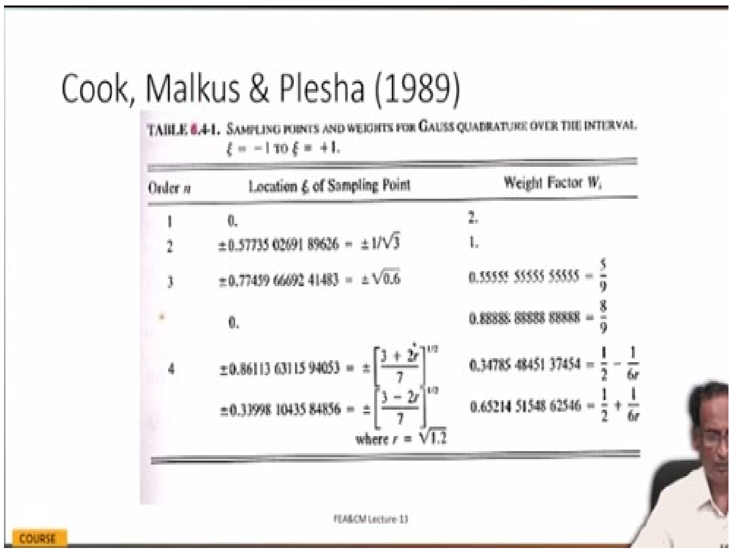


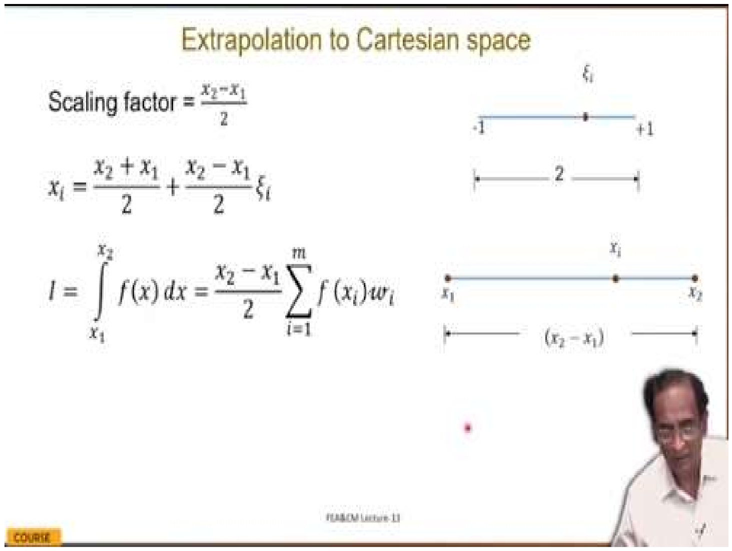
TABLE 6.4-1. SAMPLING POINTS AND WEIGHTS FOR GAUSS QUADRATURE OVER THE INTERVAL $\xi = -1$ TO $\xi = +1$.

Order n	Location ξ of Sampling Point	Weight Factor W_i
1	0.	2.
2	$\pm 0.57735\ 02691\ 89626 = \pm 1/\sqrt{3}$	1.
3	$\pm 0.77459\ 66692\ 41483 = \pm\sqrt{0.6}$	$0.55555\ 55555\ 55555 = \frac{5}{9}$
	0.	$0.88888\ 88888\ 88888 = \frac{8}{9}$
4	$\pm 0.86113\ 63115\ 94053 = \pm\left[\frac{3+2r}{7}\right]^{1/2}$	$0.34785\ 48451\ 37454 = \frac{1}{2} - \frac{1}{6r}$
	$\pm 0.33998\ 10435\ 84856 = \pm\left[\frac{3-2r}{7}\right]^{1/2}$ where $r = \sqrt{1.2}$	$0.65214\ 51548\ 62546 = \frac{1}{2} + \frac{1}{6r}$

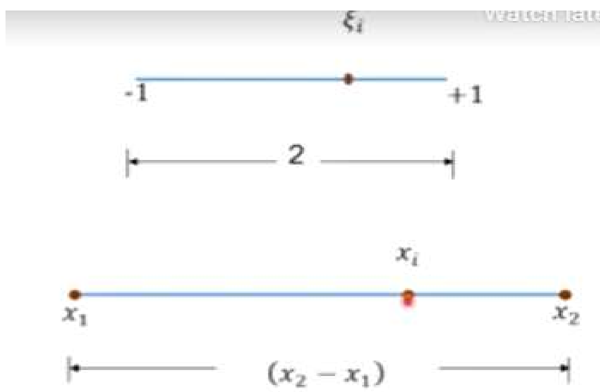
And so this particular one is photocopy from this textbook Cook, Malkus and Plesha. They have given up to fourth order four sampling point locations. And if you have four sampling points, we can exactly integrate up to seventh order polynomial, right? And so these are all the sampling point for 1-point integration it is 0 and 2. And for 2, plus or minus 1 by root 3 and 1.

