FEM & Constitutive Modelling in Geomechanics

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Lecture - 14
Classical Methods for Developing Shape Functions

Let us continue from our previous classes. And we have seen that the shape functions play the major role, because without shape functions, we cannot do any finite element calculations. We need the shape functions for interpolation and also for forming the B matrix and then for doing all the calculations like n transpose bdv or b transpose bdv and so on.

And let us see how we can define or develop the shape functions by the classical methods. By classical I mean, the methods that were originally developed, the generalized coordinate method. And later, we have this Lagrange's methods. And then, after this, we have more recent ones like isoperimetric and so on. And before we go into more advanced elements, let us look at the classical methods for developing the shape functions.

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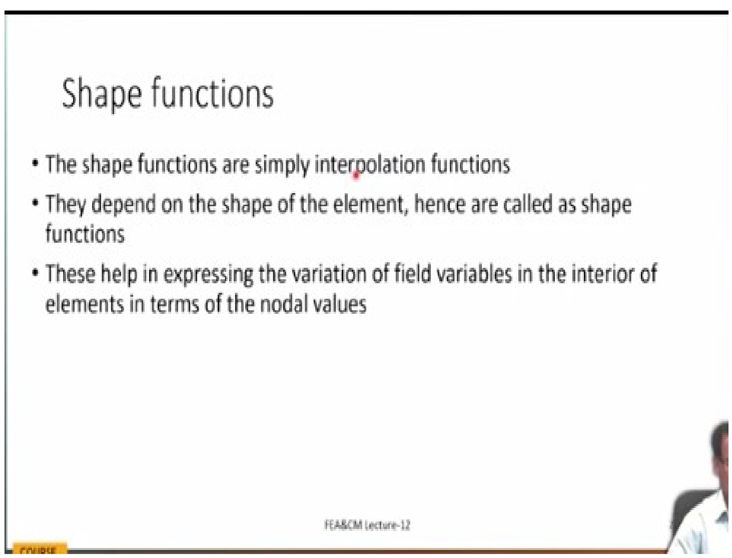

See these shape functions, they are basically interpolation functions. And they because they depend on the shape of the element, they are called the shape functions in the finite element context. And the shape functions, they express the variation of the nodal variables over the element.

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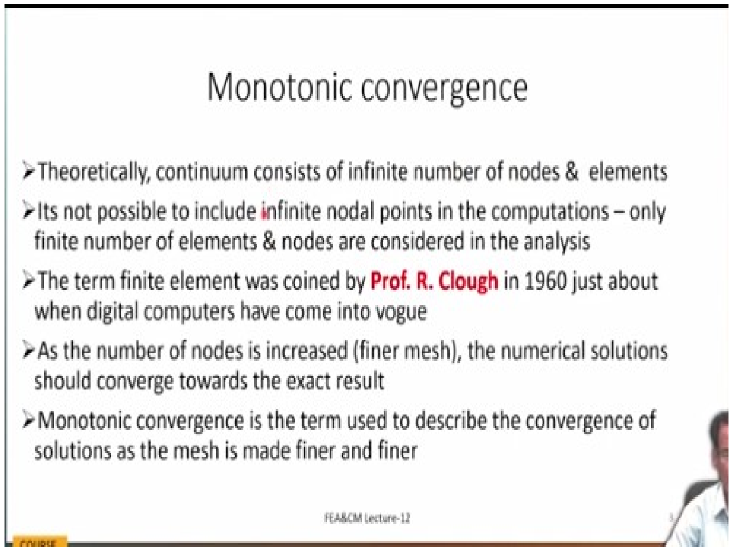

Then they also help us in getting our B matrix and so on. And theoretically, the continuum can have infinite number of nodes and elements. But it is not possible to include infinite numbers. We can only include finite number of nodes and finite number of elements. And in fact, the term finite element was coined by Professor Clough in 1960.

That is just about the time when the digital computers were coming in, and the power of the computers was also gradually increasing. In fact, in those days, it used to increase exponentially, but now it has more or less stabilized, okay? And the Professor Clough in 1960, he coined the word finite element, because actually, we can have infinite number of elements and infinite number of degrees of freedom.

But we are considering only certain number of finite elements. From that this word finite element was coined. And our requirement for monotonic convergence is as we include more and more number of elements by decreasing the size of the elements, we should or finer mesh, our numerical solution should be tending towards the exact result. And that is called as the monotonic convergence.

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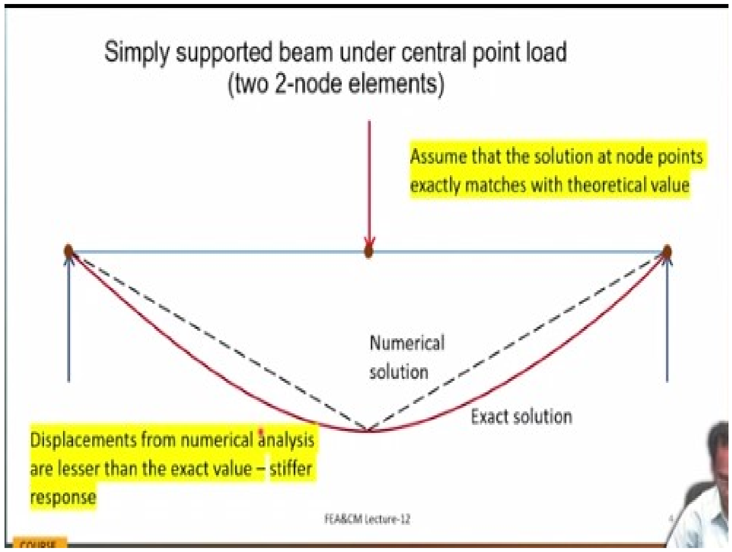

And so let us see what exactly we mean by comparison between our numerical result and then the, and then our theoretical result, okay? Let us see it in the context of simple, simply supported beam. Let us say that we have two elements and the three nodes. And the two ends, it is a simply support, simply supported beam, so the displacement is zero.

And let us say that whatever numerical method that we have, it is exact. So at the central point also our numerical result is exactly matching with the exact result. But then what about in between, in between these two nodes? There is a large variation between the theoretical solution that is given by this red line and this dotted line. So this is the numerical error that we have.

And so actually we see that at most, in most locations, our predicted displacement is lesser than the theoretical one. So that means that our stiffness is overestimated. Only when we have a higher stiffness, we get lower displacement. So we are inadvertently over predicting the stiffness. And that over prediction can be reduced by introducing more number of nodes in the mesh.

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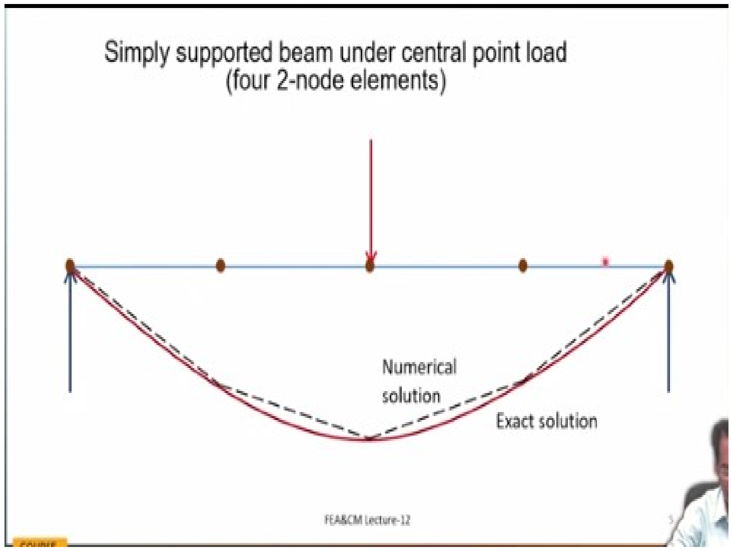

Let us see the same thing for larger number of nodes and elements. In this case, we have four elements and five nodes. And once again, let us assume that at all these nodes, we are exact, our solution is exactly matching let us say, and that is only an assumption, that may or may not be true. And in between we see that there is still some deviation between the theoretical result and then the finite element result.

But this difference is smaller compared to what we had before. Here the difference is large, whereas here it is smaller.

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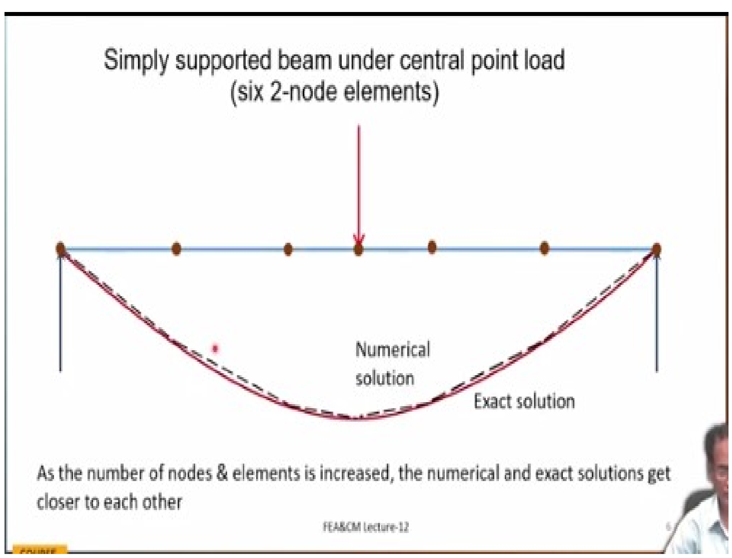

Then let us make it more finer. Let us consider six elements 1, 2, 3, 4, 5, 6. And in this case, the elements around the load are made smaller. And then here we see that, we are even more closer to the theoretical result. And, so as we are including more number of nodes and elements in the mesh, we can come closer to the theoretical result and that is what we mean by monotonic convergence and it is not an automatic process.

So we have to take some precautions so that as we include more number of nodes in the mesh, our solution is tending towards the exact result, okay?

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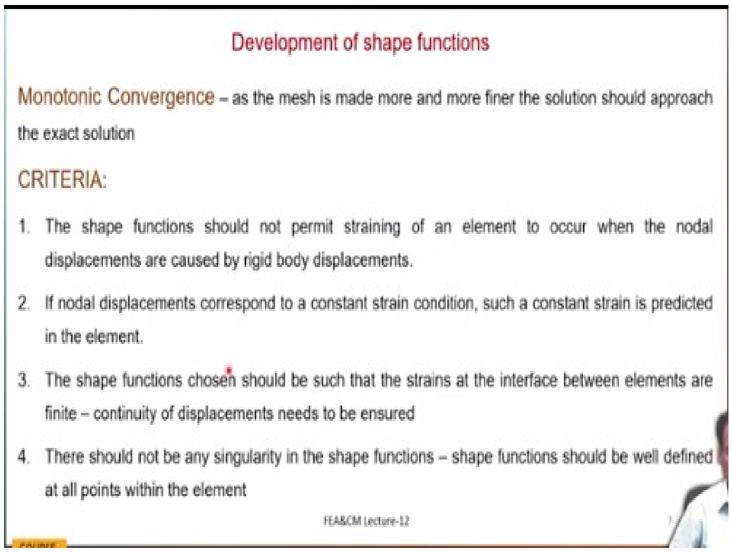

So the development of shape functions is a very important step in all the finite element analysis and we should be very systematic so that our results are also as accurate as possible. And our aim is to satisfy the requirements for the monotonic convergence. And as the mesh is made more and more finer, the solution should approach the exact solution. And the criteria are very simple.

The shape function should not permit straining of an element to occur when the nodal displacements are caused by rigid body displacements. That is what we had seen earlier. See if this pen whether it is here or here there are no strains or stresses within the element, within the pen, because it has undergone only rigid body displacement, like all the points on the pen have undergone the same displacement.

So that means that there is no relative displacement between two points on the pen or no strains. And if there are no strains, there are no stresses and that is what we mean by the first condition that if you subject a body to rigid body displacement, the shape function should be such that they should not predict any strain.

And then the second condition is if the nodal displacements correspond to a constant strain condition, we should be able to predict the same constant strain, okay? That we will see with an example. And the shape functions should be such that the strains at the interface between elements are finite and the continuity of displacements need to be ensured, okay?

Because we will come across the interface between different elements and at those points also, we need some continuity of displacements and then the continuity of strains. And there should not be any singularity in the shape functions. The shape function should be well defined at all the points within the elements.

