FEM and Constitutive Modelling in Geomechanics
Prof. K. Rajagopal
Department of Civil Engineering
Indian Institute of Technology - Madras

Lecture: 13
3-Node Constant Strain Triangle

And in the previous class we had derived the equilibrium equations for a continuum and then we have also seen how to develop our shape functions and let us continue in today's class and apply our concepts to it the simplest continuum finite element that we can think of that is the 3 node constant strain triangle. Let us we will see why we call this as a constant strain triangle as we go along.

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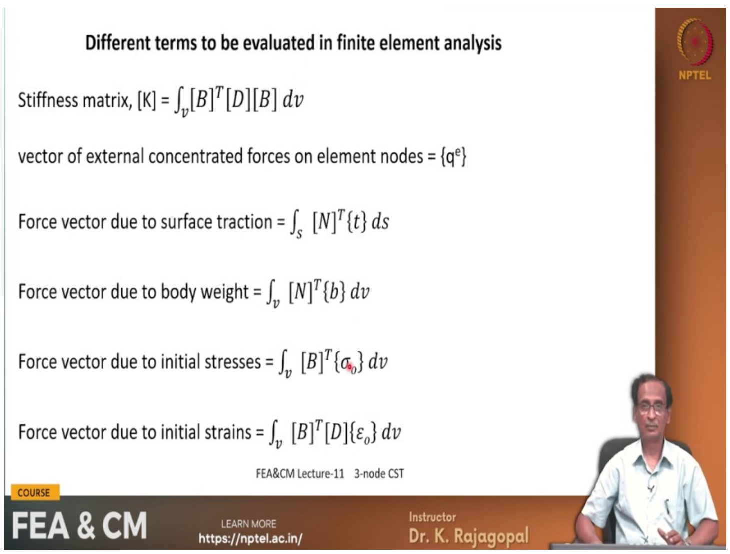

And these are all the things that we had derived in the previous class the different integrals that we need to evaluate the first is the stiffness Matrix K is B transpose D B integrated over the volume right and the this Vector of external concentrated forces and the element nodes. So, this is given this we do not need to evaluate and the force vector due to surface traction and this is because of the pressures that are acting on the surface of the element.

Stiffness matrix, [K] = $\int_v [B]^T[D][B]\, dv$

vector of external concentrated forces on element nodes = $\{q^e\}$

Force vector due to surface traction = $\int_s \; [N]^T\{t\}\, ds$

Force vector due to body weight = $\int_v \; [N]^T\{b\}\, dv$

Force vector due to initial stresses = $\int_v \; [B]^T\{\sigma_o\}\, dv$

Force vector due to initial strains = $\int_v \; [B]^T[D]\{\varepsilon_o\}\, dv$

So, this integration is done over the surface and the force Vector due to body weight and transpose B integrated over the volume of the element. And then the force Vector due to initial stresses B transpose Sigma naught and then the force Vector due to initial strains B transpose D Epsilon naught and is actually this is what we do not normally consider in geotechnical engineering because it is not possible to define the initial strains. But we can define the initial stresses because we can do some measurements.

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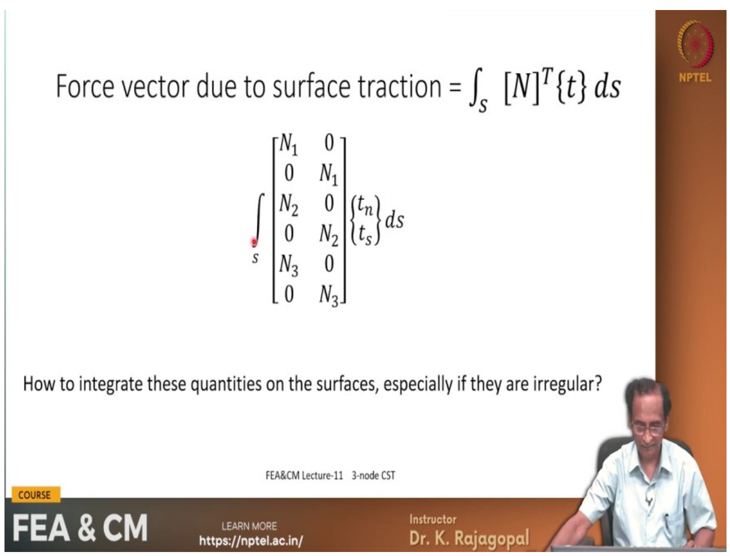

Force vector due to surface traction = $\int_s \; [N]^T\{t\}\, ds$

$$\int_s \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} t_n \\ t_s \end{Bmatrix} ds$$

And the force Vector due to the surface traction is N transpose t ds this is the N and transpose is N is N 10 N 20 and so on. Now that has 2 rows and 6 columns and then N transpose will have 2 columns under 6 rows and the T has the 2 components one is a normal component and the other is a Shear component and by integrating this over the surface we can get the we can get our surface forces but how we can do this that we will see later.

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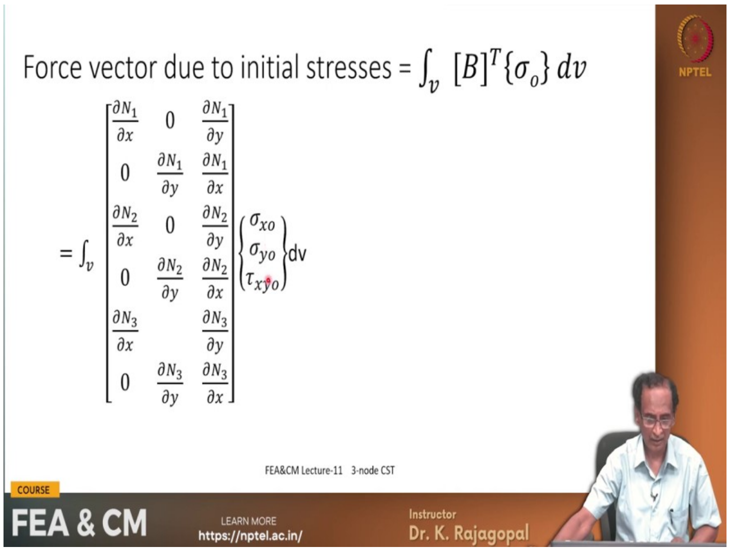