And for 3, plus or minus square root of 0.6. And 5 by 9 and 8 by 9. And with four points it is a bit more complicated. But usually we do not go beyond 3-point integration because it becomes very expensive.

(Refer Slide Time: 38:38)



And now, we derived everything in the space of -1 to +1. But then our Cartesian space is in the x 1 to x 2 or y 1 to y 2 and so on. So how do we extrapolate? So this space in the in this natural space -1 to + 1, then in the Cartesian space, you have x 1 to x 2. And any arbitrary point psi i what does it correspond to in the Cartesian space x i. So the scale factor is this length divided by this length x 2 – x 1 by 2.



That is what we call is the mapping factor. And then any arbitrary point psi 1 we can map to x i based on this x 1 being equal to -1 and x 2 being equal to +1 in the reduced scale, okay? So our x i is the average value x 2 + x 1 by 2 plus x 2 - x 1 by 2 that is half length multiplied by psi i. So actually psi values are varying from -1 to +1, right? And so -1 means x 1 and +1 means x 2.

Scaling factor = $\frac{x_2 - x_1}{2}$

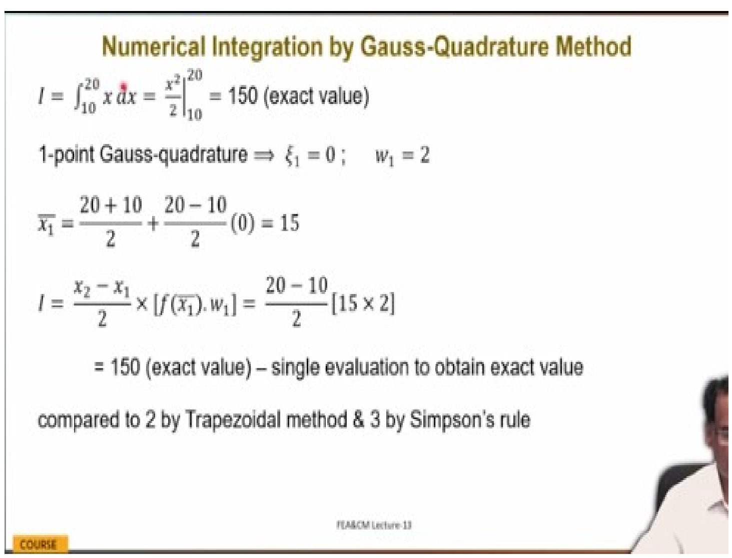
$$x_i = \frac{x_2 + x_1}{2} + \frac{x_2 - x_1}{2}\xi_i$$

$$I = \int_{x_1}^{x_2} f(x)\,dx = \frac{x_2 - x_1}{2}\sum_{i=1}^{m} f(x_i)w_i$$

And in between, if it is 0, it is exactly average value. And so we can in general write any arbitrary point psi i is mapped to a location x i in the Cartesian space as x 2 + x 1 by 2 plus x 2 - x 1 by 2 times psi i. And the integral value of this function, we can write as x 2 - x 1 by 2, that is the mapping factor multiplied by the function evaluated at different x i's multiplied by the corresponding weight factor.

So if we have one point integration, we have only psi 1 of 0 and weight factor is 2. And if you have 2-point integration psi is plus or minus 1 by root 3 and weight factor is 1 for each of them. And if you have three points plus or minus square root of 0.6 and 0, the weight factors are 5 by 9 or 8 by 9.