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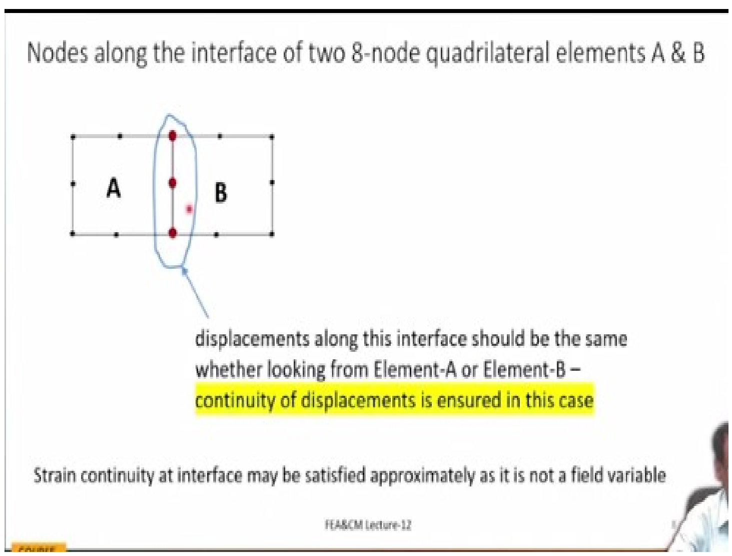

What we mean by continuity is like this. Let us consider a mesh consisting of 8-node quadrilateral elements. And let us look at two elements A and B. And these three highlighted nodes are common for both element A and element B. And if you look at the displacement at these points, whether you are looking from element A or whether you are looking from element B, we should get the same displacements.

And if we get the same displacements, that means that we are maintaining the continuity of the displacements, okay? And the displacement at these three points calculated from A will include the displacements at all these 8 points. Whereas from B, they will include all these 8 nodes, okay? And the strain continuity or the interface may not be satisfied or may be satisfied only approximately, because the strain is not our field variable.

It is only a derivative quantity. And the continuity of strain can be simulated by making the mesh as fine as possible. So if you have a very coarse mesh, we may not be able to get the strain continuity between the elements. But as the mesh is made more and more finer, we will get the strain continuity. So here we have taken both elements A and B of the same type. Both are 8-node quadrilaterals.

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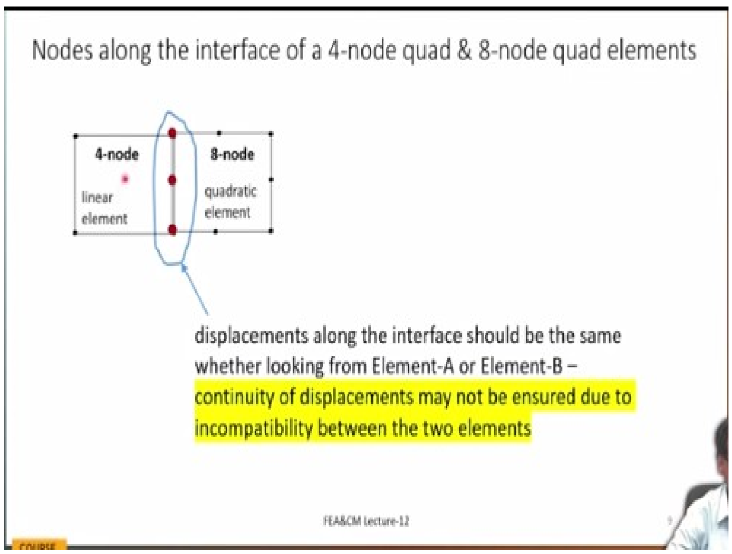

And now, let us consider an interface between a 4-node quadrilateral and an 8-node quadrilateral. Say a 4-node element is a, it has only 4 nodes. So along each line, we have only a linear function, okay? So it is a, we call it as a linear element. Whereas a 8-node element, it is a quadratic element. It is along each line, there are 3 nodes, okay?

And so the displacement that we predict from this side, from the 4-node element may not match with the displacements that we get from 8 node, 8-node element, because one is a linear function of displacements and the other is a quadratic function of displacements. So we will not be able to maintain the continuity or we call it as incompatibility between the two elements.

So one way to overcome that is by reducing the size of the elements so that we get a better representation. And invariably, we will have to do this type of thing like linking a 4-node elements to 8-node elements and so on, especially in the transition zones, okay?

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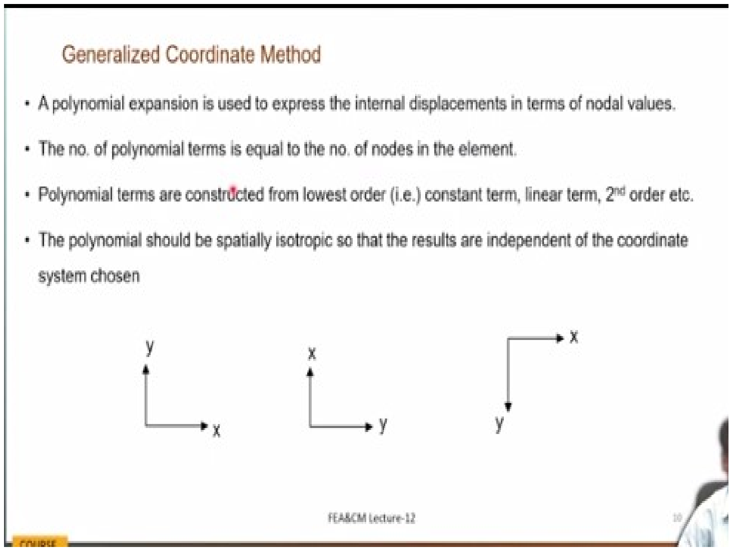

So our generalized coordinate method or the polynomial expansion used to express the internal displacements in terms of the nodal values, okay? And then the number of polynomial terms is equal to the number of nodes in the element. And then the polynomial terms are constructed from the lowest order that is the constant term, linear, quadratic, cubic and so on.

And the polynomial terms should be spatially isotropic so that the results are independent of the coordinate system, whether you take a coordinate system like this x and y, or x and y, x vertical and y horizontal, or y going down instead of going up, whatever may be the coordinate system, the results should be the same okay, numerical value of the result.

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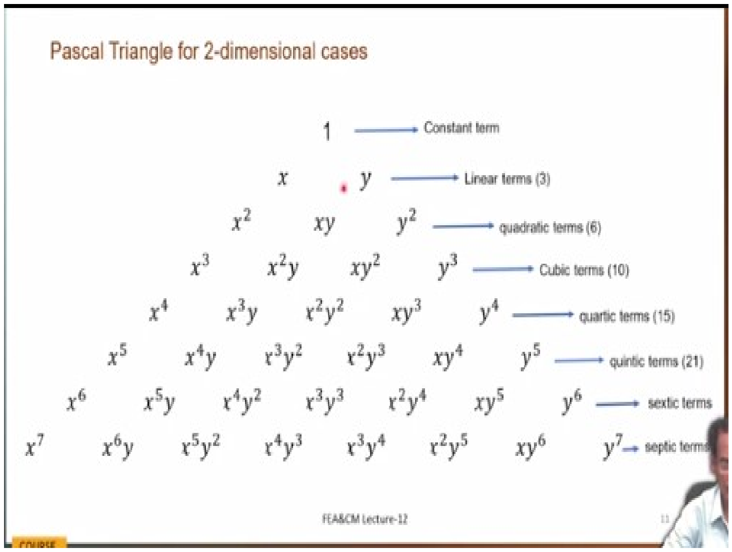

To help us in and satisfying the spatial isotropy we can take the help of this Pascal triangle that gives you all the polynomial terms of different order; constant term, linear x and y, quadratic x square xy and y square. And then cubic x cube, x square y, xy square, y cube and so on, okay?

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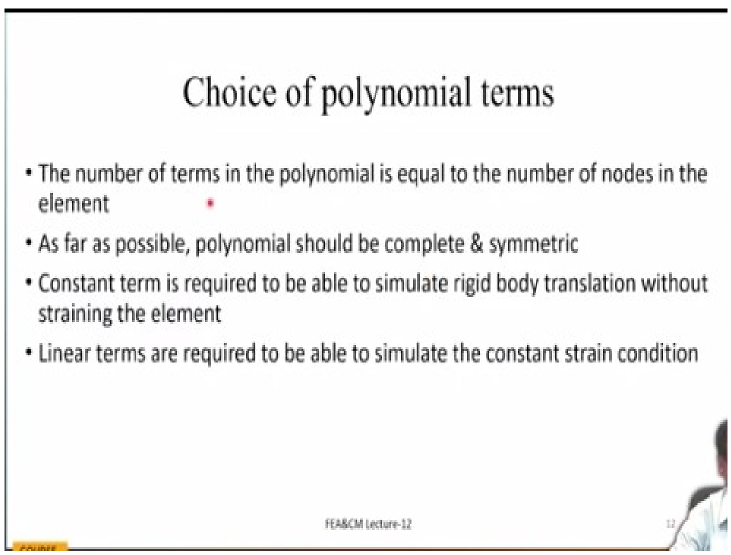

And the choice of polynomial terms, say the number of terms in the polynomial should be equal to the number of nodes in the element. And as far as possible, the polynomial should be complete and symmetric. Complete in the sense should include all the lower order terms before you start including higher order terms, okay?

And the constant term is required to be able to simulate rigid body translation without straining the element. Then the linear terms are required to be able to simulate the constant strain condition. So these three, they are observed only from experience, okay? That we will see with an example, what happens if you do not have a constant term or if you do not have a linear term and so on, okay?

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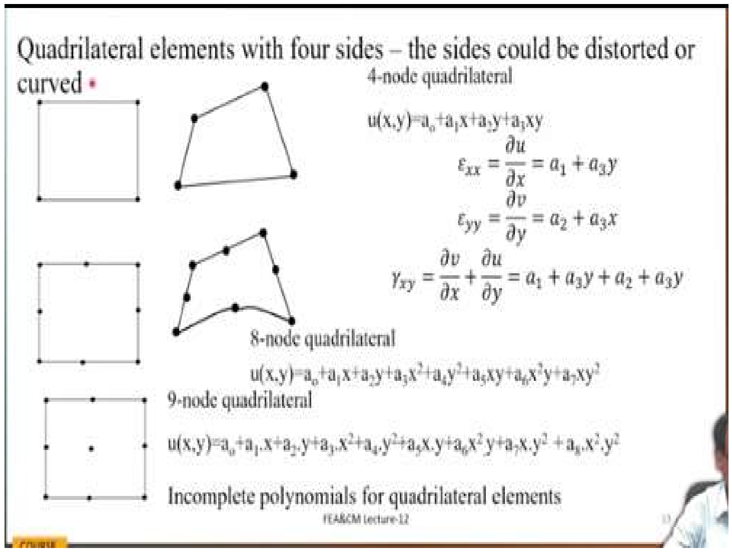
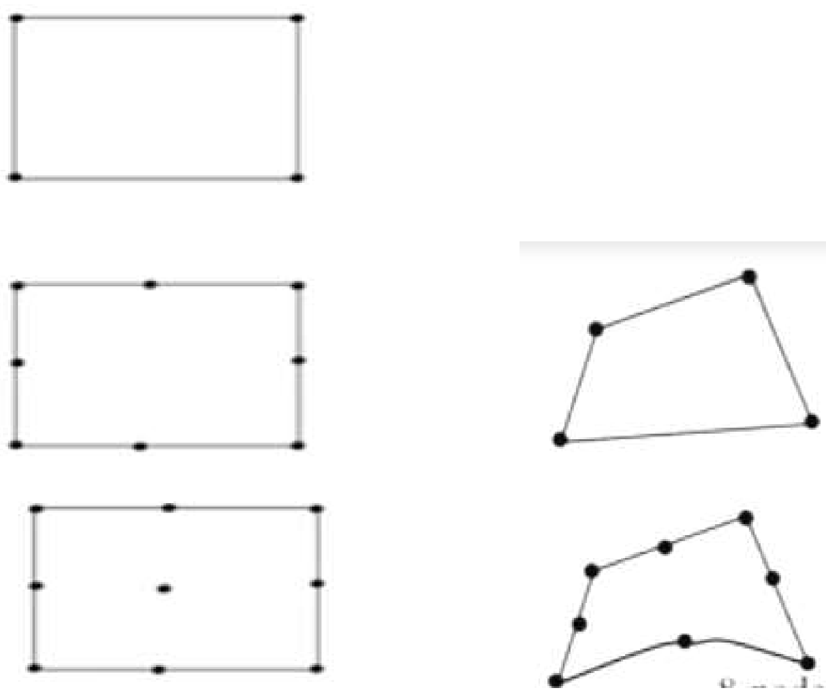


And these are some of the typical elements and then the polynomials. See you can have a four node quadrilateral like this, the rectangular element or a distorted element like this.



And the polynomial for a 4-node quadrilateral can be a naught plus a 1 x plus a 2 y plus a 3 x y. And this is a spatial isotropic expansion, because for every x there is a y, okay? And our strains are epsilon xx is doh u by doh x.

$$u(x,y)=a_o+a_1x+a_2y+a_3xy$$

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}=a_1+a_3y$$

$$\varepsilon_{yy}=\frac{\partial v}{\partial y}=a_2+a_3x$$

$$\gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=a_1+a_3y+a_2+a_3y$$

That is alpha 1 or a 1 plus a 3, a 1 plus a 3 y and epsilon yy is doh v by doh y. That is a 2 plus a 3 x and gamma xy is doh v by doh x plus doh u by doh y. That is equal to a 1 plus a 3 y plus a 2 plus a 3, sorry it should be x, okay?