Force vector due to body weight = $\int_v [N]^T\{b\}\, dv$

$$\int_v \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} dv$$

And the force Vector due to body weight and transpose B and this is our N transpose and our B and these quantities we need to integrate over the volume and then B transpose Sigma naught the B is a 3 by 6 Matrix for 3 node triangle and. So, B transpose will be 6 by 3 and sigma naught could be Sigma x naught Sigma y naught Tau xy naught and we need to integrate this.

$$\text{Force vector due to initial stresses} = \int_v [B]^T\{\sigma_o\}\, dv$$

$$= \int_v \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_1}{\partial y} \\ 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} \\ \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_2}{\partial y} \\ 0 & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} \\ \frac{\partial N_3}{\partial x} & & \frac{\partial N_3}{\partial y} \\ 0 & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_{xo} \\ \sigma_{yo} \\ \tau_{xyo} \end{Bmatrix} \text{dv}$$

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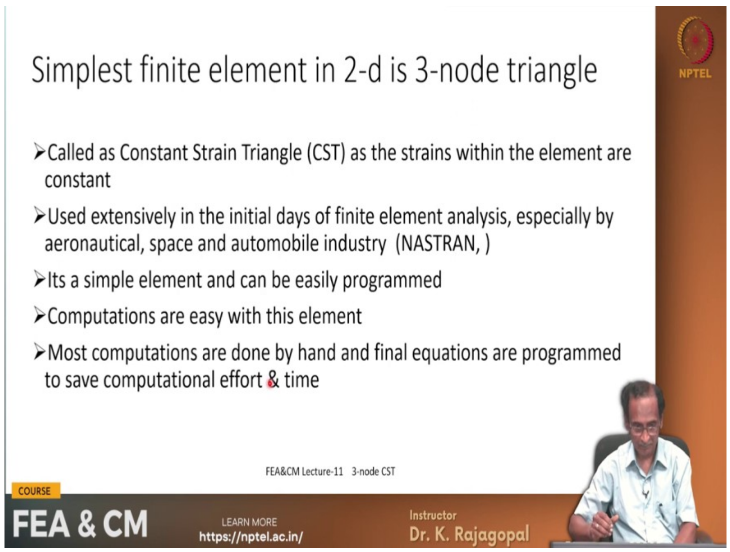

And the simplest 2 dimensional finite element is 3 node triangle and this is called as the constant strain triangle because the strains within the element are constant. And this particular element the 3 node triangle is extensively used in the initial days of finite element analysis by the aeronautical industry space and automobile industry and one program that was developed by NASTRON was very popular and in fact that has been ported in different languages and by different industry.

And the 3 node triangle became very popular because it is easy to program and the computations are also very easy that we will see in today's lecture. And then most of the computations are done by hand like we do all the product and then give the final thing as we program them the computer so that we can save the computational effort and time like we can do the same thing by computer but then it takes more effort.

Because in those days remember the memory was very small and then the computers were not this fast.

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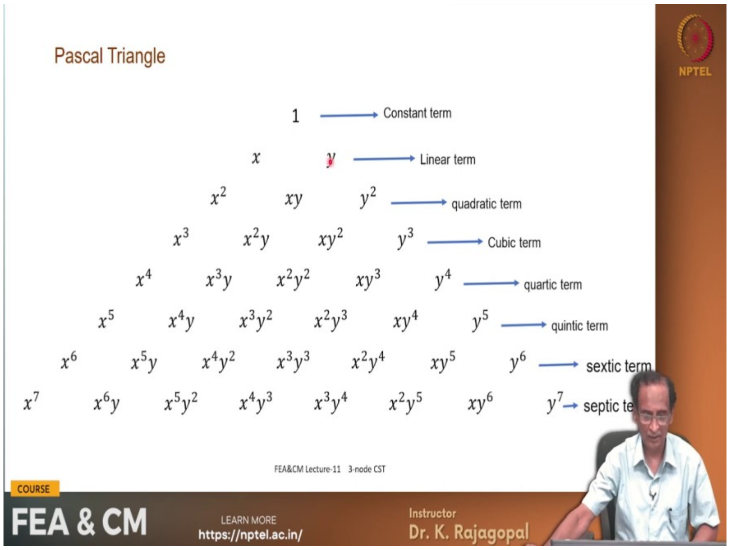

And the Pascal triangle we have seen in earlier class that helps us in choosing the polynomial terms. So, that we can have spatial isotropy and if you have a 3 node triangle the 3 terms that we can choose are this constant term 1 and then the linear terms x and y.

