FEM and Constitutive Modelling in Geomechanics
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Lecture: 12
Analysis of Continuum Systems

Let us continue from our previous lectures on this the stress state and then the equilibrium equations and then the equation for strain within a 3-dimensional continuum. And then later we have seen how to simplify the or how to consider simplified stress states in 2 dimensional plane so, that we can we can speed up our analysis. And in today's class let us continue a bit further and look at how we can describe the continuous variation of displacements within the continuum then how we can relate them to strains and then the stresses and then how to develop our equilibrium equations for a continuum.

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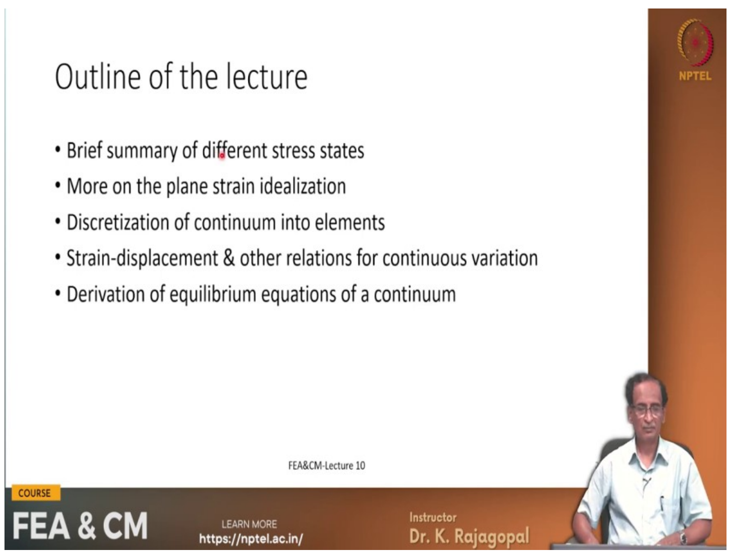

So, let us go through a brief summary of the different stress states that we had seen in the earlier classes then a bit more and the plane strain idealization and then discretization of the continuum into smaller elements. Then strain displacement and other relations and then the derivation of the equilibrium equations because that is the important step in our analysis.

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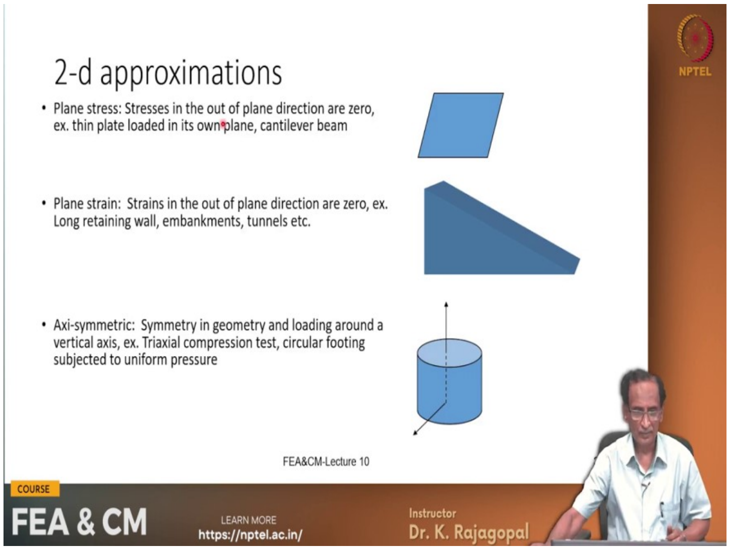

So, in the previous class we had seen these 3 2 dimensional approximations plane stress when all the strain all these stresses act within a single plane and all the outer plane stresses are 0. So, that happens in the case of a thin plate loaded in its own plane or a cantilever beam. Then we have a plane strain state where all the outer plane strains are 0. So, previously it was plane stress now all the outer plane stresses are 0.

In this case its plane strain where all the outer plane strains are 0 and this is a very common situation in a geotechnical engineering where we consider the analysis of very long retaining walls or embankments, tunnels and so on. And the other case is the axis symmetric where there is a symmetry there is a radial symmetry around the central axis. See this radial symmetry is there both in the geometry of the system and also the loading and then the resulting stress states.

So, if you have a radial symmetry like this we can just consider a 2 dimensional stress state that we had discussed earlier and then we will see a bit more in today's lecture. And this axis symmetric case is very common for circular footings under uniform pressure or our own triaxial compression tests geotechnical engineering or the consolidation test and so on.

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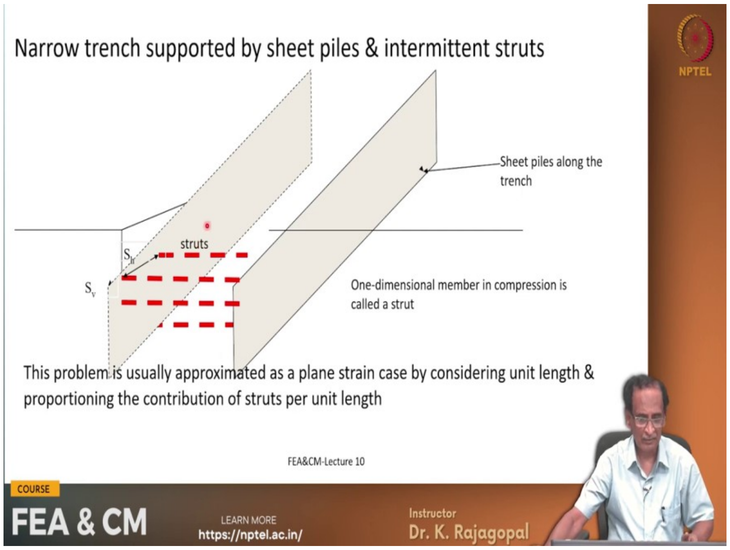

And in the previous class we had seen this example of a plane strain case where we have a deep excavation supported by continuous sheet pile walls or diaphragm walls and then in between we have the struts which are compression members which are placed at some vertical spacing and horizontal spacing. And how can we model this in a plane strain case because the sheet pile wall is not a problem because it is a continuous member but then these struts are discrete members they are placed that some horizontal spacing and vertical spacing.

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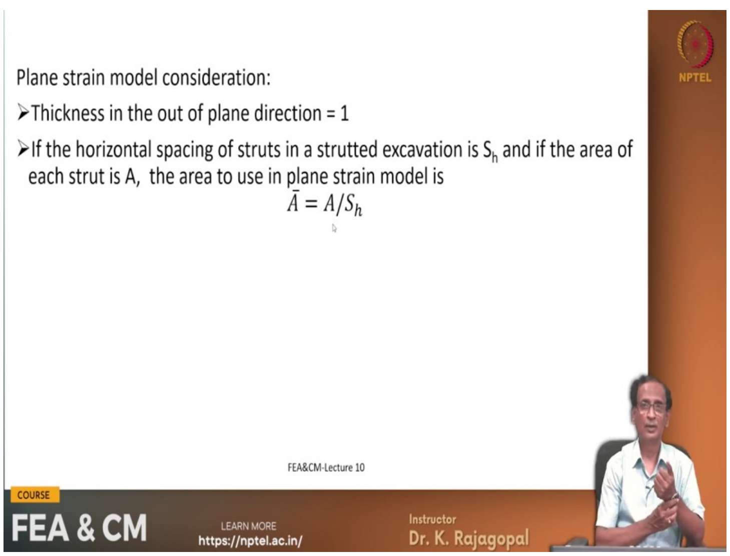

And so, our thickness for plane strain case in the outer plane direction is 1 and if your horizontal spacing of the ties in a structured excavation is as such these are the compression members it should be just bar elements is assets. And if the area of the each bar is A then the area to be used the plane strain model is A by S h. So, if the horizontal spacing of each of these struts in a next equation is Sh and if the area of each strut is a then the area that we use in our plane strain analysis is A by S h.

So, if S h is 0.5 meters then we are going to consider twice the area or if S h is 2 meters then we will consider only half the area in our analysis that is what we mean by prorating per unit length.

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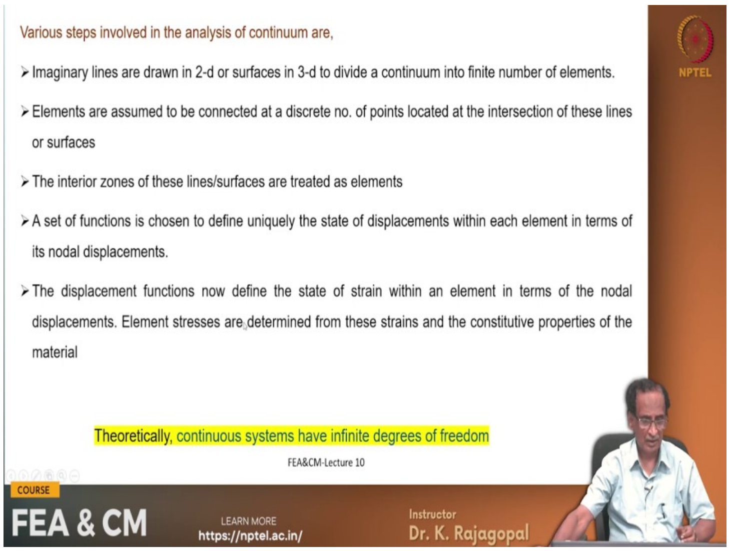

And so, now let us move on to the continuum and what are the different steps. See we are dealing with the displacement basin approach and we have seen the applications for discrete systems like the the planes truss elements and then the plane frame and so on. and it is easy to imagine the number of nodes number of elements and then and then the number of degrees of freedom. But if you have a continuum let us say you take a plate and let us say some dimensions and how many elements are there.