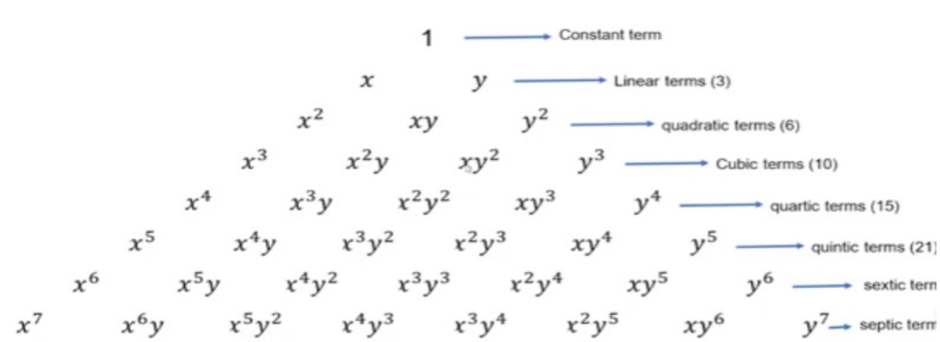
And for an 8-node quadrilateral, our polynomial can be like this, a naught plus a 1 x plus a 2 y. That is including the constant term and then the linear terms.

8-node quadrilateral

$$u(x,y)=a_o+a_1x+a_2y+a_3x^2+a_4y^2+a_5xy+a_6x^2y+a_7xy^2$$

Then the quadratic terms x square a 4 y square plus a 5 x y plus higher order terms a 6 x square y plus a 7 x y square.

Pascal Triangle for 2-dimensional cases



Actually, let us look at our Pascal triangle. So when you have a 8-node quadrilateral, we have this constant term 1 and then the linear terms x and y. Then the quadratic terms x square, x y and y square. So these are six terms and we need two more terms.

And we can include either x cube and y cube, so that we have the spatial symmetry or x square y and x y square. So how we decide whether to take x cube and y cube or x square y and x y square is very simple. See in the 8-node quadrilateral along each line, we have only three nodes, right? And if you have three data points, the maximum order of polynomial that you can fit is two.

So we can choose these two terms x square y and xy square. Then, we are satisfying the spatial isotropy, for every x there is a y. So even if you interchange x and y coordinates, it does not matter because our polynomial expansion is especially symmetric. The 9-node quadrilateral, the polynomial is like this. And for choosing the polynomial terms for higher order elements requires the observations from the Pascal triangle.

9-node quadrilateral

$$u(x,y)=a_o+a_1.x+a_2.y+a_3.x^2+a_4.y^2+a_5x.y+a_6x^2\,y+a_7x.y^2\ +a_8.x^2.y^2$$

See for 8-node quadrilateral, we have the terms up to x square y and xy square. And for the 9-node quadrilateral we can include this term, x square and y square because in the 9-node quadrilateral also along each line, we have only three nodes. So that means that maximum order of polynomial that you can fit is only two. So we can choose x square y square as our ninth term.

And as we see, all the polynomials that we have for the quadrilaterals, they are incomplete. See for the 4-node quadrilateral we have a naught plus a 1x plus a 2y plus a 3 xy but we are missing x square and y square terms. And in the 8-node quadrilateral we are missing x cube and y cube. See for the 8-node quadrilateral we included these two terms x square y and xy square but missed out an x cube and y cube.

And for the 9-node quadrilateral we have included x square y square, but we did not include x cubed y xy cube y 4 and x 4, okay? So all the quadrilateral elements, they have an incomplete polynomial.

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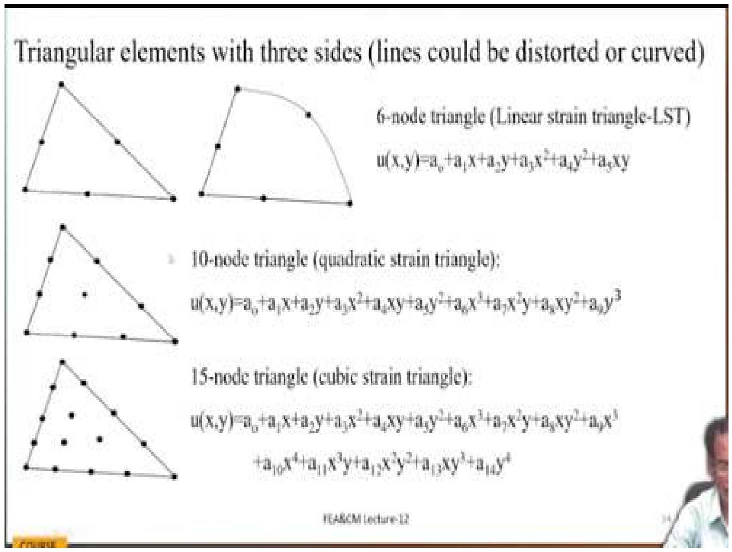
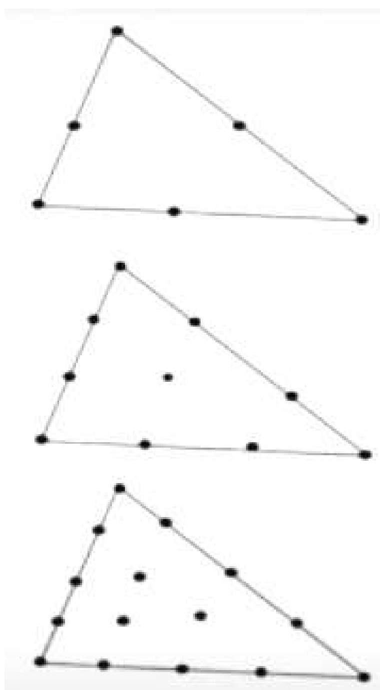
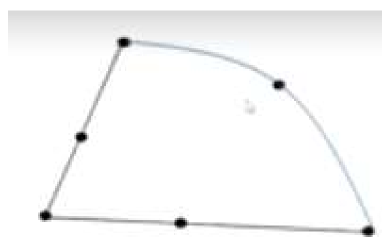


And let us look at the triangles.





The 3-node triangle we have already seen. Let us look at the 6-node triangle, 10-node and 15-node and so on. See the 6-node triangle, it need not have straight edges, we can have curved edges like this. And our 6-node triangle, we also call it as a linear strain triangle LST because the strain is linear and the polynomial expansion is a naught plus a 1x plus a 2y plus a 3x square plus a 4 y square plus a 5xy.

6-node triangle (Linear strain triangle-LST)

$$u(x,y)=a_o+a_1x+a_2y+a_3x^2+a_4y^2+a_5xy$$

And so for the 10-node triangle, it is a quadratic strain, because we have a cubic variation of displacements.

10-node triangle (quadratic strain triangle):

$$u(x,y)=a_o+a_1x+a_2y+a_3x^2+a_4xy+a_5y^2+a_6x^3+a_7x^2y+a_8xy^2+a_9y^3$$

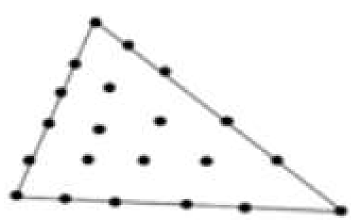
So that means that the strain will be one order less, that is x square y square. So it is quadratic. In the 15-node triangle, it is called as a cubic strain triangle, because our displacements are of the fourth order. So the strain should be of third order. And this is the expansion for a 15-node cubic strain triangle.

15-node triangle (cubic strain triangle):

$$u(x,y)=a_o+a_1x+a_2y+a_3x^2+a_4xy+a_5y^2+a_6x^3+a_7x^2y+a_8xy^2+a_9x^3$$
$$+a_{10}x^4+a_{11}x^3y+a_{12}x^2y^2+a_{13}xy^3+a_{14}y^4$$

See, if you look at the number of terms in the Pascal triangle up to any particular row, up to second row, we have three terms and up to third row we have nine terms three plus three nine. Sorry, three plus three is six, not nine, okay? And then up to fourth row, we have 10, 6 plus 4 is 10. And then in the quadratic terms in the fifth row, we have five terms 1, 2, 3, 4, 5, so 15 terms.

And if you look at the number of nodes in the triangles is 3, 6, 10, 15. And then next one will be 21 and so on.



21-node triangle:

$$u(x,y)=a_o+a_1x+a_2y+a_3x^2+a_4xy+a_5y^2+a_6x^3+a_7x^2y+a_8xy^2+a_9x^3$$
$$+a_{10}x^4+a_{11}x^3y+a_{12}x^2y^2+a_{13}xy^3+a_{14}y^4+ a_{15}.x^5+a_{16}.x^4.y +a_{17}.x^3.y^2+a_{18}.x_2.y^3+a_{19}.x.y^4+a_{20}.y^5$$

So our number of nodes in the Pascal triangle allows us to have complete polynomial like alpha naught plus alpha 1x plus alpha 2y or including alpha 2x square, alpha 3xy alpha 4y square and so on, okay? So our triangular elements, they allow us to have a complete polynomial.

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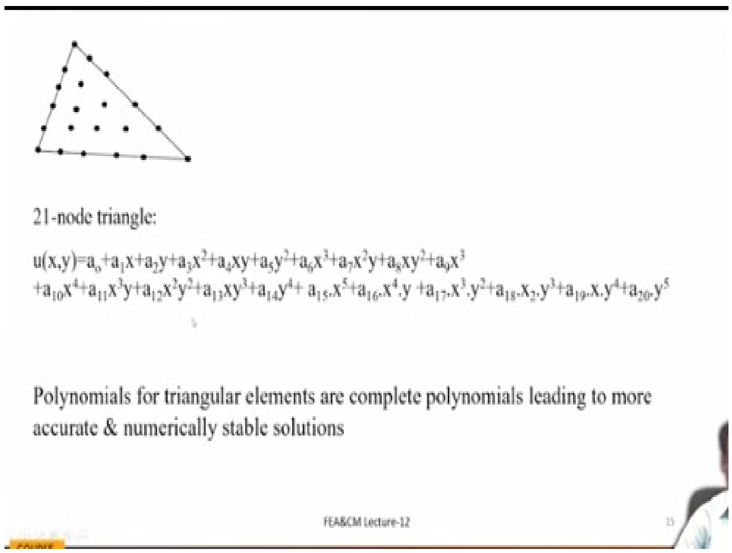

And this next one is a 21-node triangle and these triangular elements they have a complete polynomial and that gives us some advantage. In fact, we get more stable numerical solutions, because our polynomial is complete and because of that some spurious modes of displacements are not excited. Whereas we will have problems with the quadrilaterals like 4-node quadrilaterals and so on, okay?

So in the geotechnical program PLAXIS, that is one of the most popular commercial programs, they give you the option of only two elements. One is a 6-node triangle and the other is a 15-node triangle. Both are known to perform well. And depending on the nature of problem that you have, you can either choose a 6-node triangle or a 15-node triangle.

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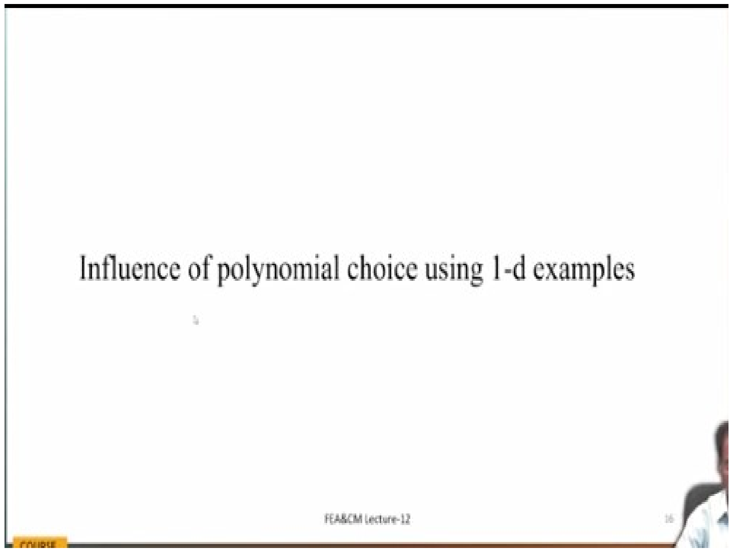

And we can look at the influence of the polynomial choice using 1 -d examples.

