Mechanics of Material
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Stresses and displacement due to torsion or inflation
Lecture – 87
Governing differential equation and solution

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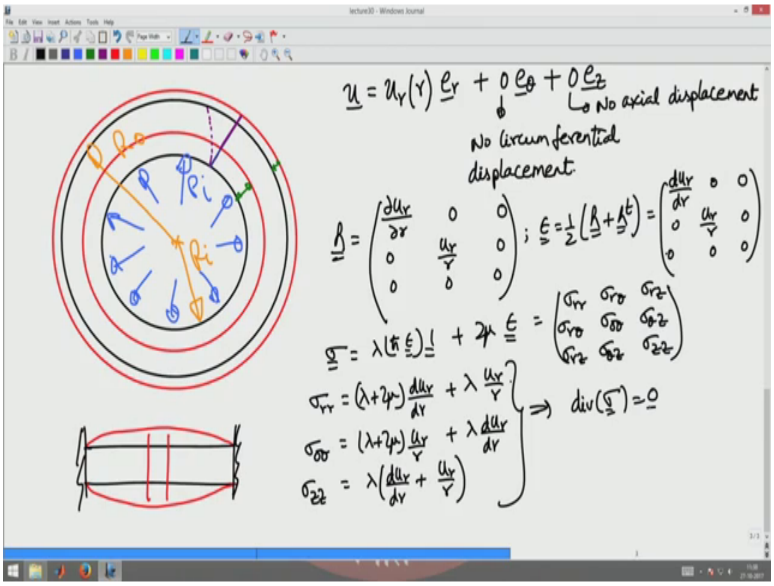

Now, you are supposed to find what this radial function u of r of r is.. Further we have run through the procedure of finding the strain, finding the stresses, finding using equilibrium equations, you find the unknown displacement component. So, now the gradient of displacement h is given by from that expression in here dou u r by dou r, that is the only component of stress which is non zero here you add here you add this is a non zero component u r as.

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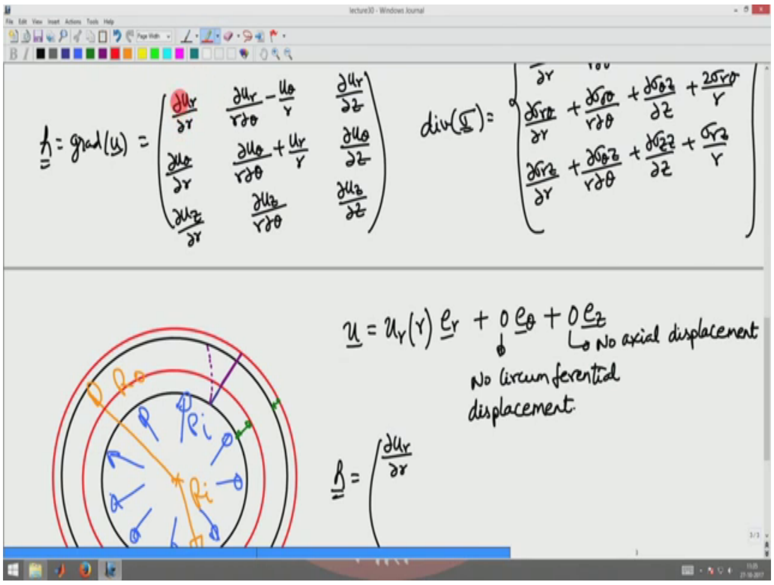

Suppose theta is not there this is 0, u theta is 0. So, this is 0 and this is also 0 because that a function of z ok.

So, you have dou u r by dou r 0 0, u theta is not there. So, all this remaining components except u r by r are 0 because there is no u theta and there is no u z in this problem. So, you have 0, u of r by r 0 0 0 0 ok. The strain corresponding to this displacement gradient would be 1 half h plus h transpose which will be same as h in this case, which will be d u r by d r 0 0 0 u r by r 0 0 0 ok. Now we will use a constellation for sigma, which is sigma times sigma is (Refer Time: 02:06) constant lambda times trace of epsilon identity plus 2 mu epsilon.