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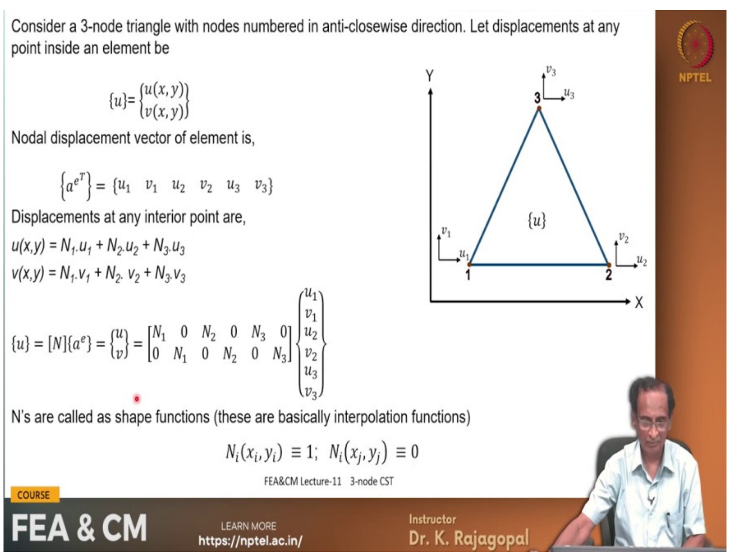

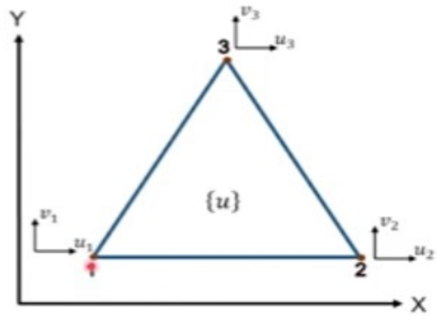

Let us look at 3 note triangle with the 3 nodes one 2 and 3 and the displacements u 1 v 1 u 2 v 2 u 3 v 3 and internal displacement is u and the nodal displacement vector is u1 v 1 u 2 v 2 u 3 v 3. And the displacements at interior points are u and v and that we can express as N 1 u 1 + N 2 u 2 + N 3 u 3 and v is N 1 v 1 + N 2 v 2 + N 3 v 3 and the u can be written as N times a e where N is our shape function Matrix and a e is the vector of nodal displacements.

$$\{u\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$

Nodal displacement vector of element is,

$$\left\{a^{e^T}\right\} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3\}$$

Displacements at any interior point are,

$$u(x,y) = N_1.u_1 + N_2.u_2 + N_3.u_3$$

$$v(x,y) = N_1.v_1 + N_2.\ v_2 + N_3.v_3$$

$$\{u\} = [N]\{a^e\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

And our ends are the shape functions basically these are interpolation functions and some of the attributes we have seen in the previous class.

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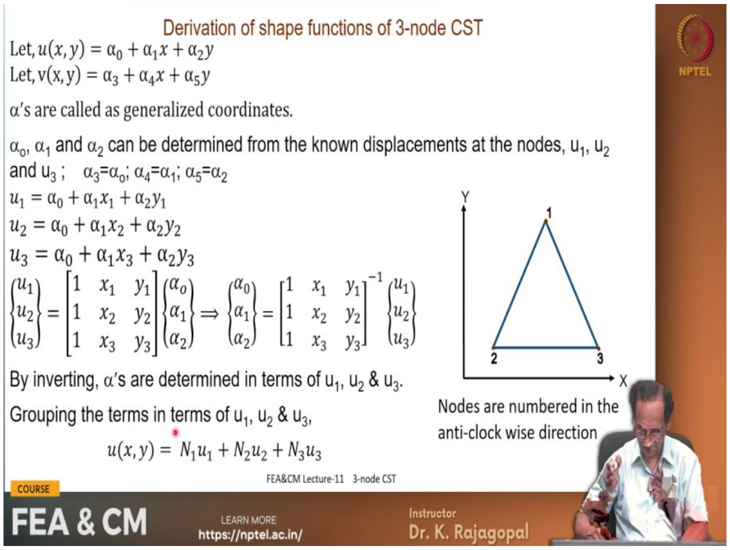

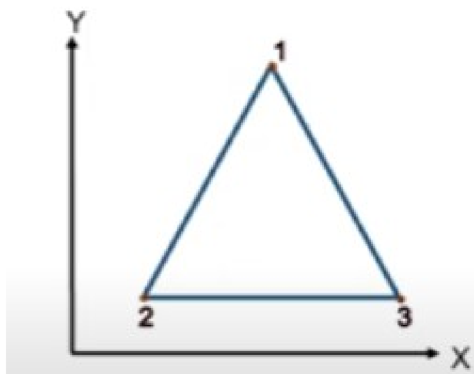

And for the 3 node element we can assume a polynomial like this u can be Alpha naught + Alpha 1 x + Alpha 2 y and v can be Alpha 3 + Alpha 4x + Alpha Phi y and these Alphas are called as generalized coordinates.

$$\text{Let, } u(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y$$
$$\text{Let, } v(x,y) = \alpha_3 + \alpha_4 x + \alpha_5 y$$

α's are called as generalized coordinates.

And actually if we derive the values we will see that Alpha 3 is exactly equal to Alpha naught and Alpha 4 is is equal to Alpha 1 Alpha 5 is equal to Alpha 2 because the shape is the same like for both u and v it is the element is the same except you have the u and v terms.

$$\alpha_3=\alpha_0;\ \alpha_4=\alpha_1;\ \alpha_5=\alpha_2$$

So, we normally write u as Alpha naught + alpha 1 x + 1 + Alpha 2 y and use the same function for both the u and v. So, at node one u 1 is Alpha naught + alpha 1 x 1 + Alpha 2 y 1 and u 2 at node 2 is all u 2 is Alpha naught + Alpha 1 x 2 + Alpha to y 2 and at node 3 u 3 is

Alpha naught + alpha 1 x 3 + Alph 2 y 3 and the matrix form we can write like this u one u 2 u 3 is the Matrix of this coordinates 1 x 1 y 1 1 x 2 y 2 1 x 3 y 3.

$$u_1 = \alpha_0 + \alpha_1 x_1 + \alpha_2 y_1$$
$$u_2 = \alpha_0 + \alpha_1 x_2 + \alpha_2 y_2$$
$$u_3 = \alpha_0 + \alpha_1 x_3 + \alpha_2 y_3$$
$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_o \\ \alpha_1 \\ \alpha_2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