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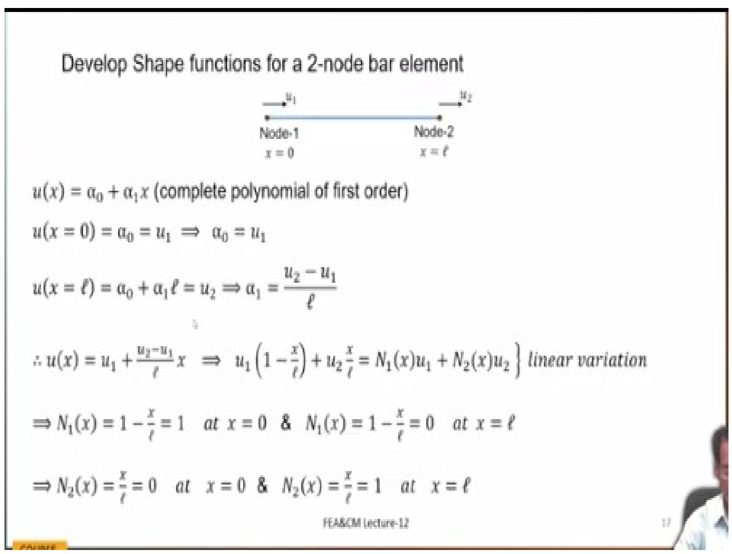
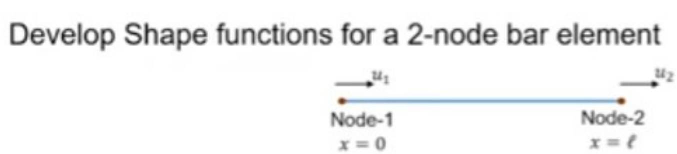


Like say if you want to do any two dimensional examples, we have to go in for finite element analysis, we cannot do by hand. And by doing some simple calculations by hand, we can see what is the effect of different orders of polynomial. Let us consider once again a 2-node bar element. It is an axial element. So we have only one displacement u. Node 1 displacement is u 1. Node 2 displacement is u 2.



And our polynomial is alpha naught plus alpha 1x. And it is a complete polynomial of the first order because we have constant term alpha naught and the linear term alpha 1

x. So if you determine the shape function that we had already done several times, our u(x) is u 1 N 1 where N 1 is 1 minus x by l plus u 2 times N 2. N 2 is x by l. And our N 1 and N 2 are either 1 or 0. 1 at their own node and 0 at the other node.

$u(x) = \alpha_0 + \alpha_1 x$ (complete polynomial of first order)

$$u(x=0) = \alpha_0 = u_1 \Rightarrow \alpha_0 = u_1$$

$$u(x=\ell) = \alpha_0 + \alpha_1 \ell = u_2 \Rightarrow \alpha_1 = \frac{u_2 - u_1}{\ell}$$

$$\therefore u(x) = u_1 + \frac{u_2-u_1}{\ell}x \Rightarrow u_1\left(1-\frac{x}{\ell}\right) + u_2\frac{x}{\ell} = N_1(x)u_1 + N_2(x)u_2 \Big\} \textit{ linear variation}$$

$$\Rightarrow N_1(x) = 1-\frac{x}{\ell} = 1 \quad at \ x=0 \ \ \& \ \ N_1(x) = 1-\frac{x}{\ell} = 0 \quad at \ x=\ell$$

$$\Rightarrow N_2(x) = \frac{x}{\ell} = 0 \quad at \quad x=0 \ \ \& \ \ N_2(x) = \frac{x}{\ell} = 1 \quad at \quad x=\ell$$

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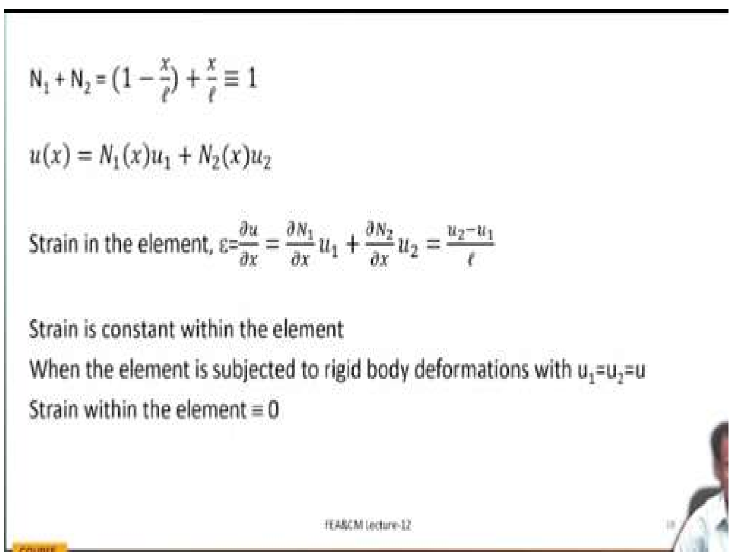

And our N 1 plus N 2 is exactly equal to 1. And the strain within the element epsilon is doh u by doh x. That is, you will get u 2 minus u 1 by l and the strain is constant within the element. And then if you subject this element to a rigid body displacement that is u 2 and u 1 are the same and the strain within the element is 0.

$$N_1 + N_2 = \left(1-\frac{x}{\ell}\right) + \frac{x}{\ell} \equiv 1$$

$$u(x) = N_1(x)u_1 + N_2(x)u_2$$

Strain in the element, $\varepsilon = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x}u_1 + \frac{\partial N_2}{\partial x}u_2 = \frac{u_2-u_1}{\ell}$

So if you subject this element to a rigid body displacement will predict a 0 strain. And if you apply a strain corresponding to or displacements corresponding to constant strain, your strain is constant within the element. So that means that this polynomial that we have assumed for the 2-node bar element will satisfy our requirements for the monotonic convergence.

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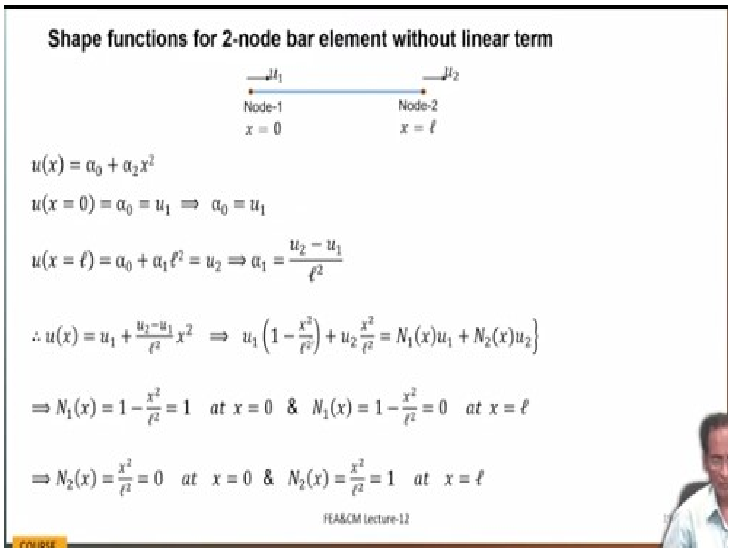

And let us repeat the same problem. But now, let us do without the linear term.

Shape functions for 2-node bar element without linear term

Let us take two terms, u(x) as alpha naught plus alpha 2 x square. So I have neglected the linear term alpha 1 x, but included the second order term alpha 2 x square, okay? So if you go through the process, our N 1 is 1 minus x square by l square and N 2 is x square by l square.

$$u(x) = \alpha_0 + \alpha_2 x^2$$

$$u(x=0) = \alpha_0 = u_1 \implies \alpha_0 = u_1$$

$$u(x=\ell) = \alpha_0 + \alpha_1 \ell^2 = u_2 \implies \alpha_1 = \frac{u_2 - u_1}{\ell^2}$$

$$\therefore u(x) = u_1 + \tfrac{u_2 - u_1}{\ell^2} x^2 \implies u_1\left(1 - \tfrac{x^2}{\ell^2}\right) + u_2 \tfrac{x^2}{\ell^2} = N_1(x) u_1 + N_2(x) u_2 \}$$

$$\implies N_1(x) = 1 - \tfrac{x^2}{\ell^2} = 1 \quad at \ x = 0 \ \ \& \ \ N_1(x) = 1 - \tfrac{x^2}{\ell^2} = 0 \quad at \ x = \ell$$

$$\implies N_2(x) = \tfrac{x^2}{\ell^2} = 0 \quad at \ \ x = 0 \ \ \& \ \ N_2(x) = \tfrac{x^2}{\ell^2} = 1 \quad at \ \ x = \ell$$

And our N 1 is 1 at x is equal to 0, and zero at x is equal to l. And similarly, N 2 at x is equal to 0 is 0, and at x is equal to l, that is at its own node, it is 1, okay?

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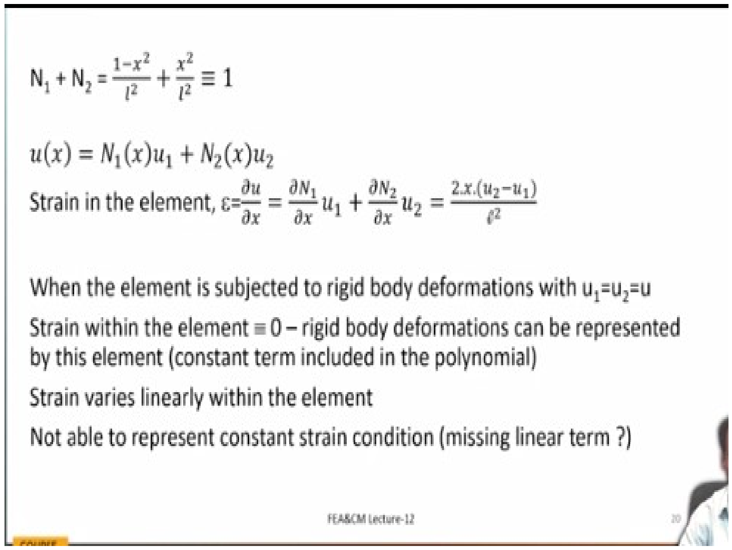

And our N 1 plus N 2 is exactly equal to 1. And then, now let us calculate the strain. Epsilon is doh u by doh x, that is doh N 1 by doh x u 1 plus doh N 2 by doh x u 2. That is 2x times u 2 minus u 1 by l square. So if you subject this body to rigid body displacements, that is u 2 is equal to u 1, you will get zero strain. But then, you see the strain is varying linearly within the element.

$$N_1 + N_2 = \frac{1-x^2}{l^2} + \frac{x^2}{l^2} \equiv 1$$

$$u(x) = N_1(x)u_1 + N_2(x)u_2$$

Strain in the element, $\varepsilon = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 = \frac{2.x.(u_2 - u_1)}{\ell^2}$

So if you subject this body to some constant strain like let us say, you set the u 1 to 0 and u 2 to some value, you will see that the strain is varying linearly along the element because it is a function of x. So we are not able to represent the constant strain condition and because it is missing the linear term. And so we can say that this element may not satisfy the, does not satisfy the monotonic convergence requirement.

So if you use this element, we may not get good results, we may not get the monotonic convergence, okay?

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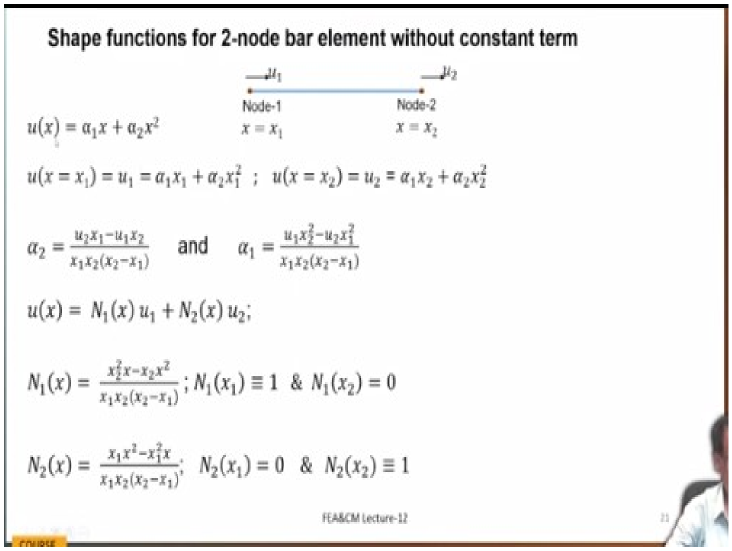
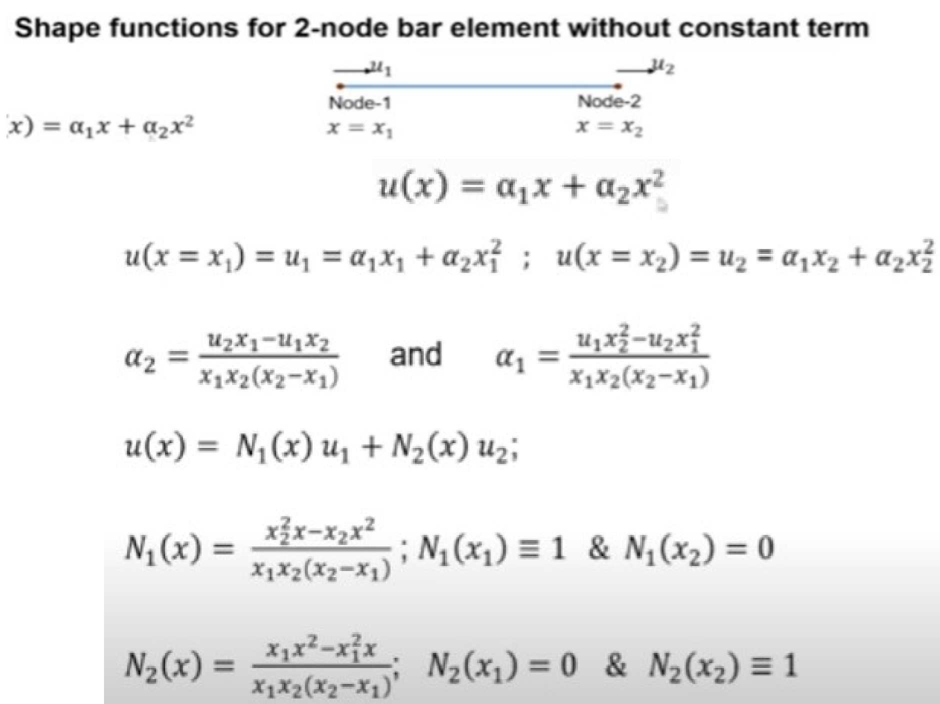


So now, let us do one more attempt. Now, without the constant term, the u(x) is alpha 1 x plus alpha 2 x square, okay? And we get some functions like this.