(Refer Slide Time: 41:18)



So now, let us apply this Gauss quadrature method for different polynomials. Let us say, let us take the first order polynomial, integral of x dx integrated from 10 to 20 that is 150. And let us apply one point Gauss quadrature. Because if your order of polynomial is 1, the number of sampling point that you require is 1 plus 1 by 2 that is 1, okay? So for the first order polynomial psi 1 is 0 and the weight factor is 2.

$$I = \int_{10}^{20} x\,dx = \frac{x^2}{2}\Big|_{10}^{20} = 150 \text{ (exact value)}$$

1-point Gauss-quadrature $\Rightarrow$ $\xi_1 = 0$; $w_1 = 2$

$$\overline{x_1} = \frac{20+10}{2} + \frac{20-10}{2}(0) = 15$$

$$I = \frac{x_2 - x_1}{2} \times [f(\overline{x_1}).w_1] = \frac{20-10}{2}[15 \times 2]$$

= 150 (exact value) – single evaluation to obtain exact value

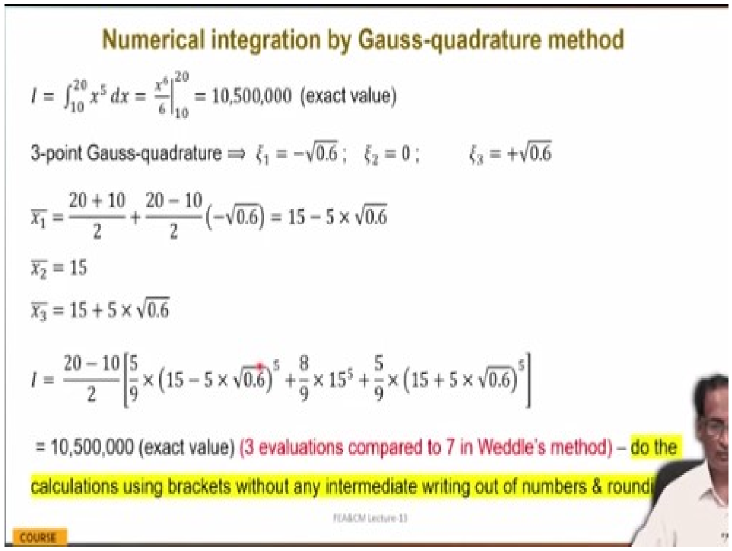
compared to 2 by Trapezoidal method & 3 by Simpson's rule

So the corresponding location in the Cartesian space, corresponding to psi 1 of 0 is x 1 bar is the average, in fact 20 + 10 by 2, plus 20 - 10 by 2 times zero, that is 15. And our integral value is x 2 – x 1 by 2 multiplied by the function evaluated at x 1 bar, multiplied by w 1, okay? So the x 2 – x 1 by 2, that is the mapping factor or the scale factor 20 - 10 by 2.

And the function at x 1 bar is 15 because the function itself is x multiplied by the w 1 2. And it is exactly equal to the value 150. And the advantage that we gained here is we used only one single computation, one function evaluation. Whereas if we apply the trapezoidal rule, you need to evaluate the function at x 1 and x 2. And if you use Simpson's rule, you need to evaluate at three points.

Whereas here, the Gauss quadrature method, we had just simple 1-point integration or one point evaluation, and then we got the exact value.

(Refer Slide Time: 43:25)



And let us apply this to a fifth order polynomial integrated from 10 to 20 that is 10,500,000. And this is the exact value. And if you have fifth order polynomial, we require three data points, three points for evaluation, psi of minus square root of 0.60 plus square root of 0.6. Then the weight factors are 5 by 9, 8 by 9 and 5 by 9, okay? And our x 1 bar is 15 minus 5 times square root of 0.6.

$$I = \int_{10}^{20} x^5\,dx = \frac{x^6}{6}\Big|_{10}^{20} = 10{,}500{,}000 \text{ (exact value)}$$

3-point Gauss-quadrature $\Rightarrow$ $\xi_1 = -\sqrt{0.6}$; $\xi_2 = 0$; $\xi_3 = +\sqrt{0.6}$

$$\overline{x_1} = \frac{20+10}{2} + \frac{20-10}{2}(-\sqrt{0.6}) = 15 - 5 \times \sqrt{0.6}$$

$$\overline{x_2} = 15$$

$$\overline{x_3} = 15 + 5 \times \sqrt{0.6}$$

$$I = \frac{20-10}{2}\left[\frac{5}{9} \times \left(15 - 5 \times \sqrt{0.6}\right)^5 + \frac{8}{9} \times 15^5 + \frac{5}{9} \times \left(15 + 5 \times \sqrt{0.6}\right)^5\right]$$

= 10,500,000 (exact value) (3 evaluations compared to 7 in Weddle's method)

And our x 2 bar is 15. And our x 3 bar is 15 plus 5 times square root of 0.6. And our function is x to the power 5. So our integral value is x 2 minus x 1 by 2, that is 20 - 10 by 2, multiplied by 5 by 9 times the function at psi 1, okay? And 8 by 9 times the function value at psi 2. And then sorry, 8 by 9 times the function value at psi 2 plus 5 by 9 times the function value at psi 3.