Depending on your imagination you can consider very large number and theoretically a continuum can have infinite degrees of freedom an infinite number of elements but obviously we cannot consider all the infinite numbers. So, what we do is we draw some imaginary lines in the 2 dimensional case or imaginary surfaces in 3 dimensions to divide a Continuum into finite number of elements because we cannot really handle infinite number like infinite number of elements or number of nodes and so on.

And to divide this continuum into a certain number of manageable number we draw some lines in the 2 dimensional problems or surfaces in 3 dimensions. And we assume that elements are connected at discrete number of points located at the intersection of all these lines that we draw or the surfaces that we draw and the interior of these lines are surfaces are treated as elements. And then we choose a set of functions to describe the variation of the displacements within each element.

Because once we define the nodal points we know we are going to solve for them but once we get the displacements of these nodal points how do we find the displacements in the interior of an element because now we are dealing with the continuum so, that means wherever you see there is material. And so, we need some methodology to determine the displacements in the interior of the elements.

And now once you choose some function say this function can define the state of strain within an element like once you have a function for the displacements derivative of that will give you the strain. And so, these strains will be in terms of nodal displacements then once you get the the strains we can get the stresses.

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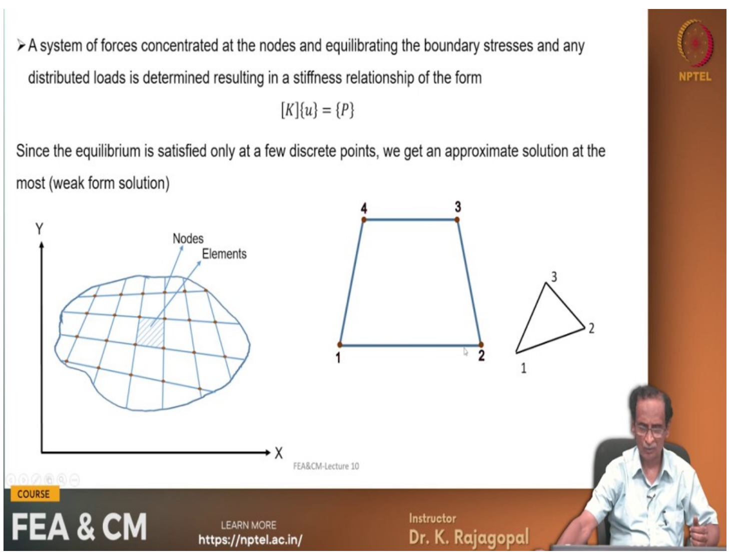
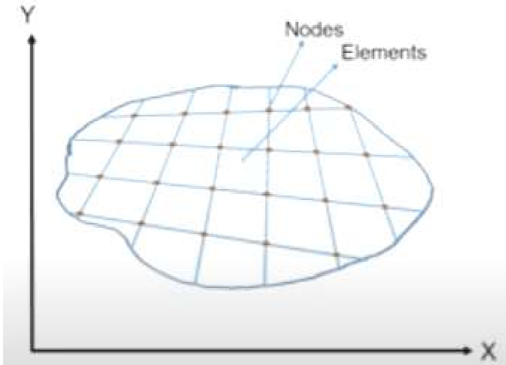


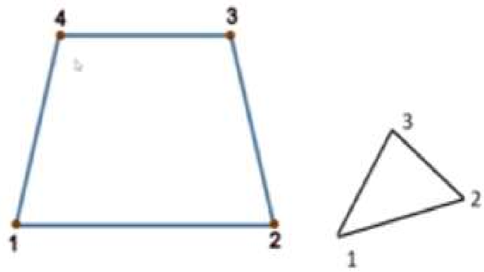
And so, what we do is then we develop our equilibrium equations as internal force is equal to external force if K is the stiffness Matrix K multiplied by a displacement Vector u should be exactly equal to the applied force P.

$$[K]\ \{u\} = \{P\}$$

And in the process we are going to satisfy this equilibrium only at certain number of degrees of freedom or number of points not everywhere. So, the type of solution that we get we call this as a weak form solution it is the solution is not exactly valid at all the points but approximately valid.



So, if you have a 2 dimensional space like this what we do is we draw some arbitrary lines and these lines could be straight lines or curved lines or anything it depends on the imagination of the user and the intersection of all these lines are treated as a as a node and the area in between these intersecting lines is treated as an element. So, here I have highlighted the nodes and then also I highlighted the elements. So, this is a continuum and theoretically it can have infinite number of elements or infinite number of nodes.



But we have divided that into a certain number of nodes by drawing these lines and the more lines that we draw the more realistic your model will be. And so, more closure your solution will be to the real solution and these elements they could be of different shape like they can be quadrilateral elements or triangles or some distorted shaped elements like this. And so, these triangles they have they will have minimum 3 nodes whereas quadrilaterals they will have four nodes and so on.

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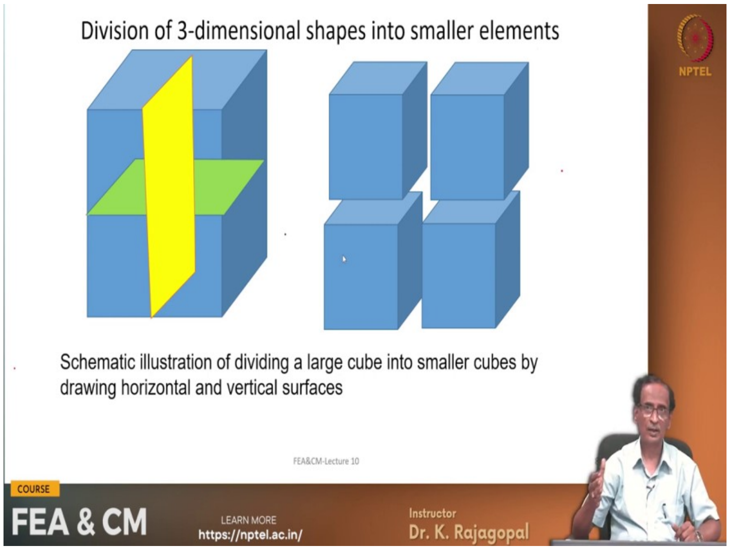
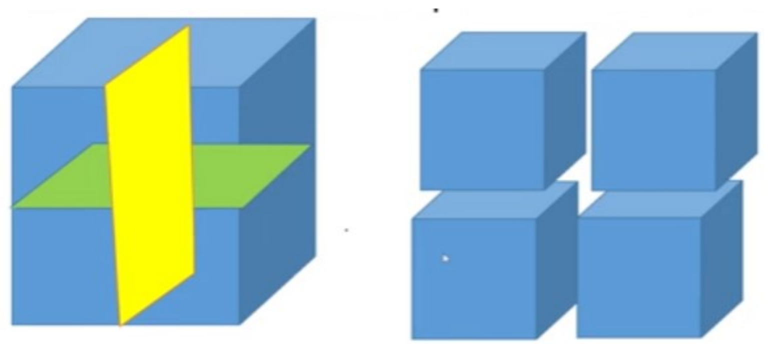


Then if we have a 3-dimensional shape we draw the surfaces let us say; you consider a large Cube like this we have drawn 2 surfaces 1 is yellow surface and then the there is a green surface and then we divide this into four cubes like 1 2 3 four.



Schematic illustration of dividing a large cube into smaller cubes by drawing horizontal and vertical surfaces

And so then if you want you can draw more number of surfaces and divide this continuum into smaller number of elements then once we draw once we get these elements we can identify the nodes as the points at the intersection of all these lines or surfaces.

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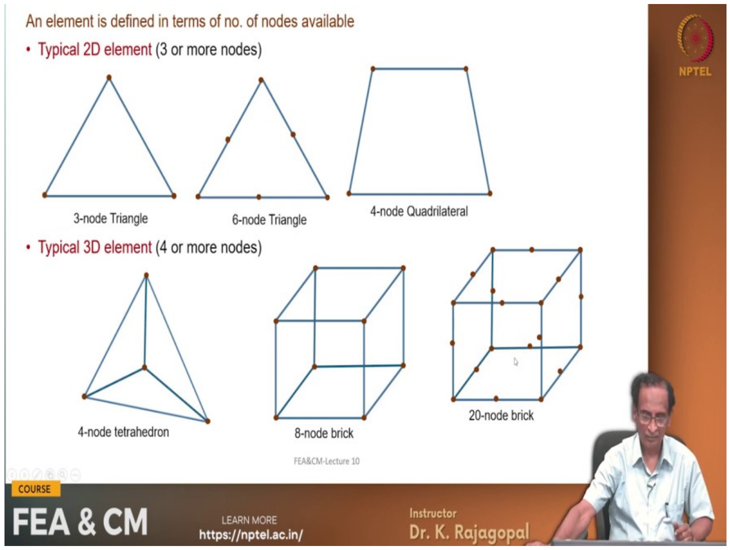

An element is defined in terms of the number of nodes within the element with number of nodes that are available. And typically we can have either 2D elements or 3D and the 2 dimensional elements we need minimum 3 nodes to define a 3 dimensional element like this 3 node triangle or a 6 node triangle and then four node quadrilaterals and actually depending on the on the shape that we have we can decide the number of nodes.