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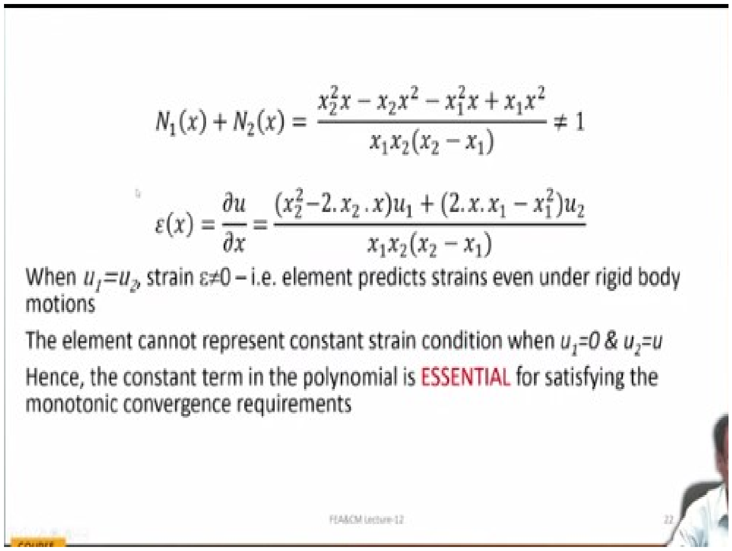

And then our N 1 plus N 2, we see that it is not equal to 1. And our strain within the element is doh u by doh x is also a bit complicated function.

$$N_1(x) + N_2(x) = \frac{x_2^2 x - x_2 x^2 - x_1^2 x + x_1 x^2}{x_1 x_2 (x_2 - x_1)} \neq 1$$

$$\varepsilon(x) = \frac{\partial u}{\partial x} = \frac{(x_2^2 - 2.x_2.x)u_1 + (2.x.x_1 - x_1^2)u_2}{x_1 x_2 (x_2 - x_1)}$$

And if you substitute u 1 is equal to u 2, the strain is not 0. So that means that the element predicts strain even when we subject the body to rigid body motions. That is because we do not have the constant term.

See if we are able to, to be able to represent rigid body displacements without developing strains, we must include the constant term. And in this particular case, we have not included the constant term. So we fail to represent the rigid body motion without strains. And this element can also represent constant strain. So if you substitute u 1 of 0 and u 2 of some value, we have this x in the equation.

So that means that your strain is going to vary linearly. So by neglecting the constant term we fail to represent the rigid body motion without straining and then the even the constant strain state. Whereas, previously when we included the constant term but missed out on the linear term, we were able to represent the rigid body motion without straining, but then we failed to represent the constant strain state.

And here, we failed in both. Both in the rigid body and also the constant strain. See you might be wondering why? See we have the linear term, so we should be able to represent constant strain condition. But then see even the rigid body motion also is a constant strain condition, because it is like whether I have the pen here or there, the strain is 0 all over, and that is also a constant strain.

It is only thing is the strain value is 0. So even the constant strain condition that we get under rigid body motion that is 0 strain that is also a constant strain. And so without the constant term, alpha naught, we will not be able to represent the constant strain conditions. And although you have this linear term, it does not help very much, because you do not have the constant term.

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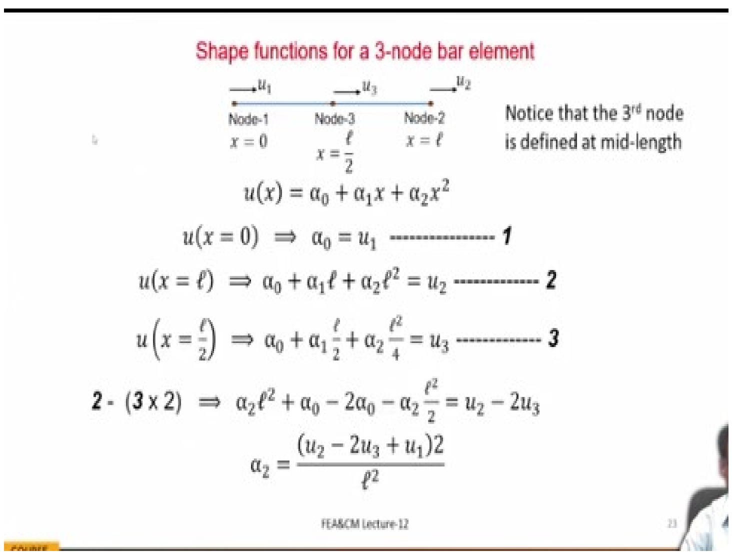
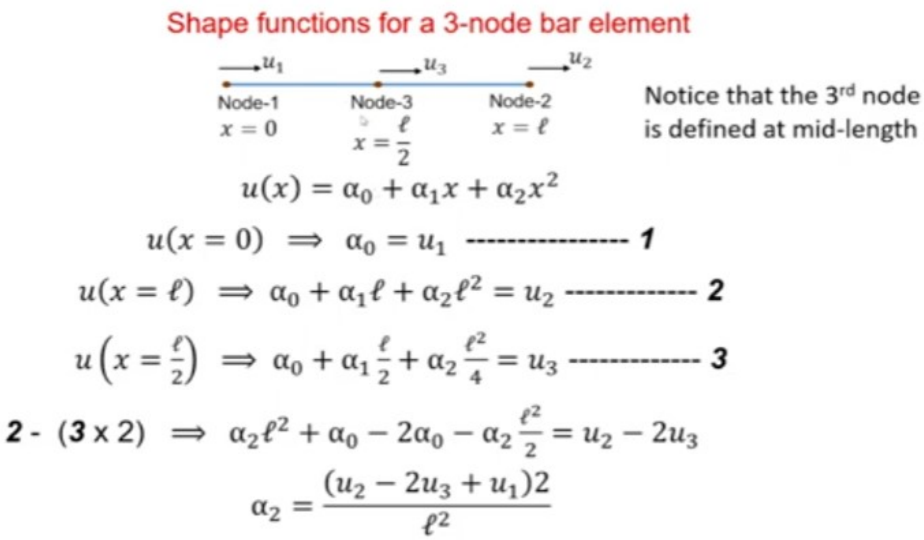


So it is very important that when we develop our shape functions, we are systematic and we include all the lower order terms. So we extend the same thing for a 3-node bar element and we assume a polynomial like this alpha naught plus alpha 1 x plus alpha 2 x square. And this is a complete polynomial, because we have all the terms up to x square, okay?



And you notice that the third node is at the mid length, node 1 and node 2. See for defining a bar element, we require minimum two nodes. And naturally, we will keep these nodes at the two ends. And if you want to place more number of nodes then you can choose some interior points. In this particular case, the node 3 is placed at mid length, but it could be placed anywhere else.

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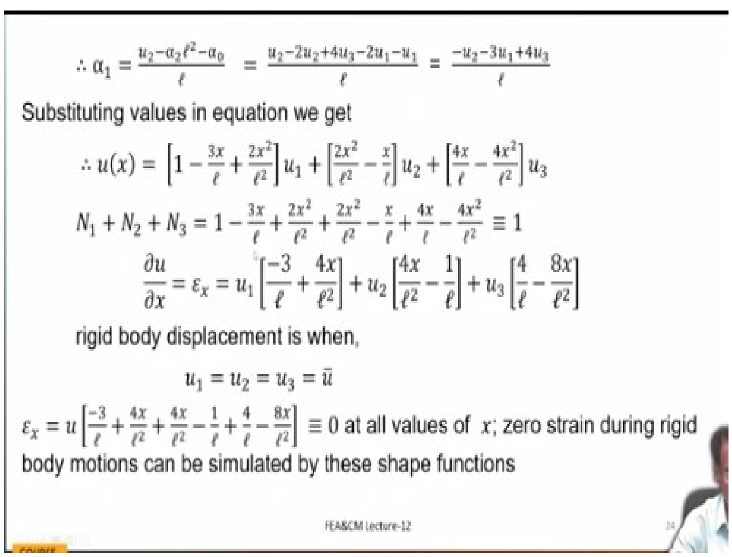

And we can determine alpha naught alpha 1 and alpha 2. And we had seen this example earlier, we see that N 1 plus N 2 plus N 3 is exactly equal to 1. And if you subject this element to rigid body displacements as u 1 is equal to u 2 is equal to u 3 is u bar our epsilon is 0. So that means that we are able to move the body as a rigid body without developing any strains.

$$\therefore \alpha_1 = \frac{u_2-\alpha_2\ell^2-\alpha_0}{\ell} = \frac{u_2-2u_2+4u_3-2u_1-u_1}{\ell} = \frac{-u_2-3u_1+4u_3}{\ell}$$

Substituting values in equation we get

$$\therefore u(x) = \left[1-\frac{3x}{\ell}+\frac{2x^2}{\ell^2}\right]u_1 + \left[\frac{2x^2}{\ell^2}-\frac{x}{\ell}\right]u_2 + \left[\frac{4x}{\ell}-\frac{4x^2}{\ell^2}\right]u_3$$

$$N_1+N_2+N_3 = 1-\frac{3x}{\ell}+\frac{2x^2}{\ell^2}+\frac{2x^2}{\ell^2}-\frac{x}{\ell}+\frac{4x}{\ell}-\frac{4x^2}{\ell^2} \equiv 1$$

$$\frac{\partial u}{\partial x} = \varepsilon_x = u_1\left[\frac{-3}{\ell}+\frac{4x}{\ell^2}\right]+u_2\left[\frac{4x}{\ell^2}-\frac{1}{\ell}\right]+u_3\left[\frac{4}{\ell}-\frac{8x}{\ell^2}\right]$$

rigid body displacement is when,

$$u_1 = u_2 = u_3 = \bar{u}$$

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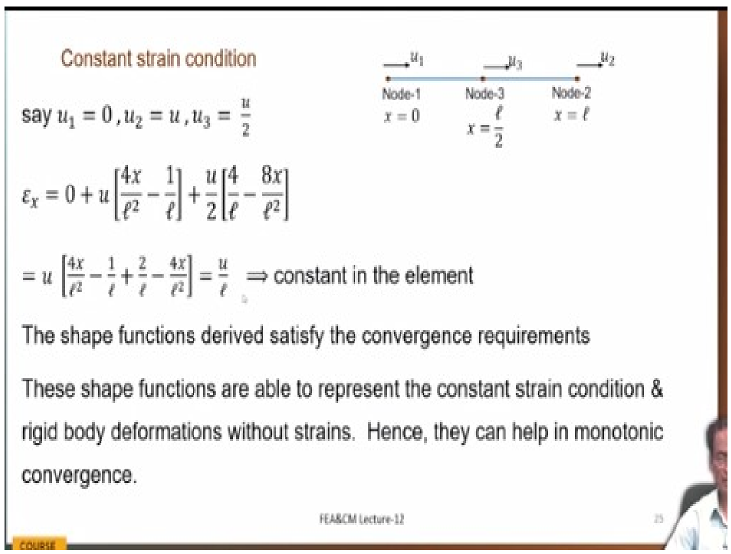
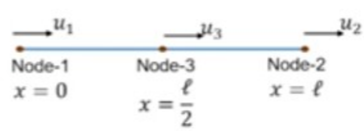


And then if you subject the element to some displacement field corresponding to constant strains that is u 1 is 0 u 2 is u and u 3 is u by 2, we do predict the strain as u by l that is constant within the element.

Constant strain condition



say $u_1 = 0, u_2 = u, u_3 = \frac{u}{2}$

$$\varepsilon_x = 0 + u\left[\frac{4x}{\ell^2}-\frac{1}{\ell}\right]+\frac{u}{2}\left[\frac{4}{\ell}-\frac{8x}{\ell^2}\right]$$

$$= u\left[\frac{4x}{\ell^2}-\frac{1}{\ell}+\frac{2}{\ell}-\frac{4x}{\ell^2}\right] = \frac{u}{\ell} \Rightarrow \text{constant in the element}$$

So by choosing the constant term and the linear term in the polynomial, we are able to represent the rigid body motion without developing strains.

And then if your displacement field is corresponding to constant strain, we are able to represent that. So this element can be satisfying the monotonic convergence requirement. So if you increase the number of elements we will be approaching the theoretical result, okay?