This will give me the non zero components or the diagonal components of the stress, which is radial stress sigma r r this will be nothing, but sigma r r, sigma r theta, sigma r z, sigma r theta, sigma theta theta, sigma theta z, sigma r z, sigma theta z, sigma z z or the cylindrical polar components of the stress and strain components are cylindrical polar components of the strain. So, I am equating this sigma r r with this equation it will be lambda plus 2 mu d u r by d r plus lambda times u r by r ok. Similarly sigma theta theta would be lambda plus 2 mu u r by r plus lambda times d u r by d r and sigma z z would be lambda times d u r by d r plus u of r by r.

Now I substitute these expressions into divergence of sigma equal to 0 ok. Now divergence of sigma is given by this expression in there. So, is given by the expression in here ok. Divergence of sigma is given by expression in here here you find that sigma r theta component is 0.

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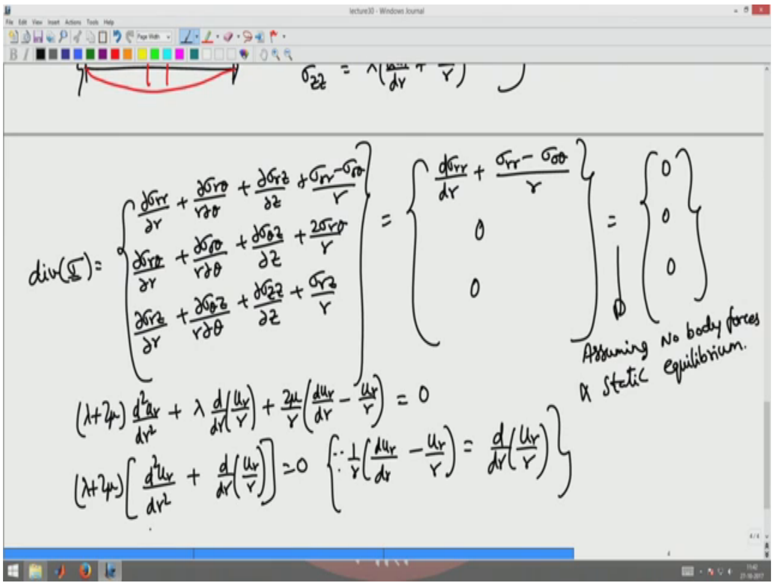

So, this goes to 0, r z 0 where sigma r r and sigma theta theta R naught 0. So, you have for in the first equation you will retain d sigma r r by d r, plus sigma r r minus sigma theta theta by r, that will be retained. The second equation you find that sigma r theta 0, sigma theta theta is not a function of theta hence this is 0, sigma theta z 0 and hence this is 0 this is also 0 in sigma r theta is 0. So, this equation trivially falls down to 0, in a similar manner the last equation sigma r z and theta z are 0, and sigma z z is not a function of z and hence this equation also falls down trivially to 0 ok. We are assuming that there are no body forces acting in the cylinder and the cylinder is in static equilibrium and hence divergence of sigma are to be equal to 0 ok; assuming no body forces and static equilibrium.

Now, what we will do next is, we will substitute for sigma r r sigma theta theta in this expression in here. Next we will substitute sigma r r and sigma theta theta into the expression here and will see what is the differential equation are getting. Let us substitute sigma r r and sigma theta theta there, I will get lambda plus 2 mu d square u r by d r square, plus lambda d by d r u of r by r plus 2 mu by r d u r by d r minus u r by r equal to 0.

Now, when rewrite this equation as lambda plus 2 mu u r by d r square plus d by d r u of r by r equal to 0. Since I observe that 1 by r d u r by d r minus u of r by r is nothing, but d by d r u of r by r since that is that I get this as this.

Now, I know from our restriction on material parameters that, lambda plus 2 mu is not equal to 0.