Then if you have more number of nodes along any line we can either can even define a curved surface. So, if you have a circle that also can be approximated the equivalent triangles or rectangles depending on the type of elements that you have in your program. And these are some of the typical 3 dimensional elements we require minimum four nodes to define a 3D shape. Here is a four node tetrahedron and the 8 node brick element or a 20 node brick element and so on.

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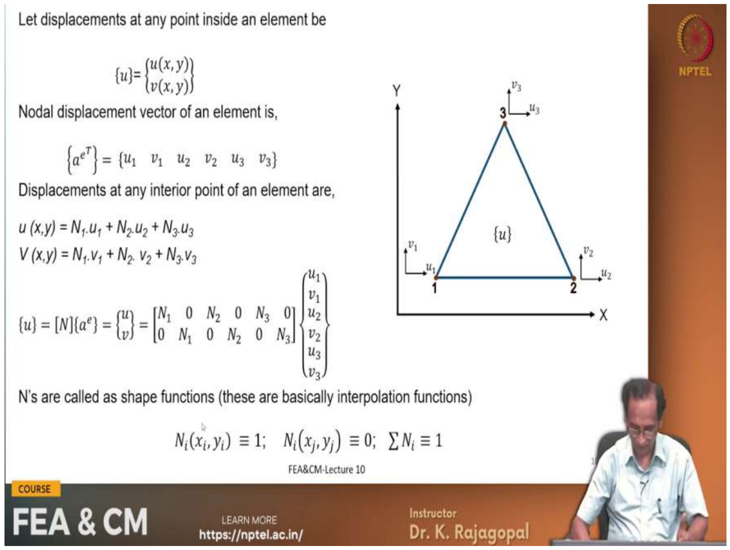

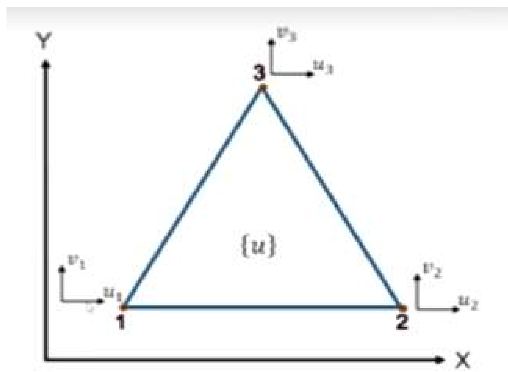

Let us say we have this x and y Cartesian coordinate system and let us define 1 triangular element like this with the nodes 1 2 and 3 defined like this. And we define these node numbers are the element level and the anti-clockwise direction that I will explain a bit later and at each of these points we have the displacements or the degrees of freedom.

Let displacements at any point inside an element be

$$\{u\}=\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$

Nodal displacement vector of an element is,

$$\{a^{e^T}\} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3\}$$

And in this case we have only displacements as the degrees of freedom certain node 1 you have 2 degrees of freedom u 1 v 1 node 2 u 2 v 2 node 3 is u 3 v 3 totally 6 degrees of freedom.

And a a e is the vector of the displacements or the element nodes. So, this is u 1 v 1 u 2 v 2 u 3 v 3 and these are the displacements or the nodal points and let us say that at any point interior we have another displacement vector u and v which is a function of x and y and in general we can write u interior displacement u in terms of the displacements or the nodal points like this N 1 u 1 + N 2 u 2 + N 3 u 3 or v is N 1 v 1 + N 2 v 2 + N 3 v 3 in here our N 1 N 2 and N3 these are called as the shape functions.

Displacements at any interior point of an element are,

$$u(x,y) = N_1.u_1 + N_2.u_2 + N_3.u_3$$
$$V(x,y) = N_1.v_1 + N_2.v_2 + N_3.v_3$$

Basically these are like our interpolation functions depending on the distance of the particular point from these 3 points the value will change. And these are shape functions they will have a value varying from 0 to 1 and the internal displacement u is we can write it as some shape function Matrix N multiplied by nodal displacement Vector a e that is u is N 1 u 1 + N t u 2 + N 3 u 3 and v is N 1 v 1 + N 2 v 2 + N 3 v 3.

$$\{u\} = [N]\{a^e\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

So, in terms of Matrix and Vector product we can write like this our shape function Matrix N is N1 0 N 2 0 N 3 0 N 0 N 1 0 and 2 0 and 3. And this is our nodal displacement vector and these ends are basically interpolation functions but we call them as shape functions in the finite element context because they depend on the shape of the element. Because they have 1 form for triangular elements and some other for rectangular elements and then depending on the on the shape like whether you have straight edges or curved edges your shape functions might change.

N's are called as shape functions (these are basically interpolation functions)

$$N_i(x_i, y_i) \equiv 1; \quad N_i(x_j, y_j) \equiv 0; \quad \sum N_i \equiv 1$$

And the property of these shape functions are their own nodes the shape function value is 1 and then at that nodes these are 0 and the sum total of all the shape functions is exactly equal to 1 these are similar to our influence functions that we come across in our structural analysis. So, if you have a moving load the reaction at different supports is expressed in terms of some interpolation functions. And the similar things we can do for our shape functions.

And we will define or we will derive what these shape functions are later on because that is 1 of our major focus in any finite element analysis.

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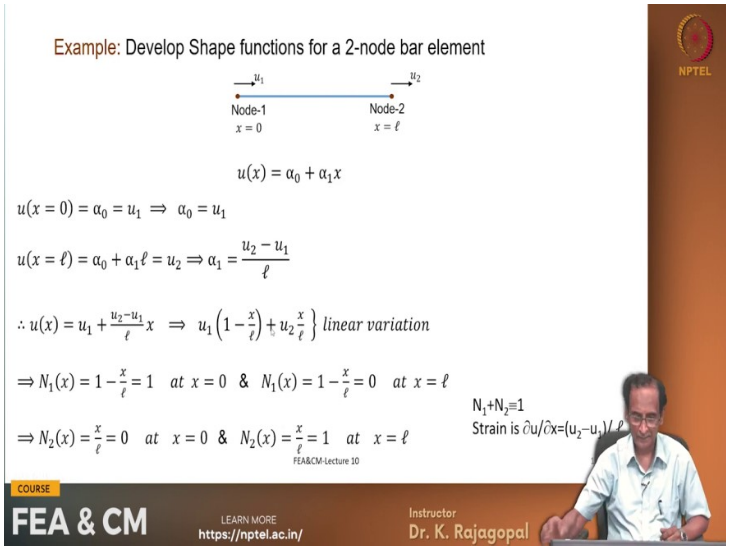

So, let us look at the shape functions for a 2 dimensional sorry the 2 node bar element and let us consider a bar element having a length of L with the Node 1 node 2 and displacement at Node 1 is u 1 displacement at node 2 is u 2 right and we Define or we develop the shape functions by different ways and 1 is called as a generalized coordinate system where we assume some polynomial.

And the number of terms in the polynomial is equal to the number of nodes in the element and then the polynomial order also we can decide based on the type of element that you have. And so, in this case since you have 2 nodes we can have 2 terms in the polynomial you can be Alpha naught + Alpha 1 x because actually this is similar to our Rayleigh-Ritz procedure where we assume the solution in terms of polynomial series the similar thing applies here.

$$u(x) = \alpha_0 + \alpha_1 x$$

And this polynomial could be anything it could be Alpha naught + Alpha 1 x or say Alpha 2x or Alpha 1 x Plus Alpha 2 y x square or anything because since you have 2 nodes you need to assume only 2 terms in this polynomial series and we will discuss how we can or what is the basis on which we can assume these polynomials. So, at say if you assume u as Alpha naught + Alpha 1 x at Node 1 that is at x is equal to 0 all u of x is Alpha naught and that is equal to u

1. $$u(x=0) = \alpha_0 = u_1 \Rightarrow \alpha_0 = u_1$$

$$u(x=\ell) = \alpha_0 + \alpha_1 \ell = u_2 \Rightarrow \alpha_1 = \frac{u_2 - u_1}{\ell}$$

So, that means that our Alpha naught is equal to u 1 and the x is equal to l; u of x is Alpha naught + alpha 1l and that is equal to u 2 at node 2. And so, our alpha 1 can be determined as Alpha 2 minus u 1 by l and server u of x you can substitute the values of alpha naught and Alpha 1 in this equation u of x is u 1 + u 2 minus u 2 minus u 1 by l times x. And by grouping all the terms under the different displacements we can write u of x is u 1 times 1 minus x by l + u 2 times x by l.

$$\therefore u(x) = u_1 + \frac{u_2 - u_1}{\ell} x \Rightarrow u_1\left(1 - \frac{x}{\ell}\right) + u_2 \frac{x}{\ell} \Big\} \textit{ linear variation}$$

$$\Rightarrow N_1(x) = 1 - \frac{x}{\ell} = 1 \quad at \ x = 0 \ \ \& \ \ N_1(x) = 1 - \frac{x}{\ell} = 0 \quad at \ x = \ell$$

$$N_2(x) = \frac{x}{\ell} = 0 \quad at \quad x = 0 \ \ \& \ \ N_2(x) = \frac{x}{\ell} = 1 \quad at \quad x = \ell$$

And we see that we have a linear variation that is corresponding to the linear polynomial that we had assumed ok. So, our N 1 is 1 minus x by l and our N 2 is x by l and at x is equal to 0 that is at node 1 your N 1 is 1 and x is equal to l our N 1 is 0 because if you substitute x of l we get 0 and similarly N 2 is x by l and that is 0 at x is equal to 0 that is at node 1 and at node 2 our value is one x over N 1 + N 2 is exactly equal to 1 that is 1 minus x by l Plus x by l that is equal to 1.