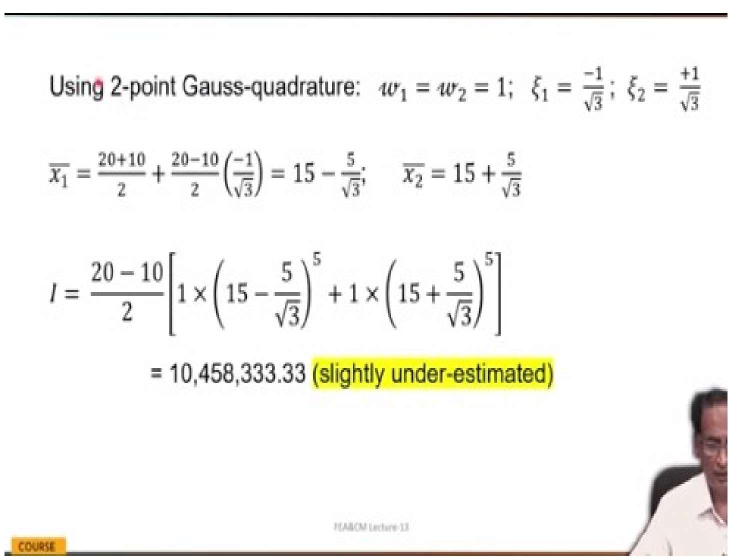
And our x 1 bar is 15 minus 5 times square root of 0.6. This is at psi 1. And at psi 2 of 0, our x 2 bar is 15. And then at the location 3 at psi 3 of square root of 0.6, this is 15 plus 5 times square root of 0.6. So if you evaluate this, you get exact value of 10,500,000. And I did not round off any of these numbers. I have just written like this. And you should also do the same thing.

In the calculator, you use your brackets intelligently and directly do this computation and do not round off any value. So if you write square root of 0.6, you will get some value. And depending on the number of digits that you use, your result might change. But in here, I did not round off any number and then I got exact value, okay? So we just needed only three sampling points that is three evaluations.

Whereas with Simpson's rule and then the trapezoidal rule we will not be able to exactly evaluate this. And when you apply the Weddle's rule, we need seven evaluations that is more than double the numbers here with the Gauss quadrature. And that is the main advantage. So whatever is the polynomial order, we can always go back and choose the number of sampling points.

So now, the question comes, what happens if you use lower number of sampling points? Say for this x to the power 5 this is the fifth order polynomial, we have used three points. But what if I use only one point or two points? Let us see what happens.

(Refer Slide Time: 46:56)



Now let us try with 2-point integration.

Using 2-point Gauss-quadrature: $w_1 = w_2 = 1$; $\xi_1 = \frac{-1}{\sqrt{3}}$; $\xi_2 = \frac{+1}{\sqrt{3}}$

$$\overline{x_1} = \frac{20+10}{2} + \frac{20-10}{2}\left(\frac{-1}{\sqrt{3}}\right) = 15 - \frac{5}{\sqrt{3}}; \qquad \overline{x_2} = 15 + \frac{5}{\sqrt{3}}$$

$$I = \frac{20-10}{2}\left[1 \times \left(15 - \frac{5}{\sqrt{3}}\right)^5 + 1 \times \left(15 + \frac{5}{\sqrt{3}}\right)^5\right]$$