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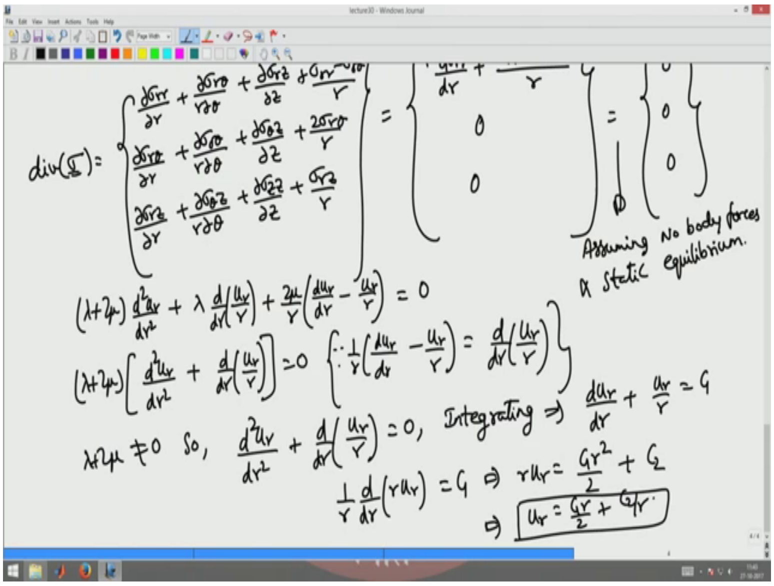

So, I require d square u r by d r square plus d by d r u of r by r to be equal to 0. Now integrating this integrating this, once I get d u r by d r plus u of r by r equal to C 1 ok. Now again I realise that I can rewrite this as d by d r u of r 1 by r equal to C 1. So, this will imply integrating this I will get r u r is C 1 r square by 2 plus C 2 ok. From here I get u of r to be C 1 r by 2 plus C 2 by r where I have to find the constant C 1 and C 2 from the boundary conditions ok.

For the problem that we are studying, let us see what the boundary conditions are.

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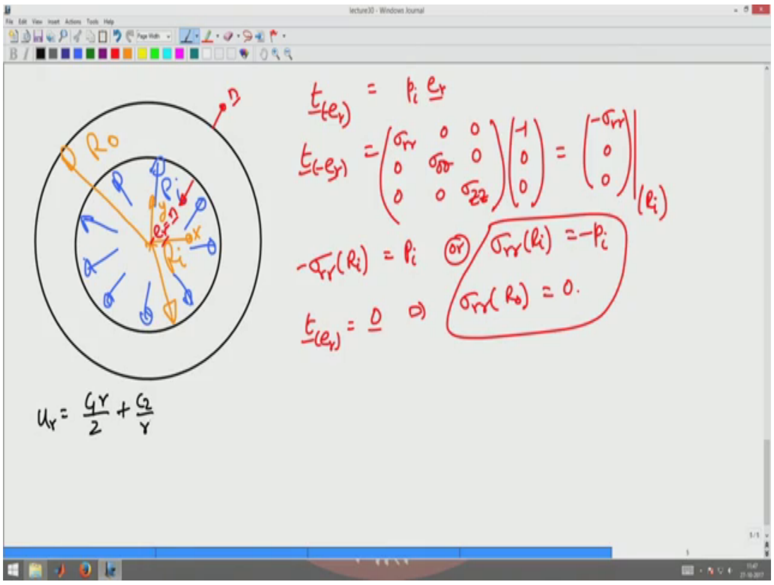

For this problem let us see what the boundary conditions are. The boundary conditions are I am applying a in pressure at the inner surface, this pressure is acting this is the out to normal to that surface is acting opposite to the outward normal to the surface. So, that is a negative stress ok. And this component of stress this normal is nothing, but e r negative of e r where e r is a radial basis for the cylindrical polar coordinate system..

So, this stress is sigma r r stress are I can more formally derive the boundary condition as t of e r the negative face is given by plus p i e r right because this acting along the e I direction ok. Now t of minus e r is, sigma r r 0 0 0 sigma theta theta 0 0 0 sigma z z this 1 is a state of stress that we got for this problem into minus 1 0 0 this will give me minus sigma r r 0 0 at the inner surface, we have already derived the inner surface that is R i.