And if you want to determine the strain within the element we just simply take the first derivative dou u by dou x and that comes to just simply alpha 1 and Alpha 1 is u 2 minus u 1 by l.

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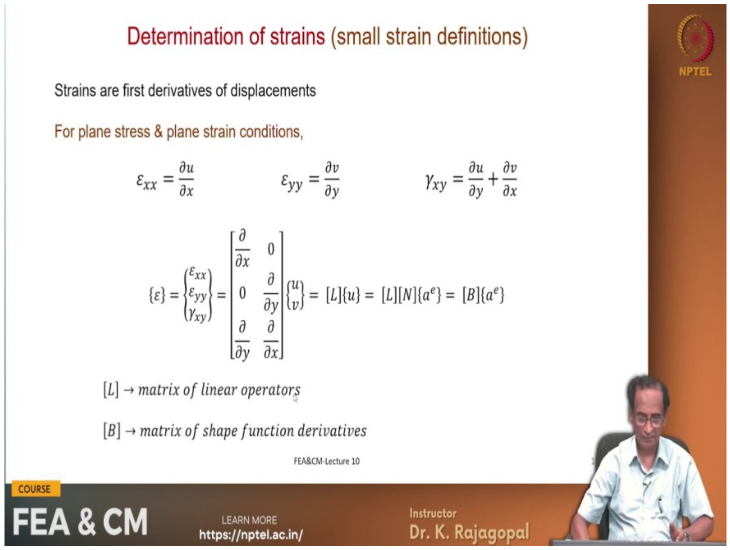

And so, the the determination of strains we have already seen the definition of strains in the previous lecture and if we use the small strain definitions for plane stress and plane strain our Epsilon xx is a dou u by dou x and our Epsilon yy is dou v by dou y and Gamma x y is dou u by dou y + dou v by dou x.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = [L]\{u\} = [L][N]\{a^e\} = [B]\{a^e\}$$

So, our Epsilon the strain Vector is Epsilon xx Epsilon yy gamma xy and actually we can write this as a product of an operator Matrix like this dou by dou x 0 0 by dou y dou by dou y and dou by dou x multiplied by displacement Vector u and v where u and v are the the displacements are the interior points.

And this Matrix we can write as l that is the Matrix of the linear operators times u and our u is actually N times a e we have already expressed that in the previous step. Here we have seen that our internal displacement to u can be written as N times a e where N is the shape function Matrix and a e is the vector of nodal displacements. So, our l times N we can is written as B. And so, our Epsilon we can write as B times a e where B is actually the shape the derivatives of the shape functions that we will see in the next step.

$[L] \rightarrow matrix\ of\ linear\ operators$

$[B] \rightarrow matrix\ of\ shape\ function\ derivatives$

Epsilon is B times a e and our B is actually it is very important Matrix the that relates the the displacements to the strains or the strains to the displacements l is the Matrix of linear operators and the B is the Matrix of shape function derivatives and B is l times N.

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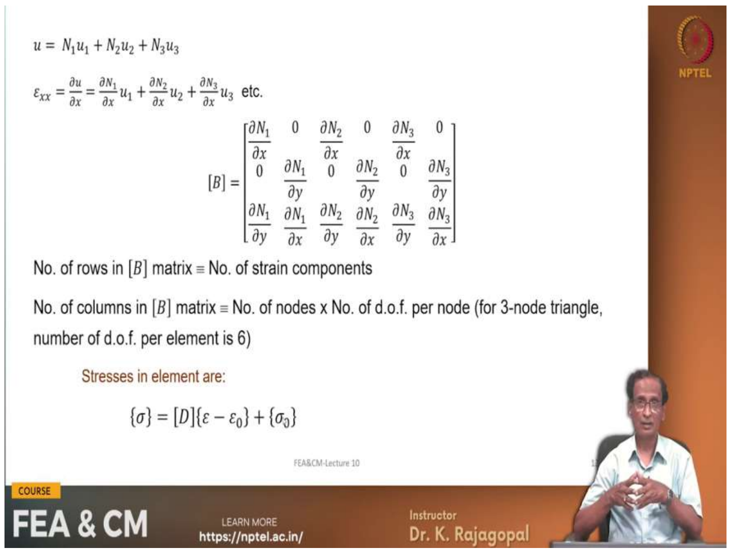

$$u = N_1u_1 + N_2u_2 + N_3u_3$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x}u_1 + \frac{\partial N_2}{\partial x}u_2 + \frac{\partial N_3}{\partial x}u_3 \text{ etc.}$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

And our u is N 1 u 1 + N 2 u 2 + N 3 u 3 and our Epsilon xx you can write as dou u by dou x that is dou N 1 by dou x u 1 + dou N 2 by dou x u 2 + dou N 3 by dou x u 3. And similarly Epsilon yy we can write as dou v by dou y that is dou N 1 by dou y v 1 + dou N 1 by dou N 2 by dou y u 2 + v 2 + dou N 3 by dou y v 3. So, this B Matrix that is the Matrix of the shape function derivatives we can write like this say the first row refers to Epsilon xx the second row refers to Epsilon yy these are the 2 normal strains.

And then the third one refers to the shear strain gamma xy. So, dou N 1 by dou x dou N 2 by dou x dou N 3 by dou x and so on. And the second row Epsilon y dou N 1 by dou y dou N 2 by dou y dou N 3 by dou y and the third row is for the shear strength gamma xy . So, the number of rows in the B Matrix is equal to the number of strain components see for the plane stress and plane strain we have only 3 strain components.

So, we have 3 rows for axis symmetric we have four strain components. So, we will have four rows and for a full 3 dimensional case we have 6 strains. So, we will have 6 rows in the B the B Matrix then the number of columns is equal to the number of nodes in the element multiplied by number of degrees of freedom. So, for a 3 node triangle we have totally 6 degrees of freedom 3 nodes multiplied by 2 that is 6.

And so, we have 6 columns 1 2 3 4 5 6. So, once you have the stresses now sorry the strains we can determine the stresses as Sigma is D times Epsilon minus Epsilon naught + Sigma naught where Epsilon naught are the initial strains and Epsilon is the is the is the strain and our displacement vector may also consist of initial displacements. And so, we get the total strain Epsilon minus the Epsilon naught corresponding to some initial strains multiplied by d plus initial stresses Sigma naught.

Stresses in element are:

$$\{\sigma\} = [D]\{\varepsilon - \varepsilon_0\} + \{\sigma_0\}$$

And the sigma naught in the geotechnical constant context could be the in-situ pressures and that depends on the geological conditions that we will see later.

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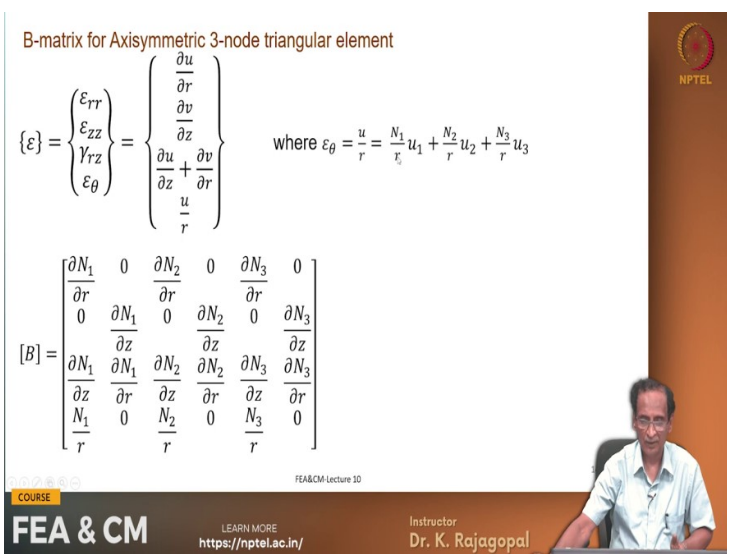

So, how do we write the B Matrix for axis symmetric case? So, for the axis symmetric case we have four strain components Epsilon rr Epsilon zz gamma rz and Epsilon Theta. Epsilon Theta is what we had seen in the previous class circumferential strain that the change in the circumference because of any displacements and this we have seen as u by r and our Epsilon Theta is a radial displacement u divided by the corresponding radial distance r and that can be written as N 1 by r u 1 + N 2 by r u 2 + N 3 by r u 3. So, the first 3 rows for the axis symmetric case are similar to our B Matrix for plane stress and plane strain except in place of x we have r and in place of y we have z.