And these are the generalized coordinates Alpha not alpha 1 and Alpha 2 and these Alphas we can get by inverting this Matrix and multiplying with the u 1 u 2 u 3. And so, we can do that and then we can group all the terms under u 1 u 2 and u 3 and then get our internal displacements as N 1 u 1 + N 2 u 2 + N 3 u 3 and if we invert this matrix and group under u 1 u 2 u 3.

$$u(x, y) = N_1 u_1 + N_2 u_2 + N_3 u_3$$

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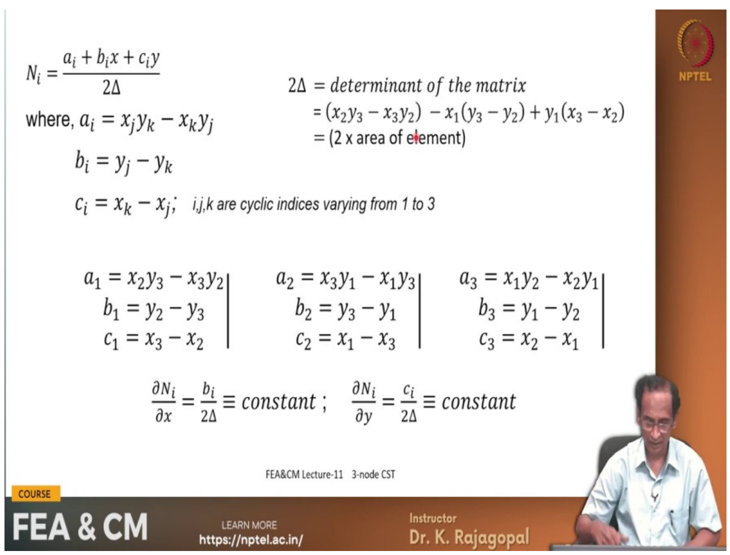

In general we can write the N i as a i + b i x + c i y by 2 Delta where our i is varying from 1 2 3, 1, 2 and 3 and a i is x j y k - y k y j is actually I can vary from 1 2 3 and j also varies from 1 2 3 K is also from 1 2 3 and it actually called as cyclic indices. So, when i is 1 j will be 2 and K will be 3. So, i is 1 2 3 j will be 2 3 1 K will be 3 2 1. And so, if you want a 1 it will be x 2 y 3 - x 3 y 1 and x 3 y 2.

$$N_i = \frac{a_i + b_i x + c_i y}{2\Delta}$$

where, $a_i = x_j y_k - x_k y_j$

$b_i = y_j - y_k$

$c_i = x_k - x_j$; i,j,k are cyclic indices varying from 1 to 3

And then similarly a 2 means x 3 y 1 - x 1 y 3. So, b i is y j - y k c i is x k - x j and so, our a 1 is x 2 y 3 - x 3 y 2 and a 2 is x 3 y 1 C when i is 2 j is 3 and the K is 1. So, that is the the relation that we have. So, we can see this a 1 b 1 c 1; a 2 b 2 c 2 and so on. And our derivative of the shape functions will give us our strain. So, if we take a dou N i by dou x that is b i by 2 Delta and it is not a function of x so, it is constant.

$a_1 = x_2 y_3 - x_3 y_2$	$a_2 = x_3 y_1 - x_1 y_3$	$a_3 = x_1 y_2 - x_2 y_1$
$b_1 = y_2 - y_3$	$b_2 = y_3 - y_1$	$b_3 = y_1 - y_2$
$c_1 = x_3 - x_2$	$c_2 = x_1 - x_3$	$c_3 = x_2 - x_1$

$$\frac{\partial N_i}{\partial x} = \frac{b_i}{2\Delta} \equiv constant \ ; \quad \frac{\partial N_i}{\partial y} = \frac{c_i}{2\Delta} \equiv constant$$

And dou N i by dou y is c i by 2 Delta that is also constant it is not a function of x or y. So, it is constant in the space. So, within the element within this 3 node triangle wherever you evaluate you will get the same strain. And this denominator 2 Delta is the determinant of the Matrix that coordinate Matrix and if you take the determinant you get x 2 y 3 - x 3 y 2 - x 1 times y 3 - y 2 + y 1 times x 3 - x 2.

$$\begin{aligned} 2\Delta &= determinant\ of\ the\ matrix \\ &= (x_2 y_3 - x_3 y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2) \\ &= (2 \text{ x area of element}) \end{aligned}$$

And this is equal to 2 times the area of the element that we will see with the numerical example a bit later.

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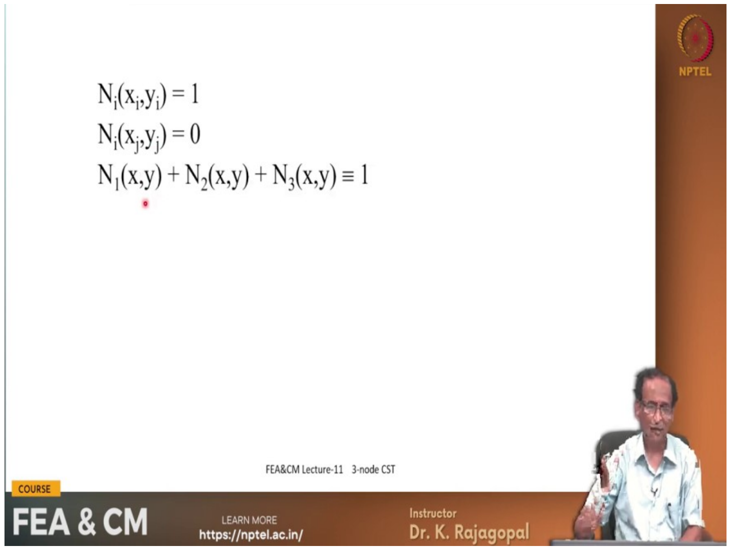