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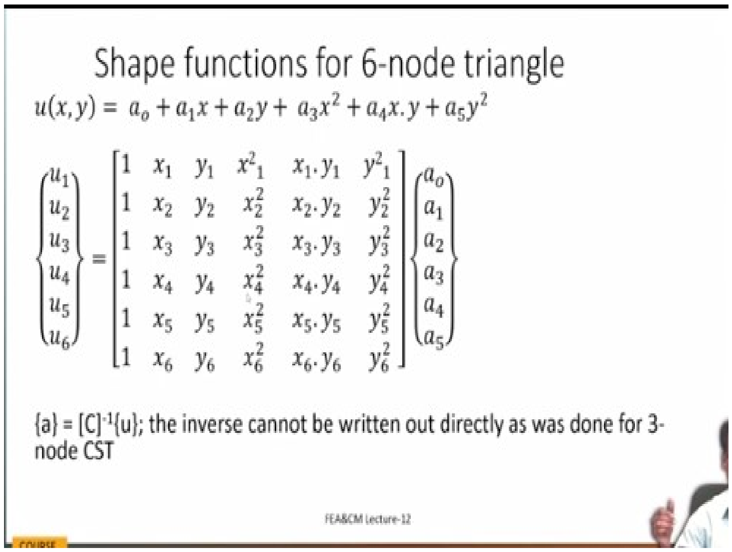

So now, let us continue for higher order elements like a 6-node triangle. So earlier, we had derived the shape functions for a 3-node triangle. And if you remember, the 3-node triangle, we got the coordinate matrix, a 3 by 3 matrix and that we can easily invert analytically.

Shape functions for 6-node triangle

$$u(x,y) = a_o + a_1x + a_2y + a_3x^2 + a_4x.y + a_5y^2$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x^2{}_1 & x_1.y_1 & y^2{}_1 \\ 1 & x_2 & y_2 & x_2^2 & x_2.y_2 & y_2^2 \\ 1 & x_3 & y_3 & x_3^2 & x_3.y_3 & y_3^2 \\ 1 & x_4 & y_4 & x_4^2 & x_4.y_4 & y_4^2 \\ 1 & x_5 & y_5 & x_5^2 & x_5.y_5 & y_5^2 \\ 1 & x_6 & y_6 & x_6^2 & x_6.y_6 & y_6^2 \end{bmatrix} \begin{Bmatrix} a_o \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix}$$

And after we invert, we have grouped all the displacement terms under u 1, u 2, u 3.

{a} = $[C]^{-1}${u}; the inverse cannot be written out directly as was done for 3-node CST

And then came out with our shape functions N i is alpha naught alpha i plus a plus b i x plus c i y divided by 2 delta and so on. But here, when you have a 6-node triangle, your coordinate matrix is a 6 by 6 matrix. So we cannot directly write out the inverse of this matrix.

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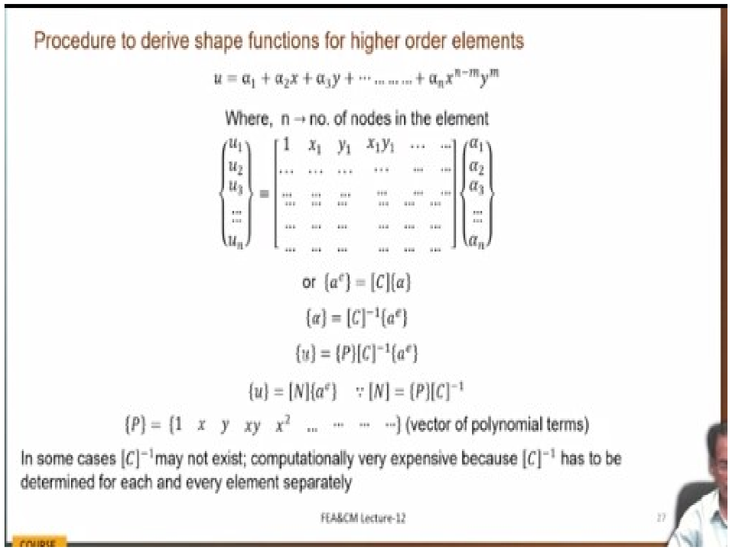

So we need some other procedure. And that procedure is explained here. Actually it is a procedure to derive shape functions for higher order elements using an automated process. So let us say that your u is alpha 1 plus alpha 2 x plus alpha 3 y and so on, okay? And our u is this coordinate matrix C, multiplied by this alpha. Alphas are the generalized coordinates.

So our alpha can be determined as C inverse a e, and our u can be P, that is the polynomial series 1, x, y and so on, okay? And so our N can be P times C inverse, okay? And the P is the polynomial vector of polynomial terms 1, x, x square and so on; y, y square and so on. And the C is the inverse of this matrix. But only problem here is C inverse could be very difficult to obtain.

Procedure to derive shape functions for higher order elements

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \cdots \dots\dots\dots + \alpha_n x^{n-m} y^m$$

Where, n → no. of nodes in the element

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{Bmatrix}$$

$$\text{or } \{a^e\} = [C]\{\alpha\}$$

$$\{\alpha\} = [C]^{-1}\{a^e\}$$

$$\{u\} = \{P\}[C]^{-1}\{a^e\}$$

$$\{u\} = [N]\{a^e\} \quad \because [N] = \{P\}[C]^{-1}$$

$\{P\} = \{1 \quad x \quad y \quad xy \quad x^2 \quad \dots \quad \cdots \quad \cdots \quad \cdots\}$ (vector of polynomial terms)

In some cases $[C]^{-1}$ may not exist; computationally very expensive because $[C]^{-1}$ has to be determined for each and every element separately

Like let us say if you are doing, if you are using a 21-node triangle, you need to invert 21 by 21 triangle. And if you have some 1000 elements in the mesh, you will be spending lot of time on just simply inverting these matrices. And after you invert the matrix, then you get your shape functions and then proceed with the rest of the analysis.

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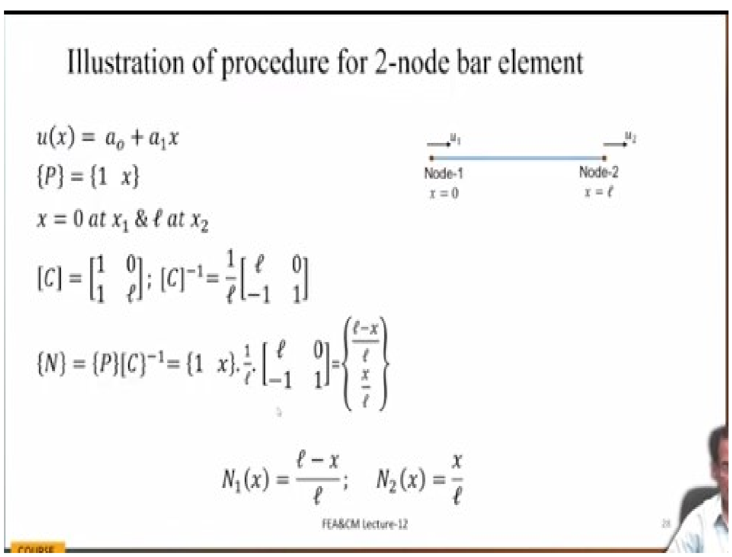

So let us illustrate this process for a 2-node bar element a naught plus a 1 x and this is the coordinate matrix 1, 0, 1, l.

Illustration of procedure for 2-node bar element

$u(x) = a_o + a_1 x$

$\{P\} = \{1 \quad x\}$

$x = 0 \; at \; x_1 \;\&\; \ell \; at \; x_2$

$[C] = \begin{bmatrix} 1 & 0 \\ 1 & \ell \end{bmatrix}; \; [C]^{-1} = \frac{1}{\ell}\begin{bmatrix} \ell & 0 \\ -1 & 1 \end{bmatrix}$

$\{N\} = \{P\}[C]^{-1} = \{1 \quad x\}.\frac{1}{\ell}.\begin{bmatrix} \ell & 0 \\ -1 & 1 \end{bmatrix} = \begin{Bmatrix} \frac{\ell - x}{\ell} \\ \frac{x}{\ell} \end{Bmatrix}$

$$N_1(x) = \frac{\ell - x}{\ell}; \quad N_2(x) = \frac{x}{\ell}$$

The C inverse is 1 by l; l, 0, -1, 1. And the N is P times C inverse. P is the matrix of this polynomial terms 1 and x. And our C inverse is l, 0, -1, 1. And then the C inverse is this. And then we have this shape function and this product will give you the shape functions. And N 1 is l minus x by l and N 2 is x by l.

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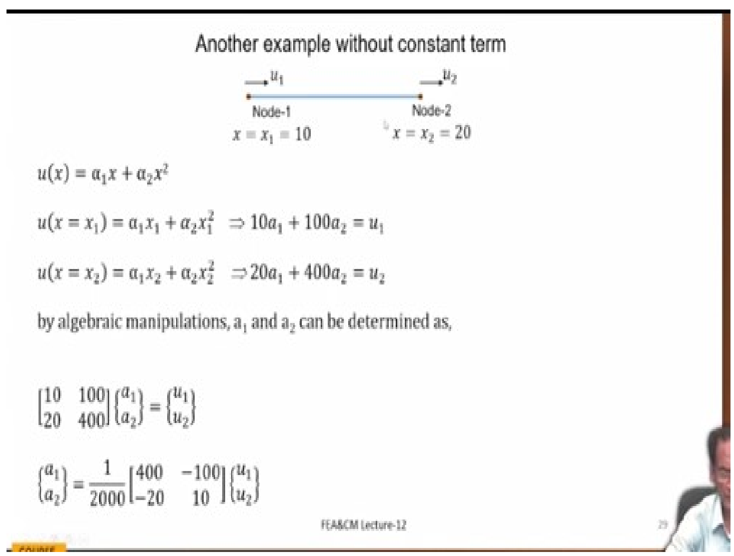

And we can look at one more example without the constant term. Let us take x 1 at 10 and x 2 of 20 so that if you have an x of 0, then it becomes difficult to find both alpha 1 and alpha 2. So because of that, the element is translated so that x 1 is not 0, okay? So our u at x is equal to x 1 is this and the u at x is equal to x 2 is u 2, that is this.

Another example without constant term

$$u(x) = \alpha_1 x + \alpha_2 x^2$$

$$u(x = x_1) = \alpha_1 x_1 + \alpha_2 x_1^2 \quad \Rightarrow 10a_1 + 100a_2 = u_1$$

$$u(x = x_2) = \alpha_1 x_2 + \alpha_2 x_2^2 \quad \Rightarrow 20a_1 + 400a_2 = u_2$$

by algebraic manipulations, a_1 and a_2 can be determined as,

$$\begin{bmatrix} 10 & 100 \\ 20 & 400 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{2000} \begin{bmatrix} 400 & -100 \\ -20 & 10 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

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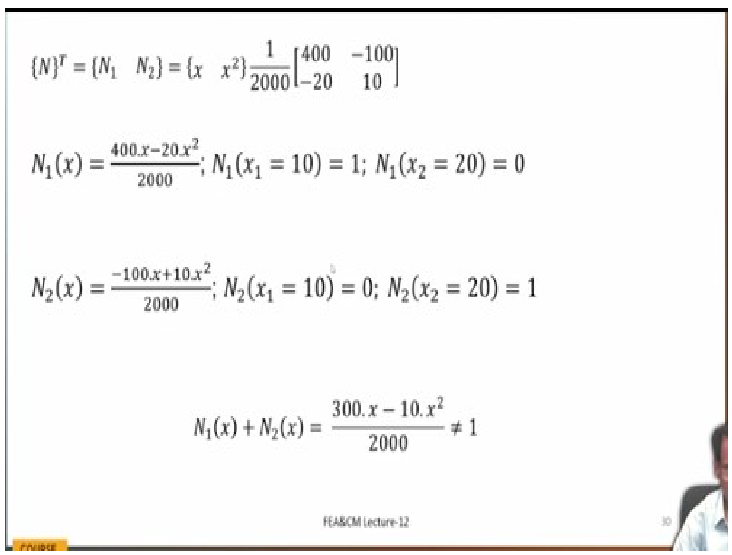

And by solving we get our N 1 as 400 x minus 20 x square by 20,000. And N 2 is minus 100 x plus 10 x square by 20,000.

$$\{N\}^T = \{N_1 \quad N_2\} = \{x \quad x^2\}\frac{1}{2000}\begin{bmatrix} 400 & -100 \\ -20 & 10 \end{bmatrix}$$

$$N_1(x) = \frac{400.x - 20.x^2}{2000};\ N_1(x_1 = 10) = 1;\ N_1(x_2 = 20) = 0$$

$$N_2(x) = \frac{-100.x + 10.x^2}{2000};\ N_2(x_1 = 10) = 0;\ N_2(x_2 = 20) = 1$$

$$N_1(x) + N_2(x) = \frac{300.x - 10.x^2}{2000} \neq 1$$

And then if you add up N 1 and N 2, you do not get 1. You get some other value. So that means that you cannot assure the monotonic convergence with this type of element.

