Mechanics of Material
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Introduction and Mathematical Preliminaries

Lecture – 03
Part 1
Concept of Force
Representation of Vector

In the last class we saw some introduction to matrix algebra or linear algebra is how to add 2 matrices, how to multiply 2 matrices and then we moved on to initial notations wherein we saw what a dummy index is, what a free index is and then we moved on to vector algebra or in we saw some basic vector operations like vector addition, scalar multiplication and then dot product, cross product and scalar triple products. And we understood the geometrical meaning. In particular we understood that vectors a dotted line segment for us is the geometric law object as a set of numbers. Now in this class we will see how we can transform a geometric law object into a set of numbers so that we can do algebra with it.

You need this because in today's world we are using computers or calculators to do the calculations which can work on only numbers, but cannot work on geometrical objects. So, we need to convert a geometrical object into a some format when you can do algebra with it that is in the some representation which has numbers using which we can do some calculations. It is similar to representing numbers using a basis 1 2, 0 to 0 or representing numbers using only 0 and 1 it is a binary format use a computer uses. So, there is an equivalence between these 2 representation of number system right. Similarly we will see today how you can represent the geometrical objects of a vector using some numbers.

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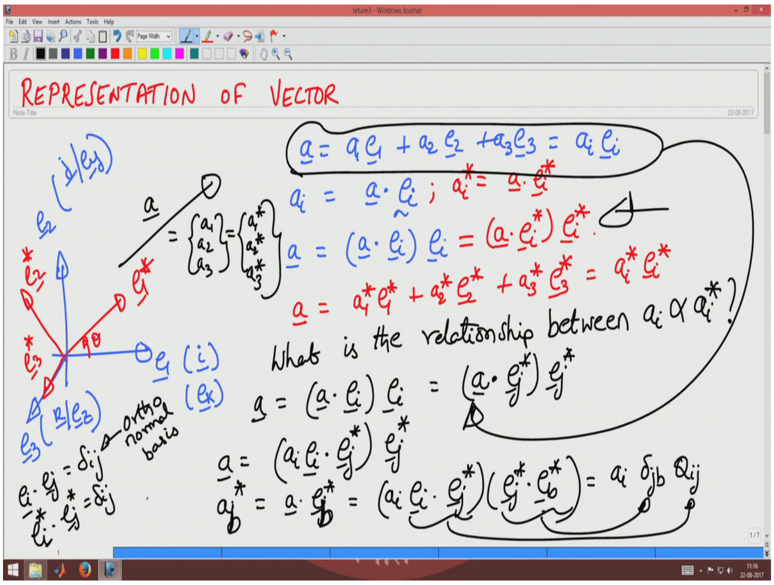

So, basically in the last class we saw that this that line segment is what we called as a vector let us denoted by a have to underscore. Now let us say that I am defining 3 other dotted line segments called e 1 which you would have denoted by i vector or which you can denote equivalently by and e x these are difference symbol that are used they represent this dotted line segment. And another dotted line segment e 2 which will call it as which you might be using the symbol j in a previous courses or you can generated by e y which will use c 2 and e y are notation there will use frequently in this course. Another third dotted line segment which is e 3 which will denoted by k or e z.

Now, this dotted line segment a you want to represent using this e 1, e 2, e 3 as the 3 dotted line segments you want to use those 3 dotted line segments and represent a. What does this mean? It is like representing numbers say ninety nine as nine into ten plus nine there is a representation of 99 right similarly you want to represent this a using this 3 dotted line segment e 1 e 2 e 3. So, when you are right a has a 1 e 1 plus a 2 e 2 plus a 3 e 3 or subsequently this can be written as a i e i, this is a reason why I am using 1 2 3 as the index was the basis vectors rather than x y z are i j k for the basis vectors.

As we said in the last class i is represented twice and dense it is called as a dummy index and dense it as we sound from 1 2 3 whose what I will one says from now on. Now how do I find this a i? I can find a i my projecting a onto the basis vectors e i, till they are underscore for me means the same thing I am not destination till day from underscore sometimes I use till day sometimes I use underscore. So, this e i is a basis vector. So, a i is the projection of a onto e i that is the meaning of a i the component of the vector a i. Now let us combining these 2 I can write a as a dotted with e i times e i.

This is a general representation of a vector using 3 as a dotted line segments. So, any vector you can represent using 3 dotted line segments. Why do need 3 dotted line segment because a spaces 3 dimensional, because space or to be 2 dimensional that maybe recommended 2 basis vectors which might be x and y or e x c y or e 1 e 2 and so on. So, you are representative vector as this.