And our N i is written as a i + b i x + c i y by 2 Delta and our requirement is N i at x i y i should be equal to 1 and you can substitute x i and y i in place of x and y and we will see that indeed that the shape function value is 1 and N i evaluated at some other node x j y j will be 0 and the sum total of all the 3 shape functions N 1 + N 2 + N 3 is equal to one.

$$N_i(x_i,y_i) = 1$$
$$N_i(x_j,y_j) = 0$$
$$N_1(x,y) + N_2(x,y) + N_3(x,y) \equiv 1$$

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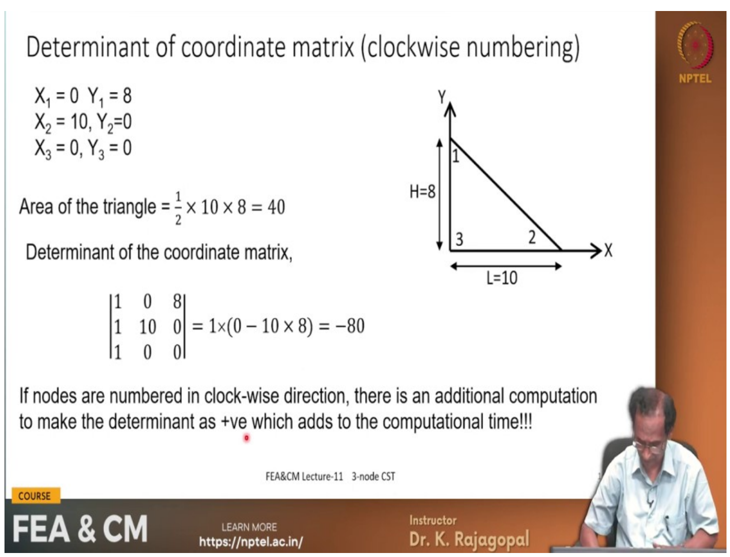
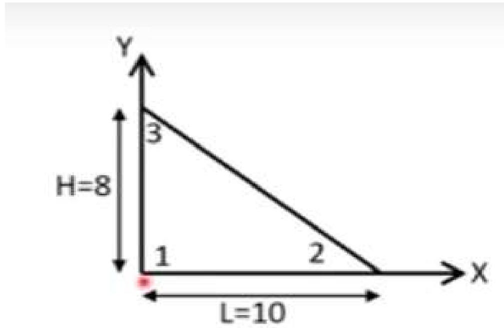


And let us determine the determinant of that coordinate matrix let us take a simple right angle triangle 1 2 3 length of 10 and the height of 8. and let us give some coordinate values.



Now let us take Node 1 at origin x 1 y 1 is 0 and x 2 is 10 y 2 is 0 and x 3 is 0 y 2 is 8. And so the area of this triangle is half the base times height that is equal to 40. And the determinant of this coordinate Matrix 1 x 1 y 1 1 x 2 y 2 1 x 3 y 3, x 1 y 1 are zeros x 2 is 10 and y 2 is zero x 3 is 0 and y 3 is 8.

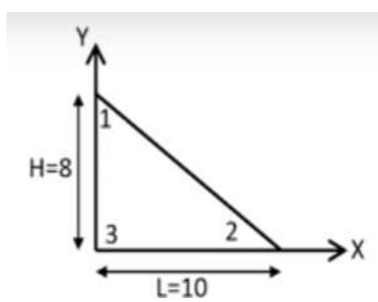
$$X_1 = Y_1 = 0$$
$$X_2 = 10,\ Y_2 = 0$$
$$X_3 = 0,\ Y_3 = 8$$

Area of the triangle $= \frac{1}{2} \times 10 \times 8 = 40$

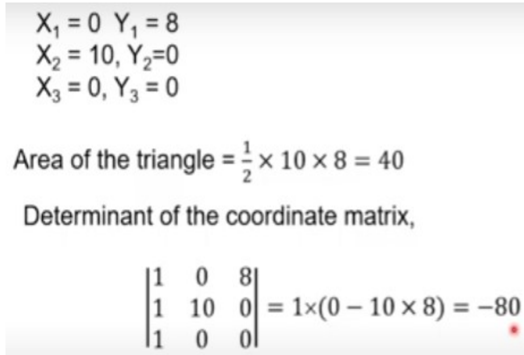
Determinant of the coordinate matrix,

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 10 & 0 \\ 1 & 0 & 8 \end{vmatrix} = 10 \times 8 = 80 = 2 \times \text{area of triangle}$$

So, this is the determinant that is 80. that is equal to twice the area of the triangle and this is with anti-clockwise numbering that is node 1 node 2 node 3 this is called as anti-clockwise we are going from x to y in the anti-clockwise direction



and let us consider what happens if you have a clockwise numbering 1 2 and 3. So our x 1 is 0 y 1 is 8 x 2 is 10 y 2 is 0 and node 3 is at origin x 3 and y 3 are zeros and the determinant of the coordinate Matrix 1 x 1 y 1 1 x 2 y 2 1 x 3 by 3 that comes to -80.



Previously we had 80 + 80 for anti-clockwise numbering but when we numbered the nodes in the clockwise manner we get a negative value and the determinant is actually related to the area of the element. So, we want a positive value because obviously our area cannot be negative. So, if the number the nodes are numbered in the anti-clockwise direction automatically the determinant we see is a positive quantity.

But if we number the nodes in the clockwise direction we get a negative value and to make it as a positive value then we need to write it as minus- of the determinant that is minus of minus 80 will be plus 80 and that adds an additional computation which adds to the time and because we may be having millions of elements in our mesh. And so to come to save the time up front we can number the nodes in the anti-clockwise direction rather than the clockwise direction so, that we can at least save that much time.