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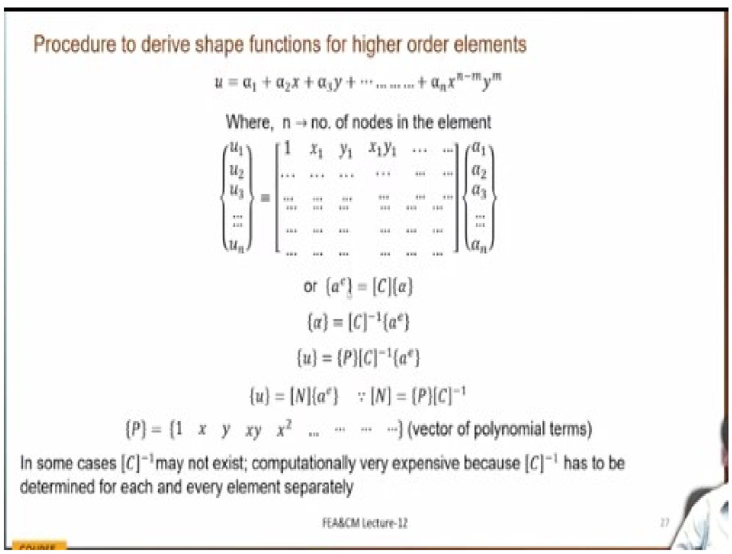

And then getting the shape functions using the generalized coordinate method is also very tedious, because we have to go on inverting the so many matrices. Sometimes these matrices are fully populated. So inverting that will take lot of time.

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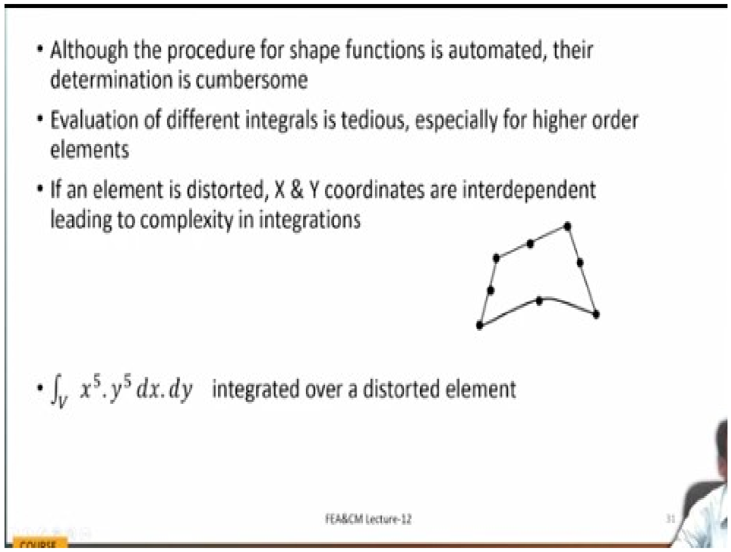

So although the procedure for finding the shape functions is automated, their determination is cumbersome and the evaluation of the integrals is also tedious. So if you have a complicated shape like this, and I ask you to integrate, we cannot do it, because of irregular boundary. And because the two coordinates, they are not normal to each other orthogonal to each other.

Whereas previously we had seen X is along the x axis and Y is along y axis, they are perfectly orthogonal to each other, okay? And so if your shape is like this, then it is difficult, we cannot easily do the integrations.

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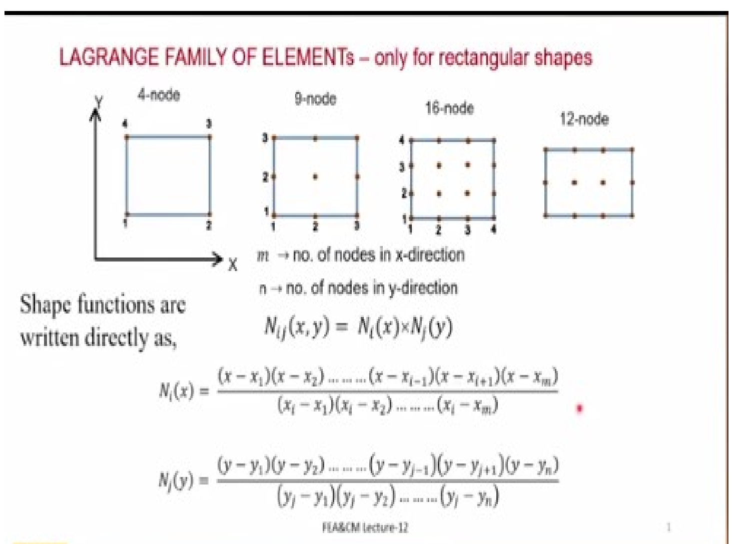
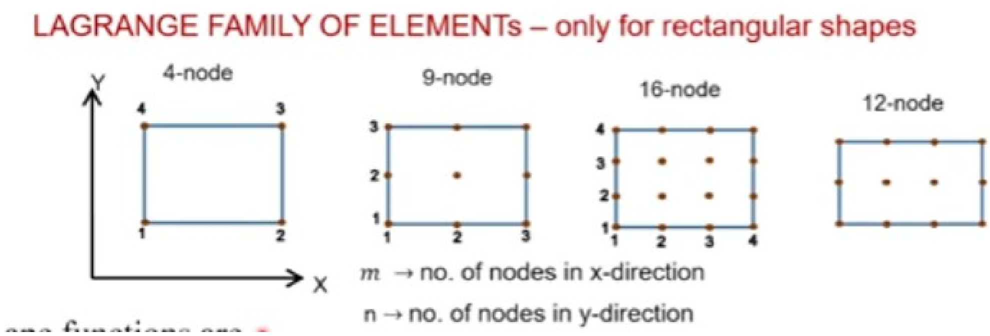


So the new family of elements that are developed are the Lagrange family of elements. And these are pure rectangular elements.



And the way their shape functions are developed is slightly different from what we have seen earlier with the generalized coordinate method. And let us consider a 4-node Lagrange element, so it has four nodes 1, 2, 3 and 4. And then 9 node, it has three nodes along the x axis, and then three nodes along the y axis.

And it could have either four nodes along the x axis and four nodes along y axis. And it is not necessary to have the same number of nodes along both x and y axis. So here we have a 12-node Lagrange element that has four nodes along the x direction, and then only three nodes in the y direction. And because of that if any quantity is moving along the x direction, the y remains constant.

And then similarly, if we are moving along the y, x remains constant. And we can write the shape functions N(x, y) as product of two separate functions, one written in terms of x and the other written in terms of y. And let us say that we have m number of nodes in the x direction, and then n number of nodes in the y direction. And then the shape function in the x direction is written as the ratio between two quantities.

Shape functions are written directly as,

$$N_{ij}(x,y) = N_i(x) \times N_j(y)$$

$$N_i(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots\ldots(x-x_{i-1})(x-x_{i+1})(x-x_m)}{(x_i-x_1)(x_i-x_2)\ldots\ldots\ldots(x_i-x_m)}$$

$$N_j(y) = \frac{(y-y_1)(y-y_2)\ldots\ldots\ldots(y-y_{j-1})(y-y_{j+1})(y-y_n)}{(y_j-y_1)(y_j-y_2)\ldots\ldots\ldots(y_j-y_n)}$$

In the numerator we have, say for i-th node, it is x - x 1 multiplied by x - x 2. You continue and x - x i - 1 times x - x i + 1. Then continuing up to x - x m, where m is the number of nodes in the x direction. And in the denominator, we take the x coordinate at the i-th node, x i - x 1 x i - x 2 and so on. And then we will not have the term x i – x i in the denominator also.

So we will have x i - x i - 1 times x i - x i + 1. And then continuing up to m-th node like this. And similarly, for j-th node in the y direction N j(y) is y - y 1 times y - y 2 and so on, up to y - y n. And in the denominator, we have y j – y 1, y j – y 2 and so on. So we see that in the numerator, we have the variables x and y whereas in the denominator, we have only constant quantities x i - x 1 and so on. And this could be a bit confusing.

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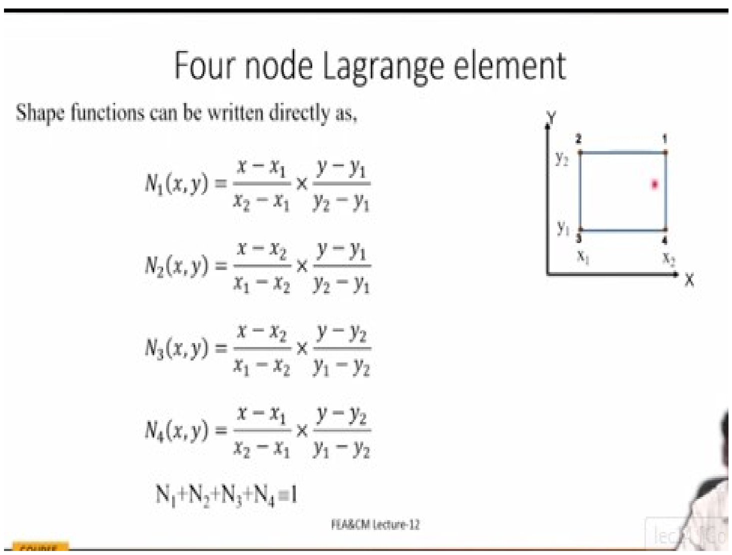
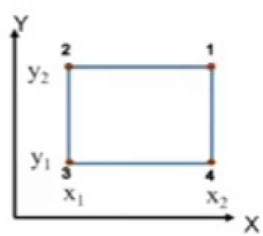


But if you look at particular elements, it becomes more simple. Let us consider a 4-node Lagrange element.



And it has four nodes 1, 2, 3 and 4. And along the x direction we have the coordinates x 1 and x 2. And along y direction, we have y 1 and y 2. And let us say that we are interested in writing the shape function for node 1.

So node 1 x of x and y is equal to x - x 1, because the coordinate at node 1 is x 2, and x - x 1 divided by x 2 - x 1. And similarly, the y coordinate at node 1 is y 2. So we have y - y 1 divided by y 2 - y 1. And similarly, the shape function for node 2 will be x - x 2 divided by x 1 - x 2 and then y – y 1 divided by y 2 - y 1 because the coordinate at node 2 is x 1 y 2. And at node 3, we have the coordinates x 1 and y 1.

Shape functions can be written directly as,

$$N_1(x,y) = \frac{x - x_1}{x_2 - x_1} \times \frac{y - y_1}{y_2 - y_1}$$

$$N_2(x,y) = \frac{x - x_2}{x_1 - x_2} \times \frac{y - y_1}{y_2 - y_1}$$

$$N_3(x,y) = \frac{x - x_2}{x_1 - x_2} \times \frac{y - y_2}{y_1 - y_2}$$

$$N_4(x,y) = \frac{x - x_1}{x_2 - x_1} \times \frac{y - y_2}{y_1 - y_2}$$

$$N_1 + N_2 + N_3 + N_4 \equiv 1$$

So we have the N 3 written as x - x 2 divided by x 1 - x 2 multiplied by y - y 2 divided by y 1 - y 2. And similarly at node 4, the coordinates are x 2 and y 1. So we have the shape function in the x direction is x - x 1 divided by x 2 minus x 1. And in the y direction it is y - y 2 divided by y 1 - y 2, okay? And directly, we have written the shape functions for all the nodes in the element.

And if you look at the sum total of all the shape functions N 1 + N 2 + N 3 + N 4, it will be exactly equal to 1. That you can check it for yourself.

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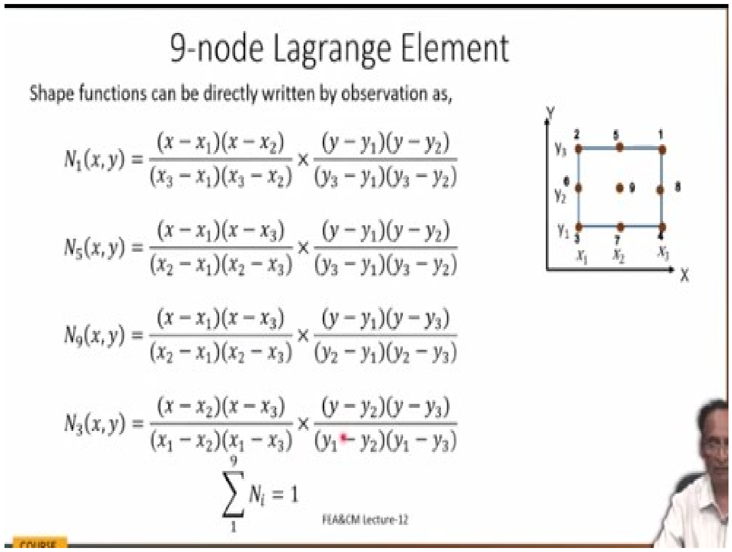
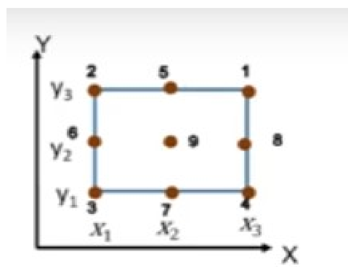


And similarly, we can extend this procedure even for higher order elements. Let us look at 9-node Lagrange element that has three nodes along the x direction and three nodes along the y direction.