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \gamma_{rz} \\ \varepsilon_{\theta} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial v}{\partial r} \\ \dfrac{u}{r} \end{Bmatrix} \quad \text{where } \varepsilon_{\theta} = \frac{u}{r} = \frac{N_1}{r}u_1 + \frac{N_2}{r}u_2 + \frac{N_3}{r}u_3$$

And then the fourth row e is our N 1 by r N 2 by R and N 3 by r and since there are no v terms we have 0s in these second fourth and 6th columns.

$$[B] = \begin{bmatrix} \dfrac{\partial N_1}{\partial r} & 0 & \dfrac{\partial N_2}{\partial r} & 0 & \dfrac{\partial N_3}{\partial r} & 0 \\ 0 & \dfrac{\partial N_1}{\partial z} & 0 & \dfrac{\partial N_2}{\partial z} & 0 & \dfrac{\partial N_3}{\partial z} \\ \dfrac{\partial N_1}{\partial z} & \dfrac{\partial N_1}{\partial r} & \dfrac{\partial N_2}{\partial z} & \dfrac{\partial N_2}{\partial r} & \dfrac{\partial N_3}{\partial z} & \dfrac{\partial N_3}{\partial r} \\ \dfrac{N_1}{r} & 0 & \dfrac{N_2}{r} & 0 & \dfrac{N_3}{r} & 0 \end{bmatrix}$$

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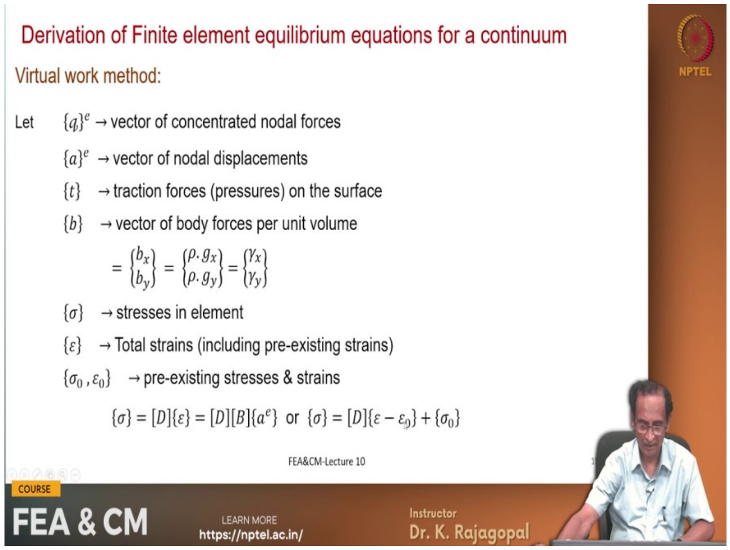

And now let us see how we can derive the equilibrium equation for a continuum because this is what we require so, that we can do all the computations. And we are going to follow the virtual work method and we have seen this virtual work method earlier for the Prismatic elements yeah that states that internal work done should be exactly equal to the external work done for equilibrium.

And we will follow that approach that will give us an easy method for establishing the equilibrium equations and let us say let us define some quantities. When we have a continuum we can have different types of loads. See one is the concentrated nodal loads let us say that that is given by vector q and with a superscript e that refers to the element because we are going to derive all these at element level and then we are going to assemble them into some global matrix corresponding to the entire enter structure.

Let $\{q\}^e$ → vector of concentrated nodal forces

$\{a\}^e$ → vector of nodal displacements

$\{t\}$ → traction forces (pressures) on the surface

$\{b\}$ → vector of body forces per unit volume

$$= \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \begin{Bmatrix} \rho . g_x \\ \rho . g_y \end{Bmatrix} = \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix}$$

$\{\sigma\}$ → stresses in element

$\{\varepsilon\}$ → Total strains (including pre-existing strains)

$\{\sigma_0 , \varepsilon_0\}$ → pre-existing stresses & strains

$$\{\sigma\} = [D]\{\varepsilon\} = [D][B]\{a^e\} \text{ or } \{\sigma\} = [D]\{\varepsilon - \varepsilon_0\} + \{\sigma_0\}$$

And a e is the vector of nodal displacements and t is the traction forces on the surface the traction force is nothing but the pressure. So, the external loading we can have as concentrated loads or uniform pressure or something like the traction force and then the other type of force that you can have is the body force vector of body forces per unit volume B is basically because of the gravitational effect on the unit weight.

So, gamma is our body force vector and in general we can have a gamma in x direction gamma in the y direction. And so at b is the body force vector is a b, b x and b y rho g x and rho g y that is gamma x and Gamma y and if our x axis is in the horizontal direction and y axis is in the vertical direction our gamma x could be just simply 0 because we do not have any gravitational force in the horizontal direction and then in the vertical Direction since the weight is acting down we will have minus gamma for gamma y and the stresses in the element and sigma.

And let us say epsilon is the total strain including the pre-existing strains and sigma naught and Epsilon naught are the pre-existing stresses and strains. So, our stress vector d we can calculate as d times Epsilon and Epsilon is b times a e and we can expand this the stress in terms of initial strain and initial stresses like this d times Epsilon minus Epsilon naught + Sigma naught right.

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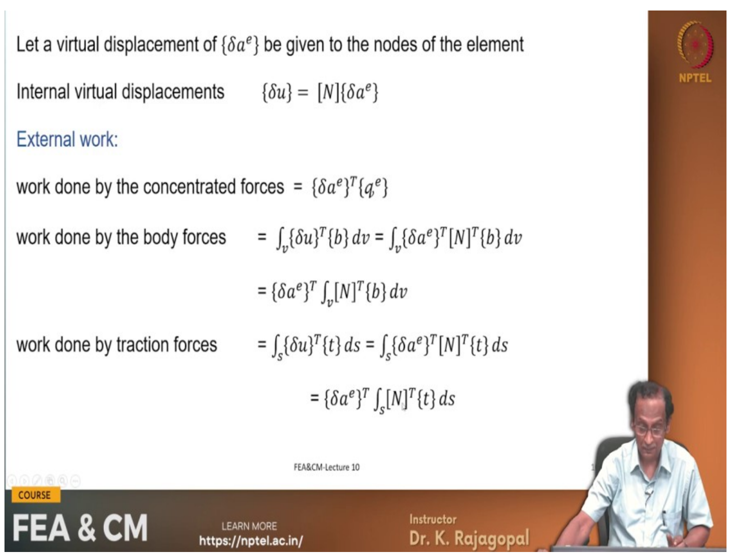

Let us say we give some virtual displacement to the system that is already under equilibrium. So, under the action of all these the body is in equilibrium and we are giving some disturbance Delta a e right are the element nodes we are applying some virtual displacement field Delta a e and because of that the internal displacements Delta u can be calculated as N times Delta a e right where our N is the shape function matrix N 1 0 N 2 0 and 3 0 0 and 1 0 N 2 0 and 3.

Let a virtual displacement of $\{\delta a^e\}$ be given to the nodes of the element

Internal virtual displacements $\{\delta u\} = [N]\{\delta a^e\}$

The number of rows in the shape function matrix is equal to the number of degrees of freedom then the number of columns is equal to the number of degrees of freedom in the element. So, the external work done the work done by the concentrated forces is Delta a transpose multiplied by q e that is the work done by the virtual displacements and then work done by the body forces is actually our body force is acting out the entire volume.

External work:

work done by the concentrated forces $= \{\delta a^e\}^T\{q^e\}$

work done by the body forces $= \int_v \{\delta u\}^T\{b\}\,dv = \int_v \{\delta a^e\}^T[N]^T\{b\}\,dv$

So, it should be integrated over the volume and Delta e u transpose b where Delta u is our internal displacements u x and u y transpose times b that is b x and b y and integrated over the entire volume and our Delta u is N times Delta a e right. So, if you take transpose of this whole thing we get Delta a e transpose and transpose b. And in here our Delta a e refer to the nodal displacements and they are not going to vary within the element.

So, we can actually bring out this Delta a e outside the integral because these are constants for this element. So, Delta a e transpose integral over volume and transpose b dv where N is our shape function Matrix N 1 0 and 0 and 2 0 and so on, b is our body Force Vector b x and b y. And similarly the work done by the traction forces these are acting on the surface of the element. So, we need to take an integral over the surface of the element Delta u transpose t and once again Delta u is N times Delta a e.

work done by the body forces $= \int_v \{\delta u\}^T\{b\}\,dv = \int_v \{\delta a^e\}^T[N]^T\{b\}\,dv$

$= \{\delta a^e\}^T \int_v [N]^T\{b\}\,dv$

work done by traction forces $= \int_s \{\delta u\}^T\{t\}\,ds = \int_s \{\delta a^e\}^T[N]^T\{t\}\,ds$

$= \{\delta a^e\}^T \int_s [N]^T\{t\}\,ds$

So, the transpose of that is Delta a e transpose and transpose t ds and once again Delta a e is constant because they refer to the to the displacements of the nodes. So, we can bring this Vector out Delta a e transpose integral over the surface and transpose t ds.

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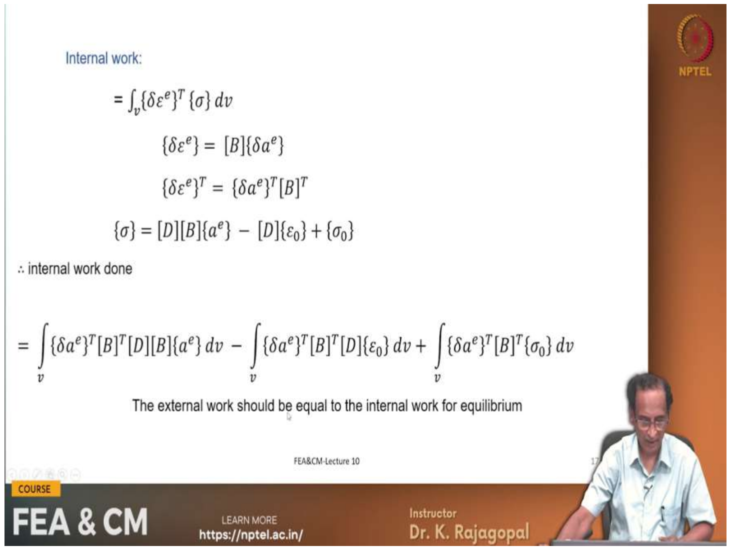

And and our internal work done is corresponding to the stresses and strains that we have inside and our internal work done is in Delta Epsilon transpose Sigma where Sigma is our stress Vector Delta Epsilon is our strain vector and our Delta Epsilon is b times Delta a e right. And our Delta Epsilon transpose if we take a transpose of this Delta u e transpose B transpose. So, actually whenever we take a transpose of a product the terms get interchanged.