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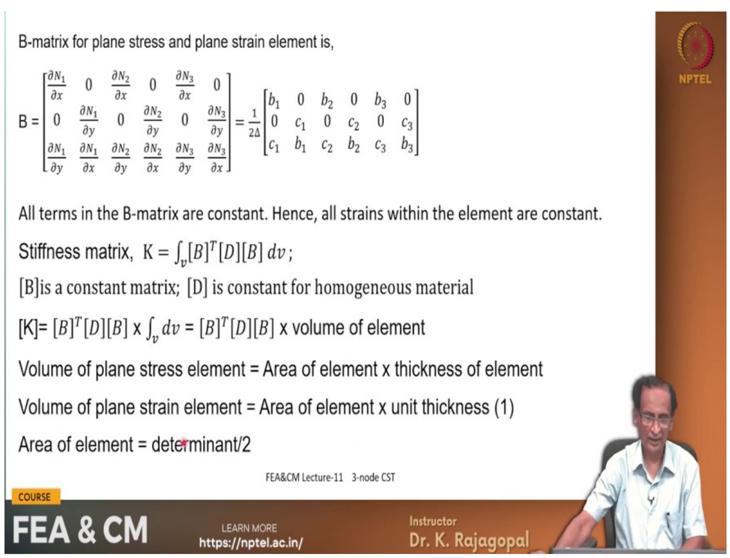

So, the B Matrix for the plane stress and plane strain elements for the 3 node triangle will be like this 3 rows corresponding to 3 strands and the 6 columns corresponding to 6 degrees of freedom and our dou N by dou x is b i by 2 Delta and dou N 1 by dou N i by dou y is a c i by 2 Delta. So, our B is one by 2 Delta B 10 B 20 and so on. And the B and the C they are absolute constants. So, our all the terms within the B Matrix are constant.

B-matrix for plane stress and plane strain element is,

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

All terms in the B-matrix are constant. Hence, all strains within the element are constant.

Stiffness matrix, $K = \int_v [B]^T [D][B]\, dv$;

[B]is a constant matrix; [D] is constant for homogeneous material

[K]= $[B]^T[D][B]$ x $\int_v dv$ = $[B]^T[D][B]$ x volume of element

Volume of plane stress element = Area of element x thickness of element

Volume of plane strain element = Area of element x unit thickness (1)

So, the strain is also constant and our equation for the stiffness Matrix K is integral B transpose D B and our B is constant and if we assume homogeneous material D is also constant. So, we can bring out the B transpose D B out of the integral and write K as integral of B sorry B transpose D B multiplied by integral dv. And this is equal to volume of the element and the volume of the element is area of the element that is the determinant divided by 2 multiplied by thickness.

See for the plane stress there is a thickness and for a plane strain it is a unit value. So, for the plane stress case the volume is area of the element multiplied by thickness of the element and volume of the plane strain element is area of the element multiplied by 1 and the area of the element itself is determinant divided by 2.

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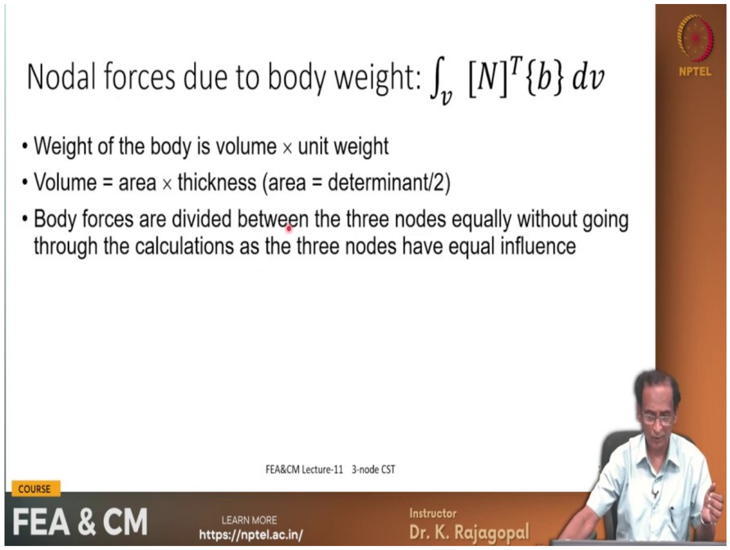

So, our nodal Force Vector N transpose B the total weight of the body is volume multiplied by unit weight and the volume is the area multiplied by thickness and the body force is distributed equally between the 3 nodes for a 3 node triangle. We do not need to really go through the calculation of this N transpose B because even if you do we will get the same thing and because each node has an equal distribution or equal influence on the stiffness of the system. So, our body force is divided equally between the 3 nodes.

Nodal forces due to body weight: $\int_v [N]^T\{b\}\,dv$

And the nodal Force Vector due to the initial stresses B transpose Sigma naught and the B and sigma naught are constant.

Nodal forces due to initial stresses: $\int_v [B]^T\{\sigma_o\}\,dv$

- As [B] and {σ} are constant over the element, their product can be evaluated without any integration

So, our we do not really need to do the integral we can bring out B transpose Sigma naught outside the integral and we will see integral dv or the volume that is equal to volume of the element once again it is a very simple calculation B is B transpose is 6 by 3 and sigma naught is is having a length of 3. So, 6 by 3 multiplied by 3 by 1 is 6 by 1. So, you get a vector of length 6 for the initial stresses.

