

**Mechanics of Material**  
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**Stresses and displacement due to torsion or displacement**  
**Lecture – 79**  
**Torsion equation**

(Refer Slide Time: 00:18)

The image shows a whiteboard with handwritten notes and diagrams illustrating the torsion of a shaft. On the left, a shaft of length  $z$  is shown with a twist angle  $\alpha z$  at the free end B. On the right, a circular cross-section of radius  $R$  is shown with a coordinate system  $(x, y)$  and a point  $(r, \theta)$ . The displacement field is derived as follows:

$$\beta R = \alpha z$$

$$\beta = \frac{\alpha z}{R} = \Omega z \quad \text{Angle of twist per unit Length}$$

$$u_x = -\beta r \sin \theta = -\Omega z y \quad u_z = 0$$

$$u_y = \beta r \cos \theta = \Omega z x \quad \underline{u} = (-\Omega z y) \underline{e}_x + (\Omega z x) \underline{e}_y$$

→ Assuming closed section ⇒ No warping.

Now, I found a displacement field, next what I want to do is I want to find the gradient of the displacement field ok.

(Refer Slide Time: 00:25)

$$\underline{H} = \text{grad}(\underline{u}) = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & -2z & -2y \\ -2z & 0 & -2x \\ 0 & 0 & 0 \end{pmatrix}$$

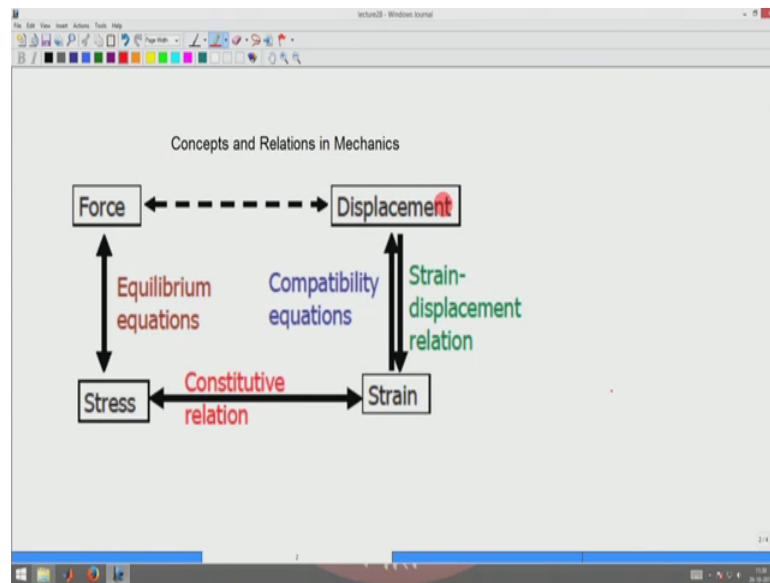
$$\underline{\epsilon} = \frac{1}{2}(\underline{H} + \underline{H}^T) = \frac{1}{2} \begin{pmatrix} 0 & 0 & -2y \\ 0 & 0 & -2x \\ -2y & -2x & 0 \end{pmatrix} \Rightarrow \text{tr}(\underline{\epsilon}) = 0$$

$$\underline{\sigma} = \lambda(\text{tr}(\underline{\epsilon}))\underline{1} + 2\mu\underline{\epsilon} = \mu \begin{pmatrix} 0 & 0 & -2y \\ 0 & 0 & -2x \\ -2y & -2x & 0 \end{pmatrix} \quad \sigma_{xz} = -\mu 2y \quad \sigma_{yz} = \mu 2x$$

Next I want to find H, which is gradient of the displacement field U similar to what we have been doing all along. So, this would be  $\frac{\partial u_x}{\partial x}$ ,  $\frac{\partial u_x}{\partial y}$ ,  $\frac{\partial u_x}{\partial z}$ ,  $\frac{\partial u_y}{\partial x}$ ,  $\frac{\partial u_y}{\partial y}$ ,  $\frac{\partial u_y}{\partial z}$ ,  $\frac{\partial u_z}{\partial x}$ ,  $\frac{\partial u_z}{\partial y}$ ,  $\frac{\partial u_z}{\partial z}$ ; if I substitute the components of displacement from here into these components, I will get 0 minus omega z, minus omega y, omega z 0 omega x 0 0 0 right ok.

Now, I want to compute the linear strain epsilon, there is one half H plus H transpose. So, that will be given by the epsilon x y term will get cancelled because its q symmetric in x y component. So, I will get it as 0 0 minus omega y 0 0 omega x minus omega y omega x 0 into half.

(Refer Slide Time: 02:16)



Now, next I have to from our scheme of things here I found a displacement field suitable for the boundary problem that I am studying. I use a strain displacement relationship to get this strain make sure I have to use the cast stimulation to get the stress, and then I have to check whether the equilibrium equation are satisfied or not that is the scheme of things now.

So, my stress expression is given by  $\lambda \text{trace } \epsilon \text{ identity} + 2 \mu \epsilon$  and I find here from here that trace of  $\epsilon$  is 0 and hence this becomes  $\mu \text{ times } 0$  minus  $\omega_y \text{ } 0 \text{ } 0 \text{ } \omega_x$  minus  $\omega_x \text{ } \omega_y \text{ } \omega_x$ .

Now, I found this stress I found in particular  $\sigma_{xz}$  component to be minus  $\mu \omega_y$  and  $\sigma_{yz}$  component to be  $\mu \omega_x$  ok. Now, I will see what happens to the equilibrium equations next the equilibrium equations are I am assuming there are no body forces and the body is in static equilibrium.