Internal work:

$$= \int_v \{\delta\varepsilon^e\}^T \{\sigma\}\, dv$$

$$\{\delta\varepsilon^e\} = [B]\{\delta a^e\}$$

$$\{\sigma\} = [D][B]\{a^e\} - [D]\{\varepsilon_0\} + \{\sigma_0\}$$

So, that is what we see here Delta a e transpose B transpose and our Sigma is d times b a e this b times a e is our Epsilon right minus d times Epsilon naught + Sigma naught. And so, our internal work done is integral Delta a e transpose times B transpose that is Delta Epsilon transpose d b a e dx dv minus integral of our work done due to the initial strains Delta a transpose B transpose d Epsilon naught Plus the work done due to the due to the initial stresses Delta e transpose B transpose Sigma naught dv.

∴ internal work done

$$= \int_v \{\delta a^e\}^T[B]^T[D][B]\{a^e\}\, dv - \int_v \{\delta a^e\}^T[B]^T[D]\{\varepsilon_0\}\, dv + \int_v \{\delta a^e\}^T[B]^T\{\sigma_0\}\, dv$$

And by the virtual work principles the work done by the external forces should be exactly equal to the internal work done.

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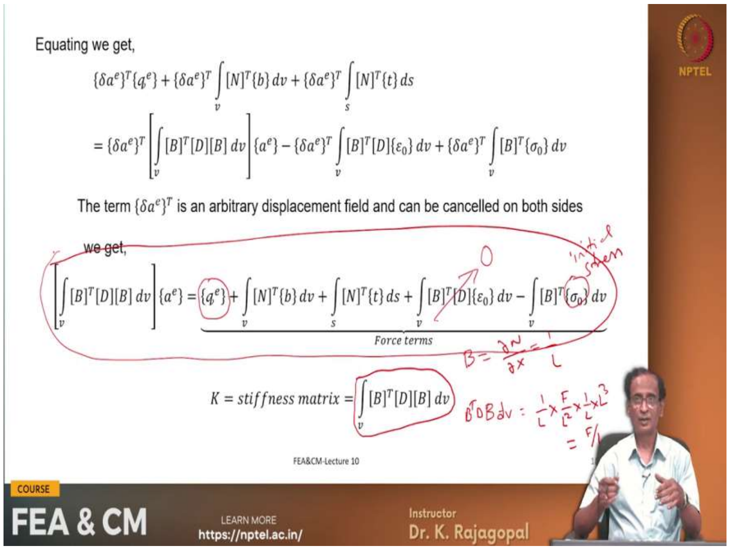

So, equating these 2 we get this inequality this is the external work done that we have calculated earlier and this is equal to the internal work done.

Equating we get,

$$\{\delta a^e\}^T\{q^e\} + \{\delta a^e\}^T \int_v [N]^T\{b\}\, dv + \{\delta a^e\}^T \int_s [N]^T\{t\}\, ds$$

$$= \{\delta a^e\}^T \left[\int_v [B]^T[D][B]\, dv\right]\{a^e\} - \{\delta a^e\}^T \int_v [B]^T[D]\{\varepsilon_0\}\, dv + \{\delta a^e\}^T \int_v [B]^T\{\sigma_0\}\, dv$$

The term $\{\delta a^e\}^T$ is an arbitrary displacement field and can be cancelled on both sides

And so, this Delta a e transpose is common on both left hand side and also on the right hand side. So, we can we can take them out we can cancel them out on both left hand side and the right hand side. So, we get B transpose D B dv and once again our a e is the vector of nodal displacements and this we can take out of the integral we don't need to keep it in the integral.

$$\left[\int_v [B]^T[D][B]\, dv\right]\{a^e\} = \underbrace{\{q^e\} + \int_v [N]^T\{b\}\, dv + \int_s [N]^T\{t\}\, ds + \int_v [B]^T[D]\{\varepsilon_0\}\, dv - \int_v [B]^T\{\sigma_0\}\, dv}_{Force\ terms}$$

So, we have this bracket large bracket integral B transpose dv integrated of the volume multiplied by a e that is equal to q e + N transpose b dv + N transpose t ds + B transpose d Epsilon naught minus integral of B transpose Sigma naught dv right. And on the right hand side we recognize that it is a force Vector in fact our q e is the vector of the applied forces and then transpose b is the vector of the force Vector due to the body weight and then transpose T is because of the surface pressures that we applied B transpose d Epsilon naught is the is the vector of nodal forces because of the initial strains.

And B transpose Sigma naught is the vector of forces because of the initial stresses. And so, actually here on the left hand side we have some quantity B transpose d dv integrated over volume and multiplied by displacement Vector a e and that is equal to the force. So, this quantity this unknown quantity B transpose D B must be stiffness right because as we recall the stiffness multiplied by displacement is equal to force.

$$K = stiffness\ matrix = \int_v [B]^T [D][B]\, dv$$

So, on the right hand side we have the force on the left hand side we have something that we do not know what it is B transpose D B integrated or volume v multiplied by displacement. So, that means that this quantity in the square brackets must be stiffness Matrix B transpose D B dv and actually it is easy to easy to see the the units for this and let us see what they are let me write the pen . So, our B is what is actually dou N by dou x and so on.

And our shape function matrix is actually it is dimensionless. So, as we have seen for the 2 node bar element N 1 is 1 minus x by l and N 2 is x by l. So, actually it has the units of 1 by l. So, our B transpose sorry we can see the units as 1 by l d is the constitute Matrix it has the units of stress force by L Square multiplied by 1 by l multiplied by volume l Cube. So, that has the units of force by l that is the force per unit displacement.

So, we see that this quantity that we have here is it has the units of this stiffness 1 by l. And so, this is the equation for the stiffness Matrix of a continuum and if you are able to evaluate this quantity then we can get our stiffness Matrix. And then we have this the vector of nodal displacements that we need to solve for on the right hand side we have all these quantities this is our q e is the vector of Applied forces N transpose B is the force Vector due to the body weight.

And N transpose T is because of the surface pressures and this Epsilon naught is the body for sorry the force Vector due to the initial strains and what is an initial strain in geotechnical engineering is very difficult to define because our soil straight may have millions of years of History and for defining the initial strains we need initial configurations that we do not know. And usually we neglect this and we do not consider initial strains because we are more interested in knowing what happens after we construct a let us say building or an embankment or a tunnel.

Because we are more interested in what happens after you construct. Before you construct the soil might have undergone lot of consolidation settlements or erosion or anything but we are not really concerned what happened in the historical past but what is going to happen after we construct. So, in general the Epsilon not is not considered I mean geotechnical engineering so, we do not usually evaluate this quantity whereas for Structural Engineers it is a very important quantity.

Because the Epsilon are not may be related to misfit of a member or because of some other reasons and this Sigma naught is actually is very important it is it is the initial stress and this could be because of our pre-existing stresses and that is a very important 1 and we can measure this quantity. So, we can perform some pressure meter test or the dielectometer tests and then and determine the in situ pressures.

The vertical pressure is the normally we estimate by based on the unit weight and then the lateral pressures we estimate by doing some field test. And so, the sigma naught is very important and Epsilon naught is normally we do not consider. And so, if you are able to calculate all these terms we will be able to do the computations for a continuum. Because these things we have done for prismatic elements for bar, beam and spring elements we have done this.

And now we need to do the same thing for a Continuum if we are able to evaluate all these quantities we should be able to solve our system of finite element equations for a continuum.

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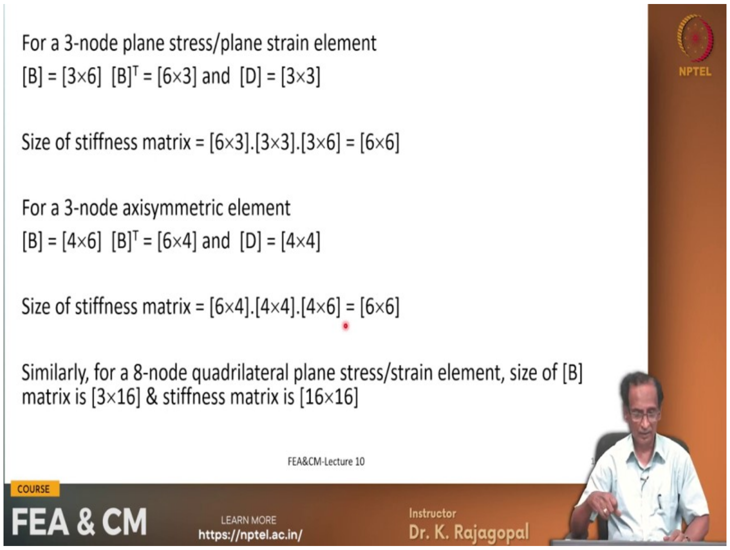

So, I think this may be the last slide oh sorry I think you have some more let me see let me just. So, let us see say for the 3 node plane stress and plane strain elements our B Matrix is a 3 by 6 Matrix 3 number of rows corresponding to 3 strains and 6 columns corresponding to 6 degrees of freedom. And our B transpose is a 6 by 3 and d is 3 by 3 and our size of the stiffness Matrix is B transpose D B.