Now how do I know that a i is say dotted with e i, that come from a definition of the dot product that you are interested in projecting a along the e ith direction. So, that is how you know that it is a projection a i is nothing, but a dotted with e i. Now this 3 dotted line segment that I chose e 1 e 2 e 3 was arbitrary right I came and do some 3 dotted line segment and chose it to represent in your dotted line segment which was say.

Now, say your friend comes about and says that he does not want to use this dotted line segments, but he wants to use something different you wants to use this which we called it as c 1 star e 2 star and say e 3 star e agrees with a direction and this is e 3 star that is as rotated the basis vector by an angle theta the anticlockwise direction to get e 1 star e 2 star and e 3 star about the z axis. So, basically now e once represent this vector a using the same vector a using the base vector e 1 star e 2 star e 3 star then what will a b, a in this red basis would be a 1 star e 1 star plus a 2 star e 2 star plus a 3 star e 3 star or in short form a i star e i star or you can write this a has a dotted with a i star e i star.

Same thing where a i star is a dot similar to this I will have a i star to be given by a dotted with e i star. So, I have is to different representations. So, there is no sanity that the number a 1, a 2, a 3 alone represent this vector that is what you are trying to understand here. The numbers if I write a as if I write a as a 1 a 2 a 3 this column vector alone does not represent a, but d 1 this column vector a 1 star a 2 star a 3 star represents the same vector a. So, there is not that only 1 set of numbers 1 2 3 represent a vector there will be n number of other 3 combinations will represent the same vector. What are those 3 combinations given a combination is what we are trying to find out now.

So, what is the relationship between, what is the relationship between a i and a i star is the question we are trying to answer now. They represent the same vector a i right, both a i and a i star represent the same vector a. So, the mounted of a has to be the same and direction related direction with respect to any other dotted line segment has to be the same. So, what does this mean? I go and substitute for from this equation I take this equation a is a dotted with e i, e i, I rewrite this as a dotted with e j star e j star j is repeating twice. So, I can replace j with k or i or a or b or c or d it does not matter is since it is repeated twice again change the similar that I use the represent number that is what I have done.

Now, for a as substitute the fact that it can be written as this is a, I use this and substitute for a e i to get a as a i e i dotted with e j star times e j star. Now then I go back to this equation where I want to find a j star would be a dotted with e j star which will be a i or let me say this is b that will be a i e i dotted with e j star e j star dotted with e b star I could not I could not have use j here because that while is this summation convention they will appear thrice on the same (Refer Time: 10:47) or equal to same which is not allowed.

So, this is this now what do I know I know that e i dotted with e j will be conic delta delta ij right from what we defined conical delta in the previous class it will be 0 if I is not equal to j and 1 if i equal to j what does this mean the basis are ortho normal, ortho normal basis. That is the mounted of each of this basis vector is 1 and their mutually orthogonal e 1 dot e 2 is 0 e 2 dot e 3 is 0 e 3 dot e 1 is 0. So, they are mutually orthogonal and the mounted of each of this vectors is 1 that is what this ortho normal basis mania.

Similarly, I have e i star dotted with e j star is delta ij. So, from that I get that this is a i delta j b for this this become delta j b and this I write it as Q ij. It is easy to see that e i dot with e j star is now delta ij because they are some they are not orthogonal.

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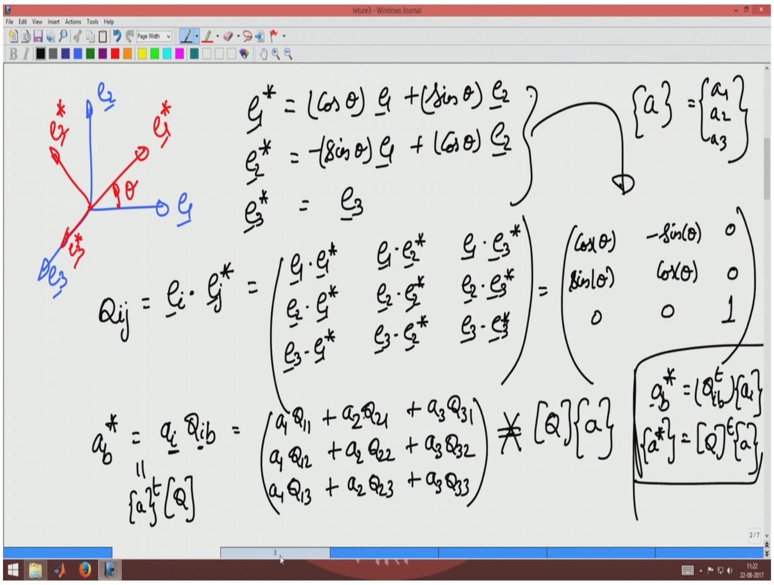

So, let us write now for the transformation that we add that may redraw the figure, the axis e 1 e 2 e 3 and I add the axis rotated by e 1 star e 2 star and e 3 star or like this because anticlockwise rotation of theta. So, now, what is e 1 star in terms of e 1 e 2? Add to project e 1 stat on to e 1 that will be cos theta e 1 plus projecting this on to this is 90 minus theta, so projecting this on to this it will be sin theta e 2. Again e 1 star e 2 star e 3 star are of length unity and hence you can write this as this, similarly e 2 star would be minus sin theta e 1 plus cos theta e 2 and e 3 star will be same as e 3.