And we have the nodes 1, 2, 3 and 4 and then 5, 6, 7, 8 and 9. And you notice that we have first numbered the corner nodes first, because for defining a rectangle, you require minimum four points and four corner points.

And then 5, 6, 7, 8 they are the intermediate points along each direction, and the ninth node is at the center of the element, okay? So at node 1 N 1(x) will be x - x 1 times x - x 2 divided by x 3 - x 1 and x 3 - x 2, okay? Because the coordinate value at node 1 is x 3 in x direction and y 3 in y direction. So in the y direction, your shape function will be y - y 1 multiplied by y - y 2 divided by y 3 - y 1, y 3 - y 2 and so on.

Shape functions can be directly written by observation as,

$$N_1(x,y) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \times \frac{(y-y_1)(y-y_2)}{(y_3-y_1)(y_3-y_2)}$$

$$N_5(x,y) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \times \frac{(y-y_1)(y-y_2)}{(y_3-y_1)(y_3-y_2)}$$

$$N_9(x,y) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \times \frac{(y-y_1)(y-y_3)}{(y_2-y_1)(y_2-y_3)}$$

$$N_3(x,y) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} \times \frac{(y-y_2)(y-y_3)}{(y_1-y_2)(y_1-y_3)}$$

$$\sum_1^9 N_i = 1$$

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And let us look at node 9 where the coordinates are x 2 and y 2. So the shape function in the x direction will be x - x 1 multiplied by x - x 3 divided by x 2 - x 1 and x 2 - x 3 and in the y direction we have y – y 1 and y – y 3 divided by y 2 – y 1 times y 2 – y 3 and so on, okay? And the sum total of all the shape functions will be exactly equal to 1, okay?

And so the advantage that we are gaining now is we can directly write the shape functions, we do not need to go through the inversion of a large matrix coordinate matrix and so on.

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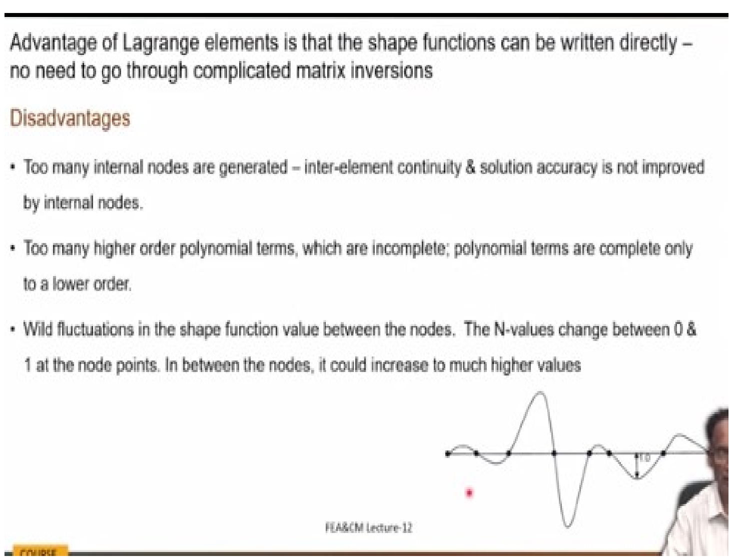

And the main advantage of these Lagrange elements is that the shape functions can be directly written without going through a complicated matrix inversion. But then, there are several other disadvantages. There are too many internal nodes like as we have seen earlier, see all these are like for a 16-node element.

These are all the internal nodes and for the 12-node element, these two are the internal nodes and for the ninth node, this is the internal node and so on, okay? And these internal nodes they do not help us in this inter-element continuity. We cannot satisfy the compatibility conditions along the edges of the elements. And so because of that the solution is not improved corresponding to the effort that you take, okay?

And we also use too many higher order polynomial terms which are incomplete and the polynomial terms are complete only to a lower order lower degree. And there could be wild fluctuation in the shape function especially if you have too many nodes. So your shape function will be either 0 or 1. 0 at all the other nodes and 1 at its own node.

But in between we have no control because if you have a very high order polynomial, there could be a lot of fluctuations like this. And so that could lead to numerical difficulties because if you are operating at some other some location in between the nodes, your shape function could be very high and that could lead to numerical problems.

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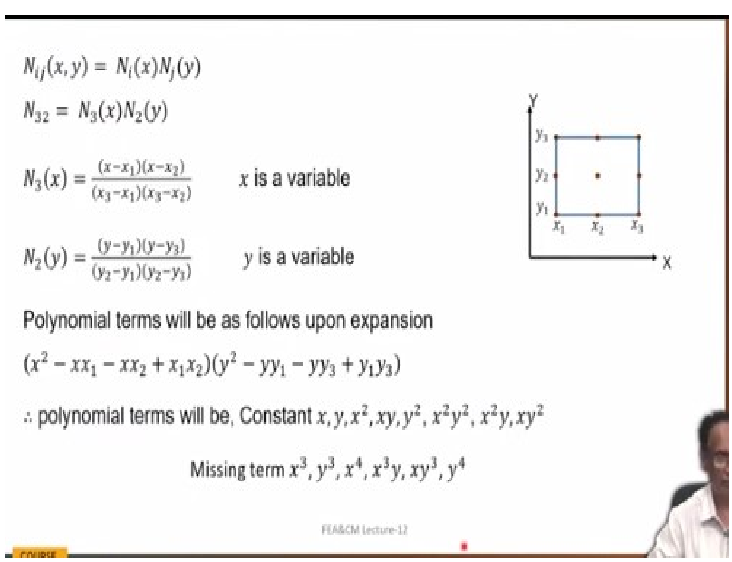
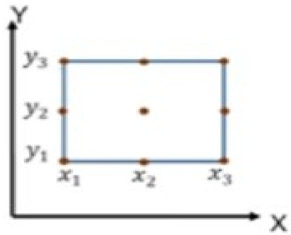


So here the, like if you look at this 9-node Lagrange element, we have seen that in the x direction for node 3 we have the shape function in the x direction is x - x 1, x - x 2 divided by x 3 – x 1, x 3 – x 2 where x is a variable and then the y is the variable in the N 2(y).



And so for this particular node N 32 that is this sorry this node, the polynomial terms if you expand and then take the product will be something like this.

$$N_{ij}(x,y) = N_i(x)N_j(y)$$

$$N_{32} = N_3(x)N_2(y)$$

$$N_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \qquad x \text{ is a variable}$$

$$N_2(y) = \frac{(y-y_1)(y-y_3)}{(y_2-y_1)(y_2-y_3)} \qquad y \text{ is a variable}$$

Polynomial terms will be as follows upon expansion

$$(x^2 - xx_1 - xx_2 + x_1x_2)(y^2 - yy_1 - yy_3 + y_1y_3)$$

We have an x square and then x times x 1, x - x times x 2 and so on. And then similarly in the y direction. So we have the polynomial terms. x 1 and x 2, these are constants and so we have a constant term x and y and then xy, y square, x square y square, x square y, xy square and so on. While we are missing some other higher order terms like x cube and y cube, see corresponding to x square y and xy square we should have x cube and y cube for our complete polynomial that we do not have.

∴ polynomial terms will be, Constant $x, y, x^2, xy, y^2, x^2y^2, x^2y, xy^2$

Missing term $x^3, y^3, x^4, x^3y, xy^3, y^4$

And corresponding x square and y square the other fourth order terms are x to the power 4, x cube y, xy cube and y to the power 4. So these are missing in our polynomial.

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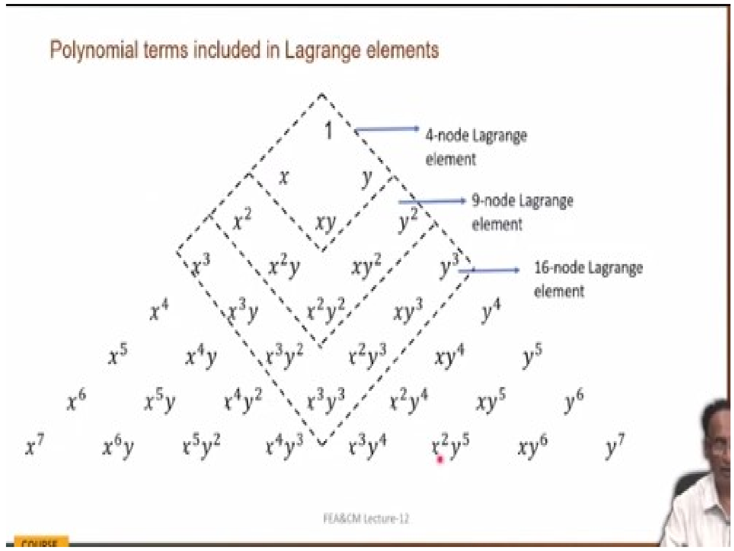
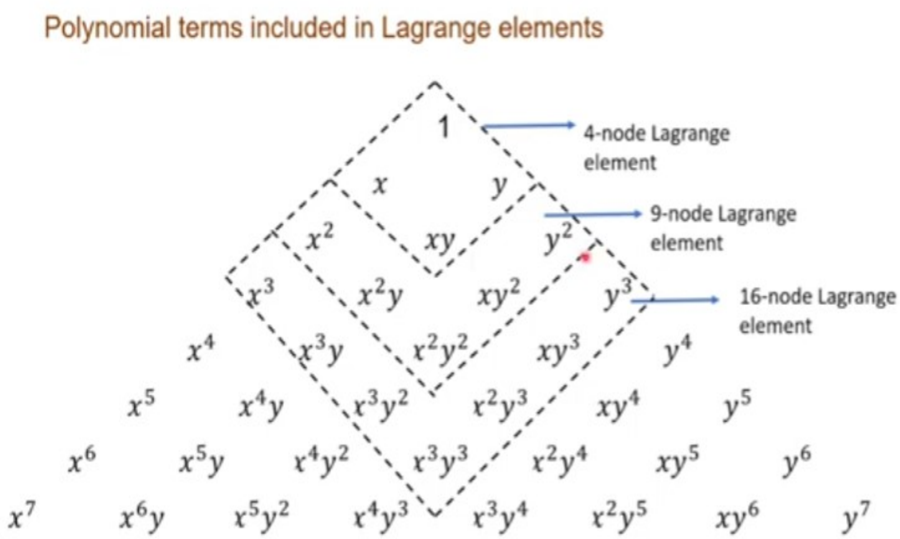


So if we look at the included terms for the Lagrange element, for the 4-node Lagrange element, we have 1.



That is the constant term x, y and xy while we are missing x square and y square. Whereas for the 9-node Lagrange element, we have all these highlighted terms while we are missing these terms x cube, y cube and then x cube y, xy cube, x to the power 4, y to the power 4, okay and so on.

Like if you extrapolate for 16-node Lagrange element we will have more number of terms but then we are also having more number of missing terms.

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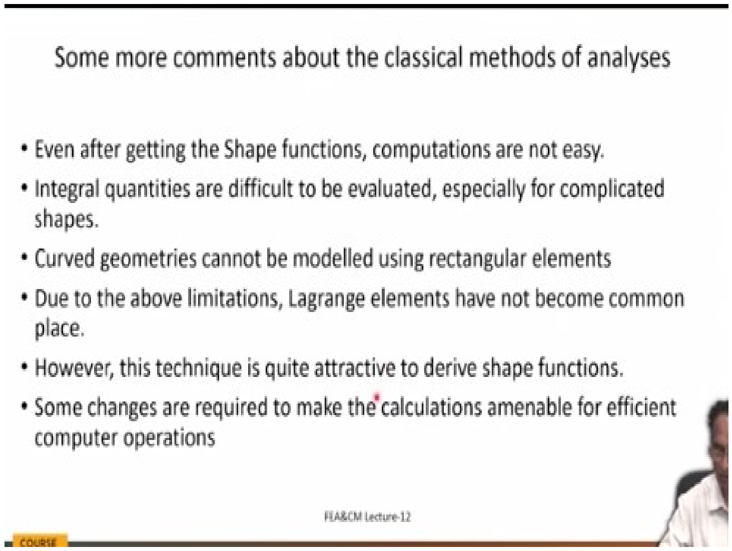

And we already know that for getting accurate finite element results, we should have a complete polynomial. And that we have seen even from the earlier methods, classical methods like the Rayleigh-Ritz and other methods. So just to summarize the comments about the classical methods or the generalized coordinate method is that even after getting the shape functions, computations are not easy.

Especially if you have an irregular geometry, a curved boundary or something, integration of the quantities is not easy. And then we cannot accommodate curved geometries and we cannot model these curved geometries with either triangular elements or rectangular elements.

And because of these problems, the Lagrange elements have not become very commonplace, although the procedure is beautiful, because we can directly write the shape functions and so on. And so this technique, the Lagrange elements have not become popular.

But then this technique is quite attractive for deriving shape functions for higher order elements and that we are going to see from next lecture onwards because we need to make some changes to make all the calculations amenable for computer operations. Like we should be able to easily program these different integrals for calculating the stiffness matrix, load vector and so on.

And that we will see from the next class onwards. And so thank you very much. We will meet in the next class.