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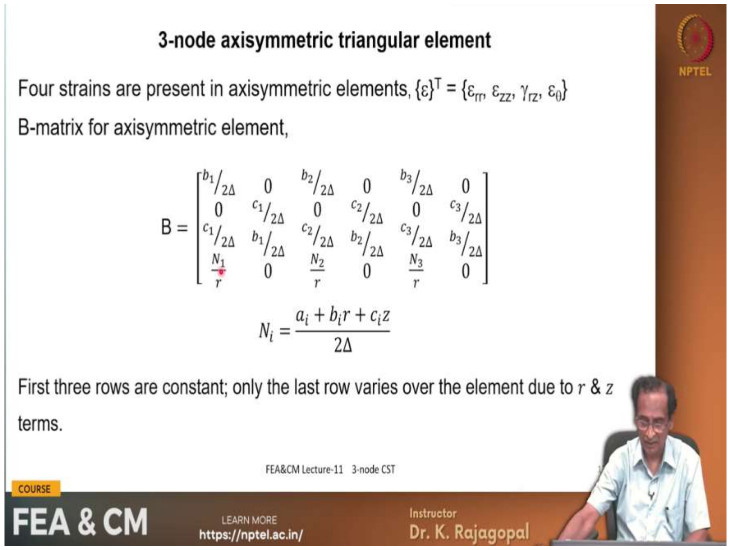

B-matrix for axisymmetric element,

$$B = \begin{bmatrix} b_1/2\Delta & 0 & b_2/2\Delta & 0 & b_3/2\Delta & 0 \\ 0 & c_1/2\Delta & 0 & c_2/2\Delta & 0 & c_3/2\Delta \\ c_1/2\Delta & b_1/2\Delta & c_2/2\Delta & b_2/2\Delta & c_3/2\Delta & b_3/2\Delta \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \end{bmatrix}$$

And so let us see for axis symmetric triangular elements we have four strain components and so, the B Matrix for the axis symmetric elements will be like this the first 3 rows are familiar B by 2 Delta 0 and so on 0 by C by 2 Delta and the third row C by 2 Delta and B by 2 Delta and the fourth row is corresponding to Epsilon Theta that is N by r k N 1 by r 0 and 2 by r 0 and 3 by r zero and our N i is a i + b i r + c i z by 2 Delta.

$$N_i = \frac{a_i + b_i r + c_i z}{2\Delta}$$

And the first 3 rows are constants as we have seen with the plane stress and plane strain only the last row is not constant because of our r and z.

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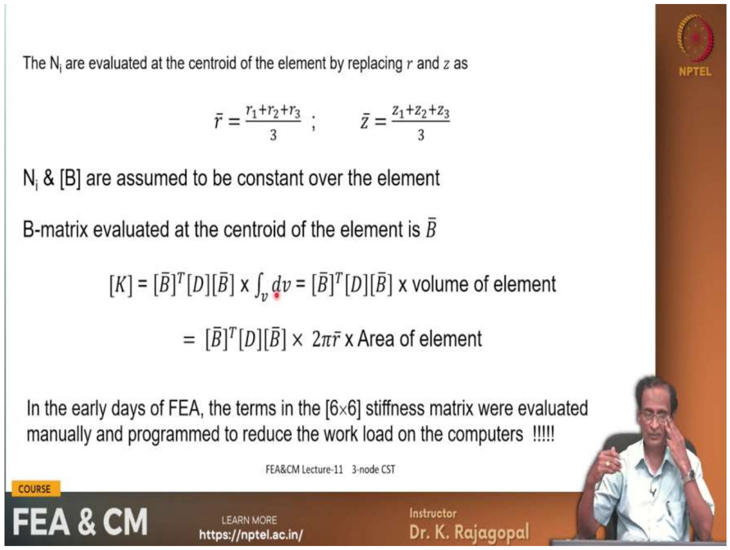

The N_i are evaluated at the centroid of the element by replacing r and z as

$$\bar{r} = \frac{r_1+r_2+r_3}{3} \quad ; \qquad \bar{z} = \frac{z_1+z_2+z_3}{3}$$

And we can make an approximation that we are going to evaluate the fourth row of the centroid of the element R Bar and z bar r Bar is r 1 + r 2 + r 3 by 3 and z bar is z 1 + z 2 + z 3 by 3 and our N and B now can be treated as constant and we can evaluate the B Matrix at the centroid of the element B bar. And our K is a b bar transpose D times B Bar multiplied by volume of the element.

B-matrix evaluated at the centroid of the element is $\bar{B}$

$$[K] = [\bar{B}]^T[D][\bar{B}] \text{ x } \int_v dv = [\bar{B}]^T[D][\bar{B}] \text{ x volume of element}$$

$$= [\bar{B}]^T[D][\bar{B}] \times 2\pi\bar{r} \text{ x Area of element}$$

And the volume of the element is a 2 pi r Bar times area of the element r Bar is the average radius and 2 pi r bar multiplied by area of the element is volume of the element and that I will illustrate the next slide. And so, in the early finite element programs see this entire 6 by 6 Matrix the programmers used to do multiplication by hand and then give all these 36 terms directly into the computer program so, that we can reduce the load on the work on the computer.

Because that was how the programs used to be developed but now we do not do that because the computers have huge memory and they are also very powerful.