Now let us find this matrix Q ij which was e i dotted with e j star. Going back to the equation you can see that Q ij is e i dot e j star right. So, you want to write what this matrix Q ij is. So, it will be e 1 dotted with e 1 star e 1 dotted with e 2 star e 1 dotted with e 3 star here it is going to be e 2 dotted with e 1 star e 2 dotted with e 2 star e 2 dotted with e 3 star e 3 dotted with e 1 star e 3 dotted with e 2 star e 3 dotted with e 3 star and for this e 1 e 2 e 3 star what you get is you get it as cos theta minus sin theta 0, sin theta cos theta 0, 0 0 1. That is what you get the Q ij matrix to be.

So, now going back to previous equation a b star is, a b star is given by this expression now, I can simplify this expression for the if j and b are not same they are going to be 0 right. So, I can replace j with b or b with j. So, let me replace j with b. So, this will be a i Q ib, Q ib. Now, you can see that there is a problem here if I define it like this what I will do is; let me come to the next page. What I add the final equation I add was this a b star equal to a i Q ib right, I add a b star is a i Q ib what does this mean I add to some the this index i and b is a; a i is a dummy index and b is a free index. So, this means I have a 1 Q 11 plus a 2 Q 21 plus a 3 Q 31 a 1 Q 12 plus a 2 Q 22 plus a 3 Q 32 a 1 Q 13 plus a 2 Q 23 plus a 3 Q 33. This is a meaning of this equation right.

Now this is not equivalent to multiplying this vector Q with this a. This means this is not equal to this, but this means this is equal to a transpose multiplied by Q. If I think of a as I think of a vector as being a 1 a 2 a 3. This is the operational gives me this initial notational, but I want to write like this. So, what do I do? I define I can write this as now a b star can be written as Q transpose ib into a i. So, in other words a star is the Q transpose a where Q is this matrix. So, we can see if I transpose this I will get the right addition of the terms here. So, this is the representation, final representation that we got for a star.

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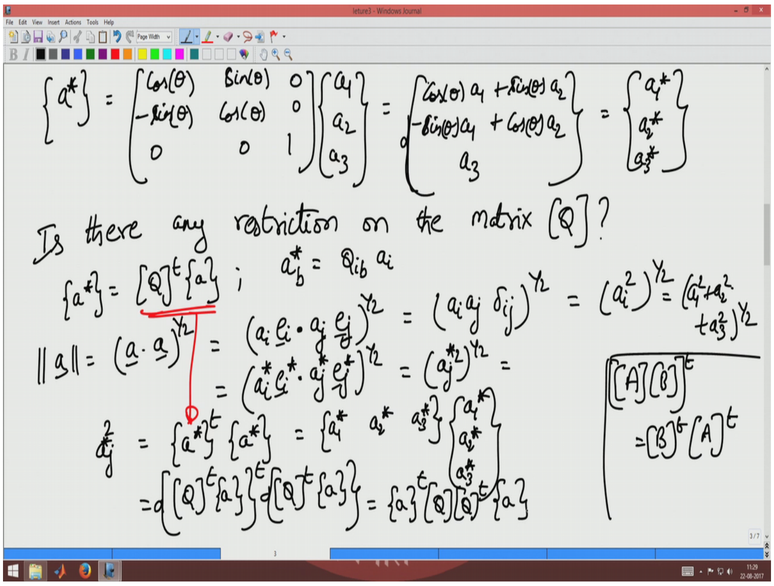

Now let us see what happens for in for the case that we are considered. So, a star would be cos theta minus sin theta sin theta cos theta 0 0 0 0 1 into a 1 a 2 a 3. So, this will be cos theta into a 1 plus sin theta into a 2 minus sin theta a 1 plus cos theta a 2 and a 3. This is a 1 star a 2 star a 3 star. So, for a planar rotation theta can be any value. So, all this set of values that you get by multiplying cos theta and sin theta with a 1 and a 2 or admissible components of the vector all this what you get a 1 star a 2 star a 3 star or also admissible components of that vector a, and represent the same vector.