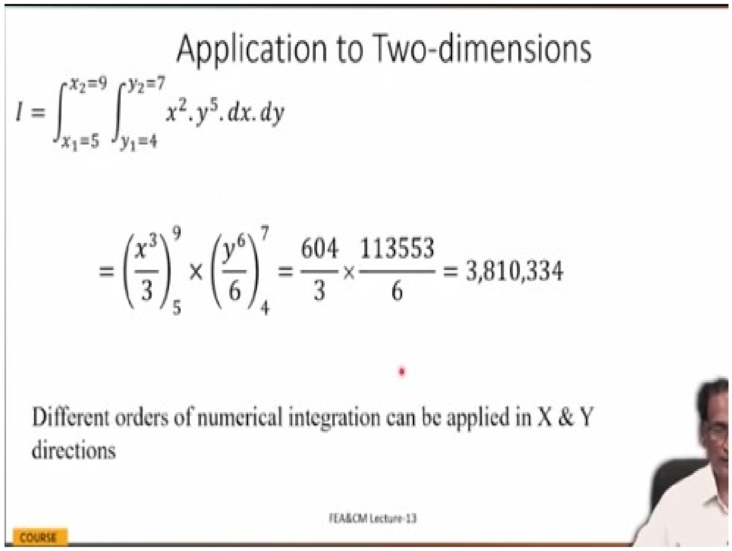
$$= 10{,}458{,}333.33 \text{ (slightly under-estimated)}$$

w 1 and w 2 are 1. Psi 1 is -1 by root 3. And psi 2 is +1 by root 3. So our locations are the Cartesian locations x 1 bar is 15 minus 5 by root 3 and x 2 bar is 15 plus 5 by root 3. And so if we do the integration, you get slightly less than 10,500,000. 10,500,000 is the exact result, but the numerical value is 10,458,333.33. And it is slightly underestimated.

And so this is the major difference between the Gauss quadrature method and the other methods. So in the Gauss quadrature if you use lesser number of data points or the integration points, your prediction is slightly on the lower side. And we can exploit this feature because when you analyze any continuum and the continuum that consists of infinite number of degrees of freedom.

And since we cannot consider infinite number, we only consider a finite number say some 1000, 10,000, 100,000. And because of that, our stiffness matrix is actually overestimated. I think I illustrated that through a simple example in the past. And so we can compensate the over stiffness because of not considering very large number of degrees of freedom by using slightly lower order of integration, okay? So that is the advantage with the Gauss quadrature method.

(Refer Slide Time: 48:56)



And this method is applicable even in two dimensions or three dimensions, okay? Let us apply this to two dimensional problem.

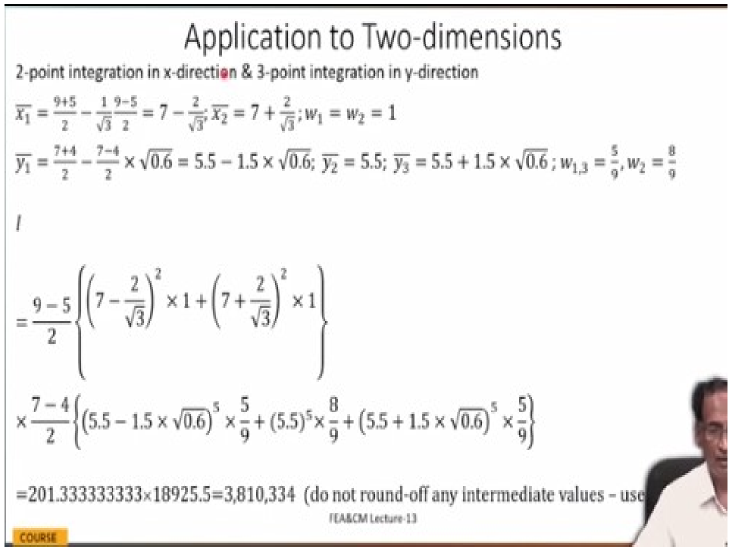
$$I = \int_{x_1=5}^{x_2=9} \int_{y_1=4}^{y_2=7} x^2.y^5.dx.dy$$

$$= \left(\frac{x^3}{3}\right)_5^9 \times \left(\frac{y^6}{6}\right)_4^7 = \frac{604}{3} \times \frac{113553}{6} = 3{,}810{,}334$$