(Refer Slide Time: 03:34)

The whiteboard contains the following handwritten content:

- At the top left, the shear stress  $\tau_{xy}$  is given as  $\tau_{xy} = \frac{\mu}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})$ .
- The stress tensor  $\sigma_{ij}$  is shown as  $\sigma_{ij} = \lambda(\nabla \cdot \underline{\underline{\epsilon}}) \delta_{ij} + 2\mu \underline{\underline{\epsilon}}$ , which is expanded into a matrix:
 
$$\sigma_{ij} = \begin{pmatrix} 0 & 0 & -2\mu y \\ 0 & 0 & -2\mu x \\ \mu y & -\mu x & 0 \end{pmatrix}$$
- The equilibrium equation  $\text{div}(\underline{\underline{\sigma}}) = 0$  is written as a system of three partial differential equations:
 
$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \end{cases}$$
- The shear stress components are identified as  $\sigma_{xz} = -\mu x y$  and  $\sigma_{yz} = \mu x^2$ .
- The torque  $T$  is calculated as  $T = \int_{-a_z}^{a_z} (\sigma_{yz} x - \sigma_{xz} y) da_z$ , which simplifies to  $T = \mu \int_{-a_z}^{a_z} (x^2 + y^2) da_z$ .
- Assuming a homogeneous body, the torque is further simplified to  $T = \mu \int_{-a_z}^{a_z} (x^2 + y^2) dx dy = 2\mu \int_{-a_z}^{a_z} r^2 da_z$ .
- The relationship between the shear modulus  $\mu$  and the polar moment of inertia  $J$  is given as  $\frac{J}{J} = \mu \Rightarrow \mu = \frac{J}{J}$ . The shear stress components are also expressed as  $\sigma_{xz} = -\frac{T}{J} y$  and  $\sigma_{yz} = \frac{T}{J} x$ .
- A note states:  $J \rightarrow$  Polar moment of Inertia.

So, the equilibrium equations will reduce to recovering balance of sigma equal to 0 which is rho sigma x x by dou x plus dou sigma x y by dou y plus dou sigma x z by dou z, dou sigma x y by dou x plus dou sigma y y by dou y plus dou sigma y z by dou z, dou sigma x z by dou x plus dou sigma y z by dou y plus dou sigma z z by dou z.

This you can see that sigma x z is not a function of z or x and hence the first equation is identically 0, the other two components are 0 the second equation sigma x y is 0 sigma y y is 0 sigma y z is not a again a function of z.

So, that is 0 the third equation sigma x z is not a function of x sigma y z is not a function of y and sigma z z is 0. So, that is also 0. So, you are recommended divergence of sigma be 0 is satisfied by this stress field now the unknown is your (Refer Time: 04:56) is per unit length omega, which you will find from the requirement that the torque is given by sigma y z into x minus sigma x z into y d a z a z ok. I substitute for the expression for the stresses into this equation to get the torque to be equal to omega times mu x square plus y square d a z a z ok.

Now, I use this expression to find the relative torque to the angle of twist omega ok. Now for the simplifications possible if I assume that the member is are the structures homogeneous assuming homogenous body, I have T to be given by mu omega; I can pull them mu outside as soon the body is homogeneous the shear modulus mu it is not a function of x y or z.

So, I move that outside that would be  $x^2 + y^2$  and  $dx dy$  is  $dA$ . This is the definition of polar moment of inertia  $J$  which is the polar moment of inertia that is  $x^2 + y^2$  is  $r^2$ . So, you can write it as  $\mu \omega \mu r^2 dA$  and this  $r^2 dA$  is the polar moment of inertia  $J$ .

So, I have now the expression  $T$  by  $J$  equal to  $\mu \omega$  now I have to find what is the effective shear stress in the cross section right; I now I know that from the expression for shear stresses (Refer Time: 07:23) I know that  $\sigma_{xz}$  is minus  $T$  by  $J$  into  $y$  and  $\sigma_{yz}$  is  $T$  by  $J$  into  $x$  and  $\sigma_{yz}$  is  $T$  by  $J$  into  $x$  ok. Now what I am interested I am not interested in the components of the shear stress, but I am interested in the magnitude of the shear stress right.

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Assuming homogeneous body,  $T = \mu \omega \int (x^2 + y^2) dA = \mu \omega \int r^2 dA$   
 $J \rightarrow$  Polar moment of Inertia  
 $\frac{T}{J} = \mu \omega$ ;  $\sigma_{xz} = -\frac{T}{J} y$  and  $\sigma_{yz} = \frac{T}{J} x$   
 $\tau = \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2} = \frac{T}{J} \sqrt{y^2 + x^2} = \frac{T}{J} r$   $\rightarrow$  Radial distance of the point in the cross-section from the center of rotation (CG)  
 $\frac{\tau}{r} = \frac{T}{J} = \mu \omega$  — Torsion equation.

So, the magnitude of the shear stress  $\tau$  is given by square root of  $\sigma_{xz}^2 + \sigma_{yz}^2$ , because these are acting in the same plane ok. So, this will be  $T$  by  $J$  root of  $y^2 + x^2$   $y^2 + x^2$  is a radial distance of the point ok.

So, this is nothing, but  $T$  by  $J$  into  $r$ , where  $r$  is the radial distance of the point in the cross section from the centre of rotation, which is usually the CG or the cross section. So, combining these equations I have  $\tau$  by  $r$  equal to  $T$  by  $J$  equal to  $\mu \omega$  ok. So, this is called as the torsion equation this called as the torsion equation.