So, what I get is, I get minus sigma r r at R i is equal to p i or sigma r r at R i is minus p i ok. Now similarly at the outer surface I am offering no traction. So, t's of e r the outer surface in normal is this normal is that that is a positive e r dash normal. So, t's of e r is 0 vector. So, this will imply that sigma r r at R naught is equal to 0 ok. I can alternatively I could operate a pressure in the outer surface you know which case what are pressure that I apply in the outer surface should come in here as a pressure that I am applying there ok, but for example, problem here I am assuming it to be 0. So, these are the 2 boundary condition that I have to now enforce ok. I found the displacement field to be I found u of r to be C 1 r by 2 plus C 2 by r.

Correspondingly now I go back and substitute this in the expression for the stresses in here the expression for the stress in here at the earlier what this stress stresses are in terms of the constant C 1 and C 2 and radial location r.

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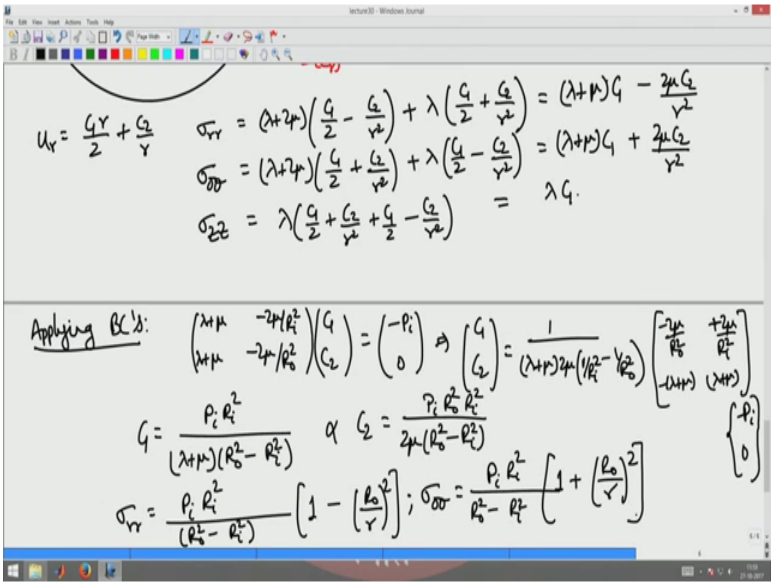

So, sigma r r now becomes lambda plus 2 mu C 1 by 2 minus C 2 by r square plus 2 mu plus lambda sorry plus lambda C 1 by 2 plus C 2 by r square which can be written as lambda plus lambda plus mu C 1 minus 2 mu C 2 by r square ok. Similarly the for sigma theta theta becomes lambda plus 2 mu C 1 by 2 plus C 2 by r square plus lambda times C 1 by 2 minus C 2 by r square, which can be written as lambda plus mu C 1 plus 2 mu C 2 by r square and sigma z z now will be lambda times C 1 by 2 plus C 2 by r square plus C 1 by 2 minus C 2 by r square which will be lambda times C 1.

Now, I apply the boundary conditions, sigma r r at r I equal to minus p i sigma r r into R naught equal to 0 to obtain the following equations applying the boundary conditions applying the boundary conditions you get lambda plus mu minus 2 mu by R i square lambda plus mu minus 2 mu by R naught square into C 1 C 2 is equal to minus p i 0.

Now, I solved this equation to get from here C 1 C 2 is 1 by lambda plus mu 2 mu 1 by R i square minus 1 by R naught square into minus 2 mu by R naught square minus 2 mu by R i square minus lambda plus mu and lambda plus mu into into minus p i 0 ok. That gives us the constant C 1 to be p i R i square divided by lambda plus mu into R naught square minus R i square. And C 2 would be 1 by 2 mu R naught square minus R i square p i R naught square R i square.