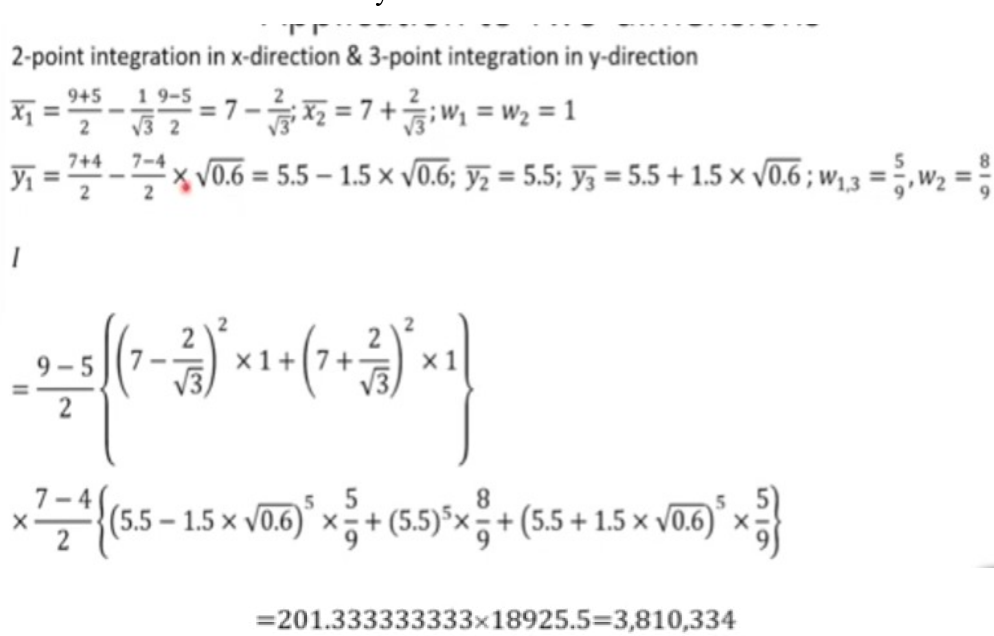
Let us say our function is x square y to the power 5. x is varying from 5 to 9, and y is varying from 4 to 7. And this exact value if you integrate is 3,810,334. And so here, you have two different orders of polynomial, 2 in the x and 5 for y.

And the advantage with the Gauss quadrature method is, you do not need to apply the same order of integration in both the methods, in both the directions. So in x direction we can apply a 2-point integration and in y direction we can apply the 3-point integration.

(Refer Slide Time: 49:55)



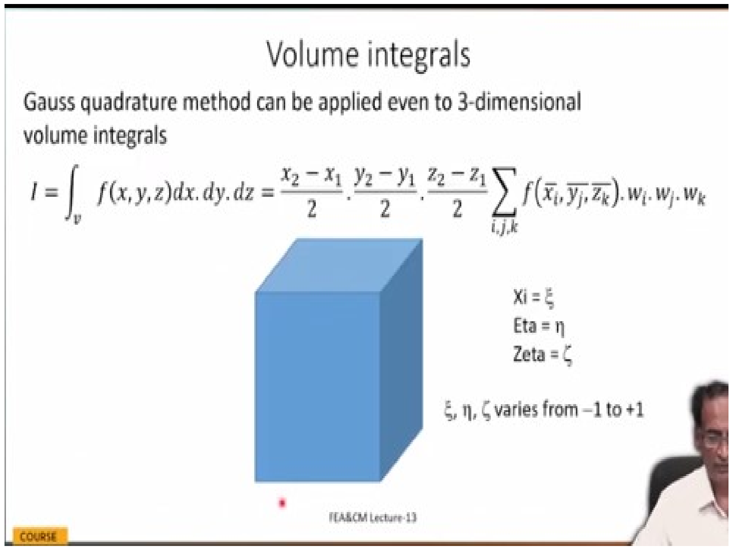
And so the sampling point locations in the x direction are plus or minus 1 by root 3. And so our x 1 bar is 7 minus 2 by root 3.



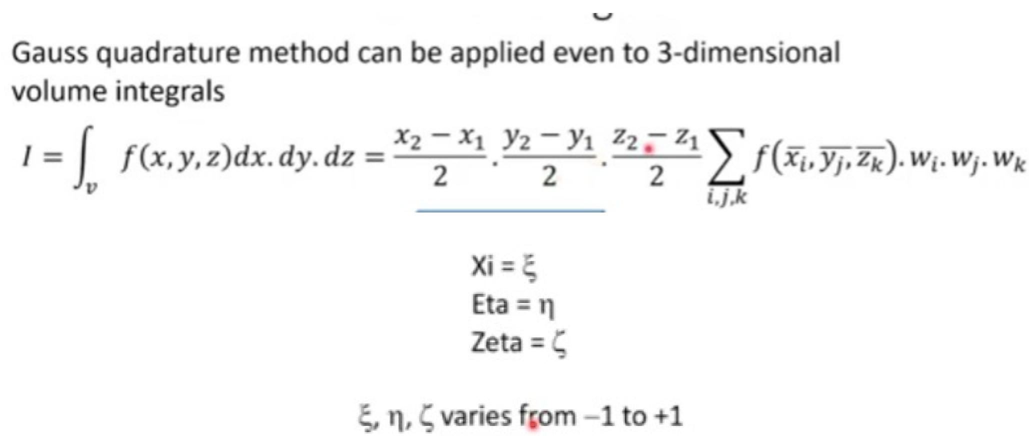
And our x 2 bar is 7 plus 2 by root 3. And the weight factors are w 1 and w 2 of 1. And in the y direction, we have three sampling points, y 1 bar, y 2 bar, and y 3 bar. Then the weight factors of 5 by 9 and 8 by 9. So if you do this integration numerically, this is what we see.