For a 3-node plane stress/plane strain element
$[B] = [3\times6]$ $[B]^T = [6\times3]$ and $[D] = [3\times3]$

Size of stiffness matrix = $[6\times3].[3\times3].[3\times6] = [6\times6]$

B transpose 6 by 3 D is a 3 by 3 and B is 3 by 6 and the total product is a 6 by 6 Matrix that is the stiffness Matrix this 6 corresponds to the number of degrees of freedom that we have in the element and for the axis symmetric element we have four strain components.

For a 3-node axisymmetric element
$[B] = [4\times6]$ $[B]^T = [6\times4]$ and $[D] = [4\times4]$

So, B is four by 6 B transpose is a 6 by four. So, our D is four by four. So, our size of the stiffness Matrix is once again 6 by 6. So, for say for an eight node quadrilateral will have 16 degrees of freedom our stiffness Matrix will be a 16 by 16 Matrix.

Size of stiffness matrix = $[6\times4].[4\times4].[4\times6] = [6\times6]$

Similarly, for a 8-node quadrilateral plane stress/strain element, size of [B] matrix is $[3\times16]$ & stiffness matrix is $[16\times16]$

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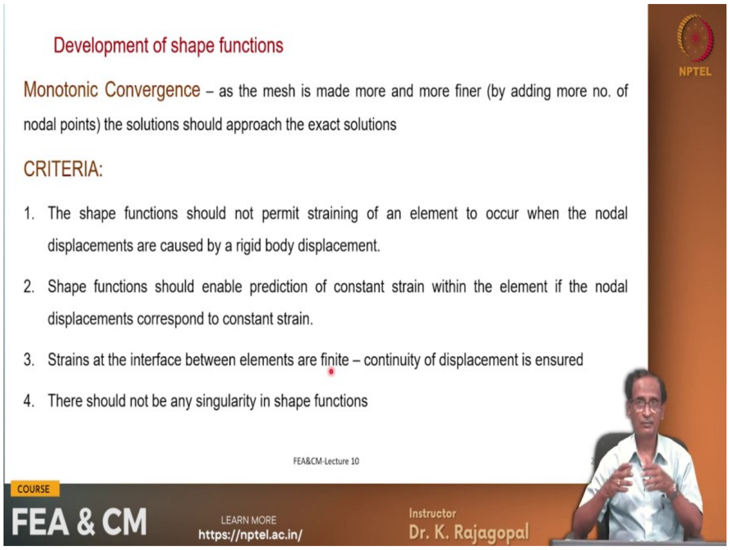

So, the next step is we need to develop our shape functions. See the shape function is the major component that we need to develop because our shape function Matrix is dependent on the shape functions and then our B Matrix is also dependent on the on the shape functions that we have N 1 and 2 and 3 and so on. And what are different criteria for determining these shape functions?

See we have to ensure that as we are making the mesh finer and finer we should be approaching the theoretical solution the exact solution and that we call as the monotonic convergence. As the mesh is made more and more finer by adding more number of nodal points the solution should approach the exact solution and that is called as the monotonic convergence. And some of the requirements are that the shape functions that we define they should not permit the straining of an element to occur when the nodal displacements are caused by the rigid body displacement.

Like for example if I take this pen if I apply some compression like this there is some force. But let us say I take this pen and simply move it from 1 place to the other that we call as the rigid body displacement that all the nodes within the element are subjected to the same displacement. And if that is so, there should not be any forces developed in the rigid body and in the body and the shape functions should enable prediction of constant strain within the element.

If the nodal displacements correspond to constant strain say for example if I compress this pen there is some strain and whatever may be the point within the pen and the strain is constant because that is what is applied on the system and our shape function should be able to predict this constant strain. And the strains at the interface between the elements need to be finite and the continuity of displacements is ensured and there should not be any singularity in the shape functions.

Say once we define the shape functions at all the points within the element they should have some finite value they cannot have some infinite value and that is what we mean by the shape functions not having any singularity.

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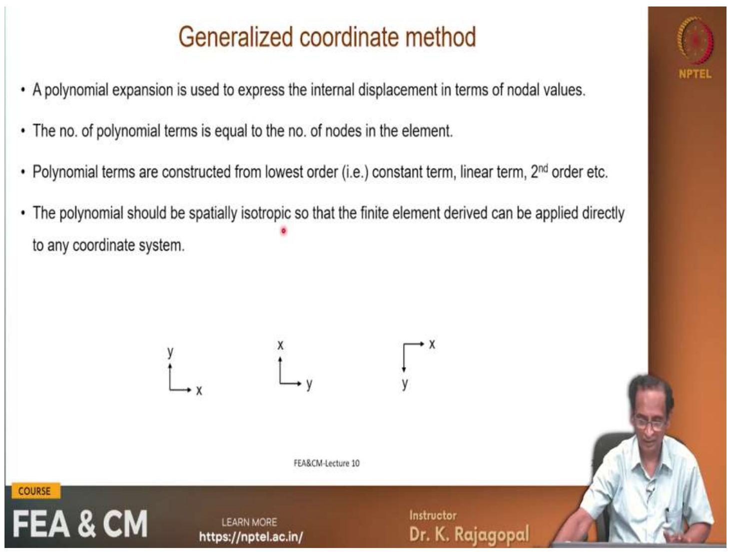
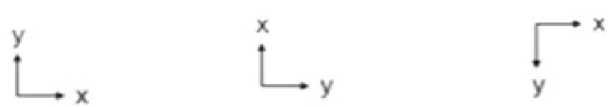


And said one of the simplest methods is the generalized coordinate method that we can use for developing our shape functions. So, we assume a polynomial to express the internal displacements in terms of the nodal values and the number of terms in the polynomial equation is equal to the number of nodes in the element. And the polynomial terms are constructed from the lowest order and as we go further we add higher order terms like first constant term next linear term and then the quadratic term cubic term and so on.



And the polynomial should be spatially isotropic. So, that the finite element derived can be applied directly to any coordinate system. So, our coordinate system is also dependent on the user say some users may prefer this coordinate system like axis along the horizontal axis y is along the vertical axis or you might even interchange somebody might want y to be horizontal x to be vertical or say we might have we might flip so, that our y is a positive downwards instead of positive upwards index is horizontal.

So, whatever may be the coordinate system that we have our result should be the same. So, if you get 10 millimeter displacement here we should get the same 10 millimeters even with other coordinate system. So, actually in this the value might change to minus 10 and here it could be + 10 because our y direction displacements are changing other than that the values themselves should not change and that is what we mean by spatial isotropic coordinate system.

Whatever may be whatever may be the coordinate system that we have we should get the same value.

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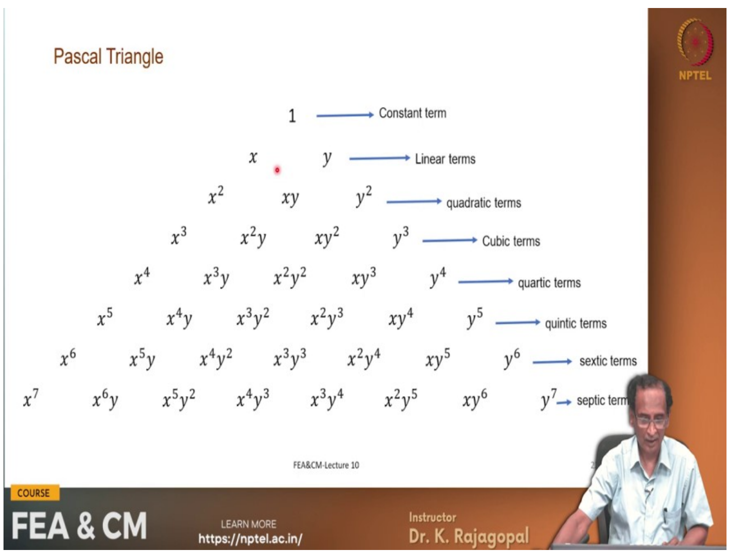

And to help us in choosing the terms we have a Pascal triangle and this particular 1 is in 2 dimensions in terms of x and y and we need to include a constant term in the polynomial and then the linear terms and then quadratic terms x square xy y Square and then the cubic terms x Cube x square y xy Square y Cube and so on. The quadric terms quintic sectic septic and so on.