Now, correspondingly I have found the constants I can write these stresses sigma r r would be p i R i square divided by R naught square minus R i square into 1 minus R naught by r whole square and sigma theta theta would be p i R i square by R naught square minus R i square into 1 plus R naught by r the whole square.

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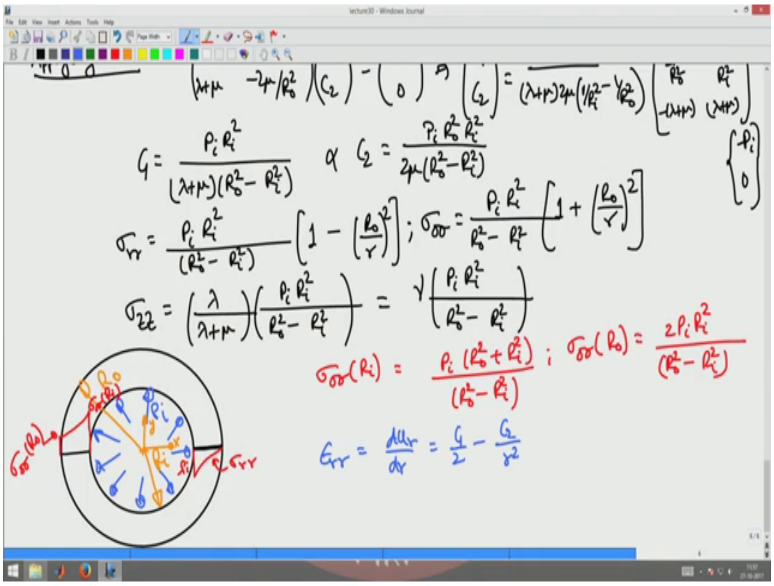

Similarly, sigma z z would be lambda by lambda plus mu into p i R i square by R naught square minus R i square.

From the definition of Poissons ratio you know that this is nothing, but Poissons ratio times p i R i square divided by R naught square minus R i square ok. Now let us plot the variation of sigma r r and sigma theta theta and sigma z z along the thickness of the cylinder ok now first lets plot the variation of sigma r r across the thickness of the cylinder ok. You know that at a inner surface when r equal (Refer Time: 00:00) i sigma r r is minus p i. So, this is minus p i and at the outer surface when r equal to when small r equal to R naught this becomes 0. So, it vary from p i minus p i to 0 like this its 1 by r square. So, it will be an hyperbolic kind of a variation there this is sigma r r variation.

On the other hand sigma theta theta now lets plot sigma theta theta variation. Sigma theta theta at R i would be p i R naught square plus R i square divided by R naught square minus R i square and sigma theta theta at R naught would be 2 times p i R i square divided by R naught square minus r I square you can see that it decreases from the value at the inner surface to the outer surface, but is not 0 and it is positive throughout the domain. So, sigma theta theta will vary like that wherein this is sigma theta theta at R i and this is sigma theta theta at R naught.

Now, the point is the hoop stress or sigma theta theta stress is tensile in nature ok. And the actual stress is constant over the entire cross section it does not vary radially that is what we got from our derivation. Now let us understand why there is actual stress and radial stress developed in the cross section ok. Now because hoop stress is tensile due to poisons effect there will be a lateral contraction along the thickness direction and the axial direction. But in the axial direction what you have done is we have fixed a length to be a constant since the length has been fixed to be constant, there should be axial tension coming in to maintain the constant length and hence you have sigma z z which is a tensile stress which is a constant stress across the cross section ok.

Similarly, you will find that epsilon r r is dou u r by dou r is not 0 ok. Dou u r by dou r is C 1 by 2 minus C 2 by r square which has some value which is not zero and hence what happens is there will be a thickness change in the cross section. So, what you have seen in this lecture is, you have seen how to analyze pressure vessels thick wall pressure vessels to be more specific we saw what the displacement field is, and we saw how to get the stresses and we saw how to get the stresses in a thick walled cylinder. We will stop here for today.

Thank you.