And once again, do not round off any numbers. And so if you do this, you will get the exact value that we get by integration 3,800,334, okay.

(Refer Slide Time: 50:55)



And we can also apply this in three dimensions. So actually, in two dimensions, we have the scale factor. I think I forgot to write the scale factor in two dimensions. But so here is the scale factor in three dimensions x 2 - x 1 by 2, y 2 - y 1 by 2, z 2 – z 1 by 2. So in two dimensions, we will be having only these two terms. In three dimensions, we have this.

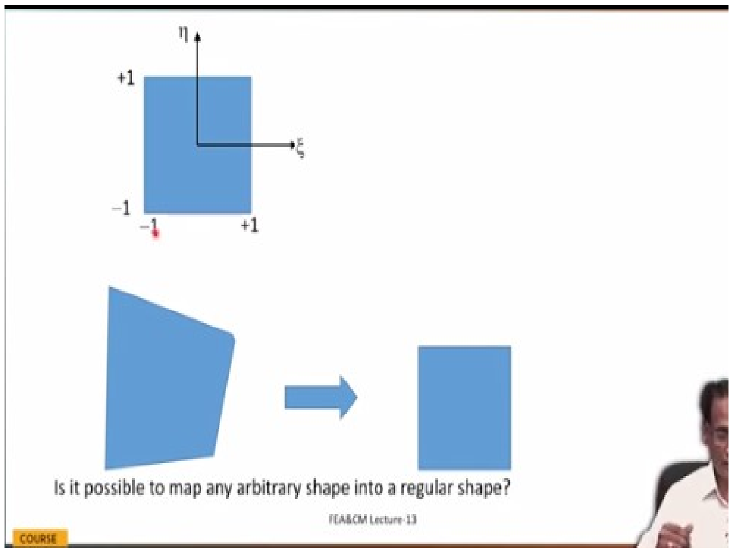


So we have three directions psi, eta and zeta each of them varying from -1 to +1. And the main advantage is whatever sampling points that you have in the psi direction, you have the same in the eta direction and also in the zeta direction. So if it is one point in psi one point in eta direction one point in zeta, it is only one point. So sampling point weight is 2.

And then with two points plus or minus 1 by root 3. And with three points plus or minus square root of 0.6 and 0 okay? So it is actually it is a versatile method and we can apply this for one dimensions, two dimensions, three dimensions, and it will give us the exact integral value. And in general, the procedure that we are going to adopt is when you need to evaluate any integral for load calculation, we use the exact integration.

And when it comes to stiffness matrix, we use a slightly lower order of integration so that our address because of not considering infinite number of degrees of freedom is slightly compensated, okay?

(Refer Slide Time: 52:50)



And the next question comes is, so we have derived our solution in the space of -1 to +1 and so, and we have seen how to map the space of Cartesian space x 1 to x 2 to this. And so, we can actually can we come out with a procedure so that any arbitrary shape can be converted into some regular shape, say square region for 2d or a cube in the three dimensional problems, can we do this?

And if we can do this, then all our problems are solved, because we can directly apply the numerical method that we had developed just now, okay? So that will be our approach in the next few classes, okay? So if you, I think this is the last slide. So if you have any questions, please write to this email and then I will respond back.

So just summarize, in this class, we had seen the different methods of a numerical integration and we have developed the Gauss quadrature method of numerical integration. And the main advantage is, depending on the order of polynomial that you have, you can choose the number of sampling points. So if you have a fifth order polynomial, we can use number of sampling points is 5 plus 1 by 2 that is 3 okay?

And if you have seventh order polynomial you can use four points and so on okay? So thank you very much. We will meet in the next class.