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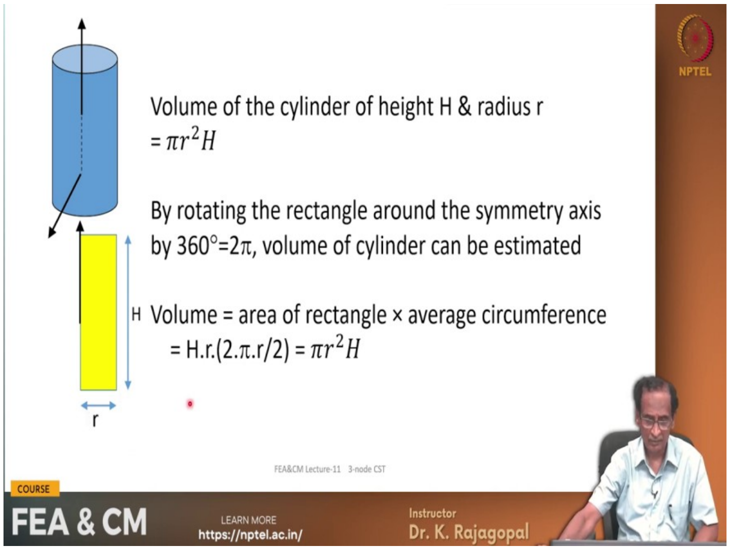
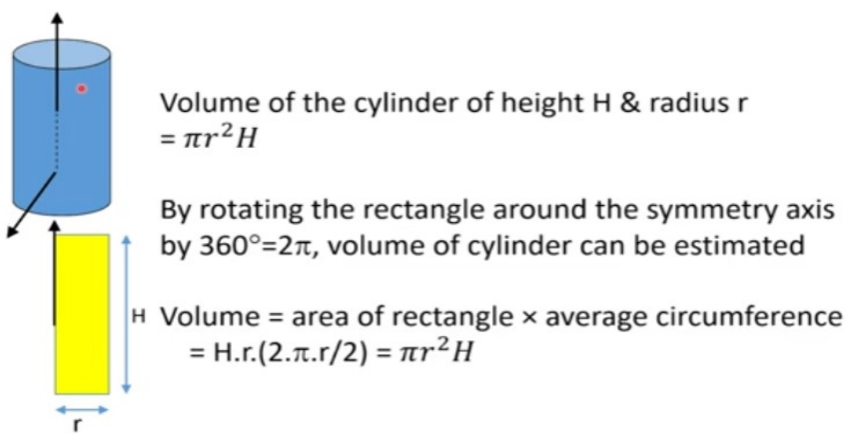


So, let us see what exactly we mean by the volume see we have this cylinder having a radius of r and the height of h. So, the volume of the cylinder is pi r square that is the area of the circle multiplied by H. And we can get the cylinder by rotating a rectangle of radius r rotating around this symmetric axis and rotating full circle by 360 degrees that is equal to 2 pi and by sweeping this around we can get the volume as area of the rectangle that is H multiplied by r is what we are seeing on the on the computer screen.



That is the area of the rectangle multiplied by average circumference that is a 2 pi r by 2 that is equal to Phi r square H that is the same as the volume of the cylinder. So, when we did this calculation that is what we have done 2 pi r Bar that is r Bar is the average radius and 2 pi r bar is the average circumference multiplied by the area of the element that will give you the volume of the element.

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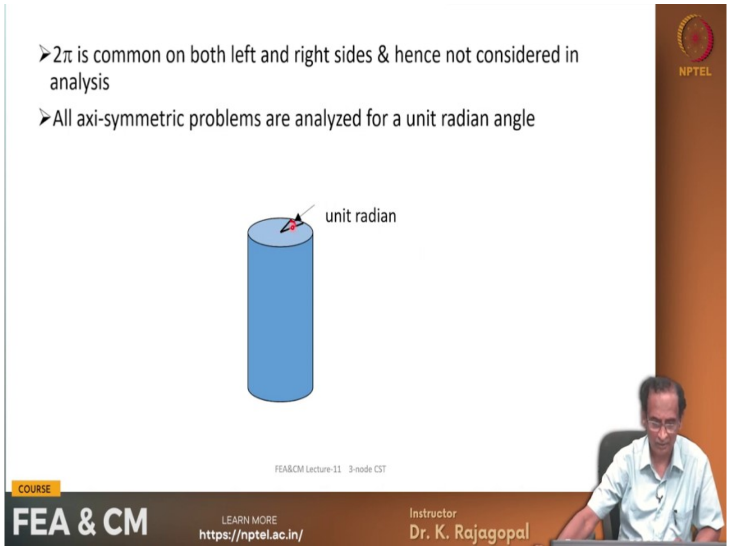

And we do all these calculations for a unit radian because our 2 pi is common and both the left hand side and the right hand side we have this 2 pi in the stiffness term and also and all the load terms. And so, we can we remove the 2 Pi on the left hand side and the right hand side and do the calculations for a unit radian say all our calculations for the axis symmetric case they are done for a unit radian not for the full circle 2 pi r but we do only for the the this r theta where theta is one.

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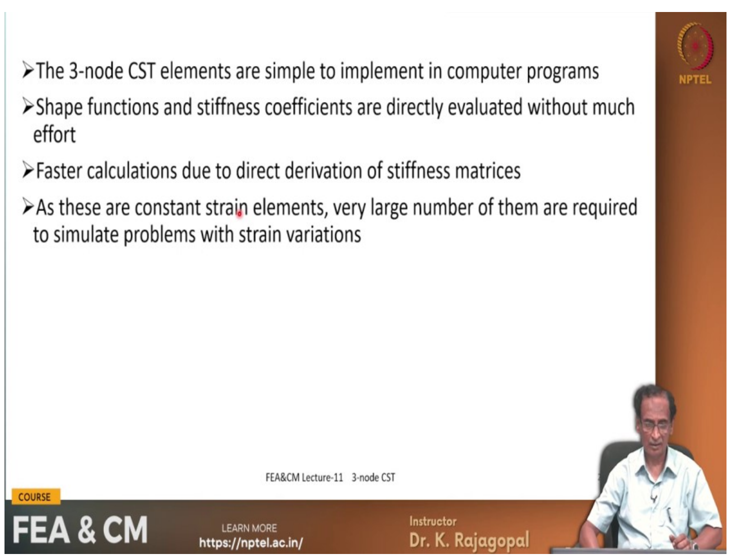

And this 3 node CST are the constant strain triangle that simple to implement in computer programs the shape functions and the stiffness coefficients are directly evaluated without much effort. And faster calculations due to direct derivation of the stiffness matrices and but only limitation is as the strain is constant within the element we require very large number of elements to simulate any problem with the strain variation.

Most of the problems there will be a gradient at the strain and if you want to represent that accurately we need very very large number of CST's constant strain triangles. So, that is a brief introduction on how we can do the computations with the 3 node triangle and if you have any questions please send an email to this address profkkrg@gmail.com. So, thank you very much.