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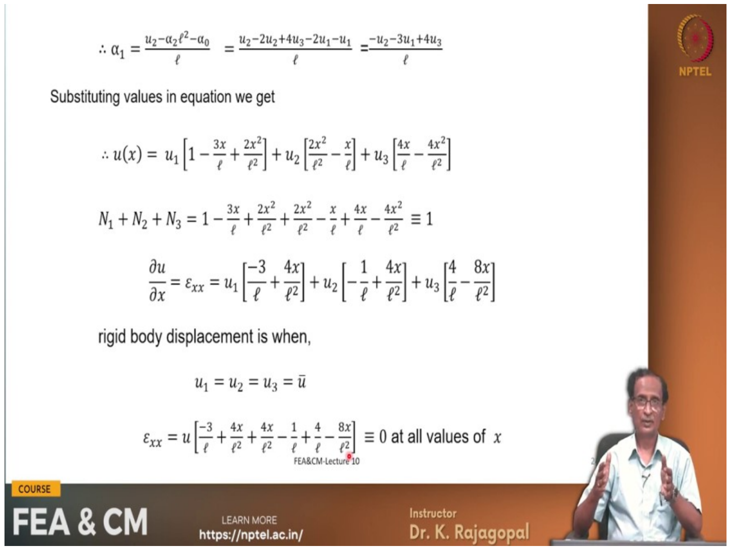
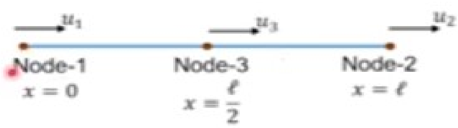


So, this and there is one example given here previously we had derived the shape functions for a 2 node bar element and let us consider a 3 node bar element with node 1 at x is equal to 0 node 2 at x is equal to l and node 3 at Mid length l by 2 and because we have 3 nodes we need 3 terms. And so, our displacement function u of x can be assumed as Alpha naught + Alpha 1 x + Alpha to x square we can have up to the quadratic term x square.



So, if you substitute x is equal to 0 u of x is Alpha naught and that should be equal to u 1 and similarly we can set x is equal to l when your u is u 2 and x is equal to l by 2 u is u by 3 and so on..

$$u(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$$

$$u(x=0) \implies \alpha_0 = u_1 \text{ ---------------- } 1$$

$$u(x=\ell) \implies \alpha_0 + \alpha_1 \ell + \alpha_2 \ell^2 = u_2 \text{ ------------- } 2$$

$$u\left(x=\frac{\ell}{2}\right) \implies \alpha_0 + \alpha_1 \frac{\ell}{2} + \alpha_2 \frac{\ell^2}{4} = u_3 \text{ ------------- } 3$$

And so, by solving we can get the 3 constants Alpha 1 Alpha naught Alpha 1 and Alpha 2 and then substituting and then grouping the terms under u 1 u 2 and u 3 we get like this our N 1 is 1 minus 3 x by l + 2 x square by l square and 2 is 2 x square by l Square minus x by L and 3 is four x by l minus 4 x square by l Square.

Substituting values in equation we get

$$\therefore u(x) = u_1\left[1 - \frac{3x}{\ell} + \frac{2x^2}{\ell^2}\right] + u_2\left[\frac{2x^2}{\ell^2} - \frac{x}{\ell}\right] + u_3\left[\frac{4x}{\ell} - \frac{4x^2}{\ell^2}\right]$$

$$N_1 + N_2 + N_3 = 1 - \frac{3x}{\ell} + \frac{2x^2}{\ell^2} + \frac{2x^2}{\ell^2} - \frac{x}{\ell} + \frac{4x}{\ell} - \frac{4x^2}{\ell^2} \equiv 1$$

$$\frac{\partial u}{\partial x} = \varepsilon_{xx} = u_1\left[\frac{-3}{\ell} + \frac{4x}{\ell^2}\right] + u_2\left[-\frac{1}{\ell} + \frac{4x}{\ell^2}\right] + u_3\left[\frac{4}{\ell} - \frac{8x}{\ell^2}\right]$$

And if you add up all the shape functions N 1 + N 2 + N 3 that should come as 1 and that is indeed 1 like if you add up all the terms you get 1. And then our strain dou u by dou x is Epsilon x that is if you differentiate u 1 times minus 3 by l + 4x by l square + u 2 times minus 1 by l + four x by l Square + u 3 times 4 by L minus 8 x by l square right and if you apply a rigid body displacement for the system.

rigid body displacement is when,

$$u_1 = u_2 = u_3 = \bar{u}$$

$$\varepsilon_{xx} = \bar{u}\left[\frac{-3}{\ell} + \frac{4x}{\ell^2} + \frac{4x}{\ell^2} - \frac{1}{\ell} + \frac{4}{\ell} - \frac{8x}{\ell^2}\right] \equiv 0 \text{ at all values of } x$$

Let us calculate strain in this system and rigid body displacement is when u 1 is equal to u 2 and u 3 is equal to u 3 and let us say that that is equal to u bar. And so, if you calculate Epsilon xx and because u 1 u 2 u 3 are constant at u Bar we can take it out and then if we see the sum total of all these bracketed terms is equal to 0. So, that means that our strain is 0 within the element if we subject the body to some rigid body displacements that is what we require as one of the conditions for monotonic convergence.

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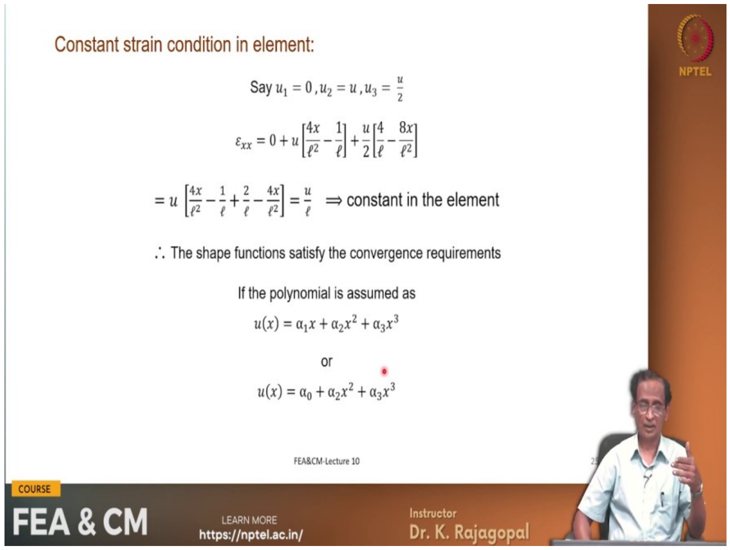
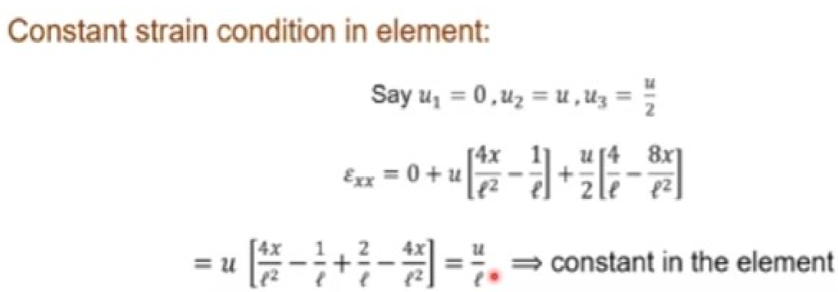


And similarly we can examine whether we can predict a constant strain state within the element. Let us say our u 1 is 0 that is the left side node is fixed and the right side node is deformed by u and the central node should defined by u 2 u by 2 right. So, our Epsilon xx is 0 + u times 4x by l Square minus 1 by l + u by 2 times 4 by l minus 8x by l Square these are the terms that we derived earlier. So, our Epsilon xx is if you take the total of this it is just simply u by l.



And that is constant in the element it is not a function of x so that means that if you are applied displacements are corresponding to a constant strain state we should be able to predict a constant strain within the element that is u by l. And so, the the shape functions that we derived they correspond to monotonically convergence monotonically convergent solution. So, we can say that our polynomial that we assumed is will ensure our convergence.

If the polynomial is assumed as

$$u(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

or

$$u(x) = \alpha_0 + \alpha_2 x^2 + \alpha_3 x^3$$

So, if you assume a polynomial like this Alpha 1 x Plus Alpha to x square + Alpha 3 x Cube or Alpha naught + Alpha to x square + Alpha 3 x Cube it is possible these are also admissible but then we may not be able to satisfy our monotonic convergence requirements and because of that our solution may not be accurate. So, that we are going to see more in the in the next lectures.

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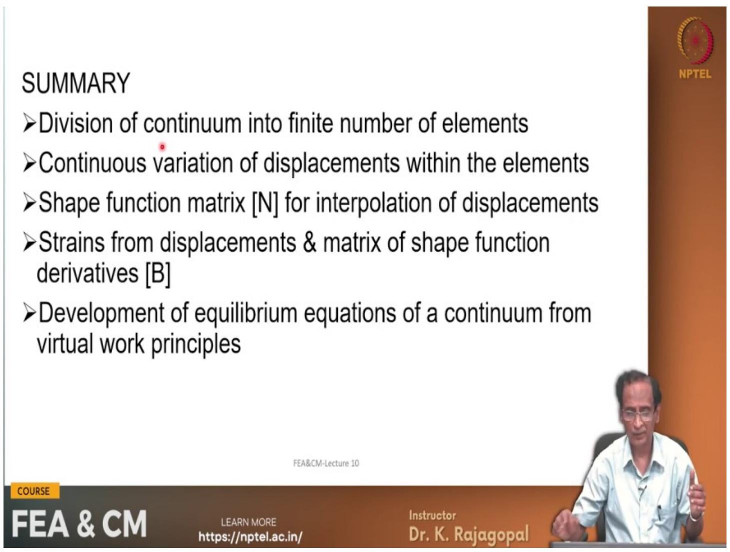

So, just to summarize we have seen how to divide a Continuum into some finite number of elements and we have seen continuous variation of the displacements within the element by developing our shape functions and the shape function Matrix N for interpolation of displacements and the strains from displacements in terms of Epsilon is B times u but B times a and our we have seen the equilibrium equations for a continuum.

So, this we will do a bit more in the next lecture. So, that we understand how we can do the analysis ok. So, thank you very much.