Now let us find is there any restriction on, is there any restriction on the matrix Q. Let us answer this question. So, basically now we add that a star is Q transpose a right this is the equation we got in matrix multiplication term this is what it is. In component terms you add a star b as Q ib a i, Q ib a i.

So, basically now what do we know if a star also represent the same vector a, if a star also represents the same vector a phase are also represents the same vector a then the mounted of this vector has to be the same. If a star also represents a, the mounted of a should remain same whether I represented using a 1 a 2 a 3 or a 1 star a 2 star a 3 star. So, I am interested in finding what you mounted is interested in finding the mounted of a that we saw in the last classes a dotted with a over a square root.

So, this will be a i e i dotted with a i a j e j square root, I use the representation of the vector as a i e i a j e j, I can be represent with j because i and j are dummy indexes. So, this will be a i a j delta ij power half again this conical delta phi is not equal to j this is going to be equal to 0. So, what I can do is I can replace is i with j this will become a i squared power half. This is a i square is I represented twice so this will be nothing, but a 1 square plus a 2 square plus a 3 square power half that is why you can cancel this 2 with this off it is some it means I have to some I from 1 2 3.

Now, let us do the same thing for this primed stared corner system a i star e i star dotted with a j star e j star power half this also I will get it as a j star squared power half and I know a j star squared is, now, how do I define a j star squared this can be written as a star transposed with a star right a star a is; form me vector is a column vector say a star transpose will give me a 1 star, a 2 star, a 3 star and this is a 1 star, a 2 star and a 3 star. So, this gives me a j square.

Now substituting this, substituting this for a star what do I get is, I get it as Q transpose a the whole transpose Q transpose a vector. So, you know that A matrix B matrix multiplied and transposed is B transpose A transpose. So, this will become a transpose Q q transpose a. If there is a j star square.

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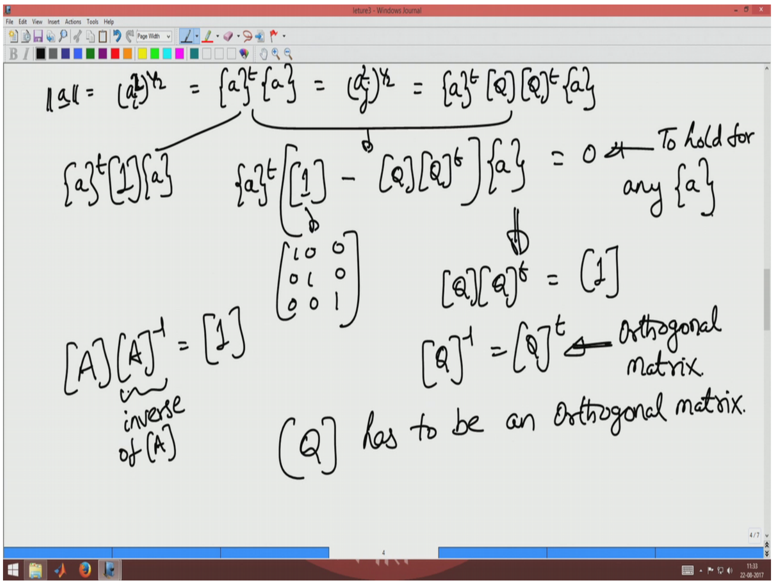

So, you have norm of a given by a i square, a i squared power half that is r nothing, but a transpose a this is nothing but a j squared power half which is nothing, but from what we found here this a j star squared we are using that expression in a to get it as a transpose Q, Q transpose a is what we got. Now from this two, we find that I can write it as a transpose into this as we equal. So, equating from those 2 I will get it as identity matrix we will see what this is in a short while Q, Q transpose times a has to be equal to 0.

Now, this identity matrix means the diagonal terms are alone 1 0 0, 0 1 0, 0 0 1. So, you can see I have done nothing if I multiply this with a i will get back a. So, what I have done is I have written this as a transpose identity matrix times a and essentially I have nothing in a just that I have introduce one more matrix here for simplification. This has to hold to hold for any a i require this as hold for any a, which implies that Q Q transpose has to be an identity matrix. What does this mean? We know that if I have a, a and a inverse then this will result in a identity matrix right this is called as inverse of a.

That will be an identity matrix which means here for Q, Q inverse is equal to the transpose of Q that is what is means inverse of the matrix is a transpose of the matrix this matrix is are called as orthogonal matrix such matrix are called as orthogonal matrix for inverse is transpose. So, there is a restriction on, this is the restriction on Q. So, Q has to be and orthogonal matrix. For it to be able to map different components of the same dotted line segment that is a restriction. So, let us go back here let us check whether this satisfy this Q satisfies this property.

Let us check whether this Q satisfies this property that it is an orthogonal matrix.

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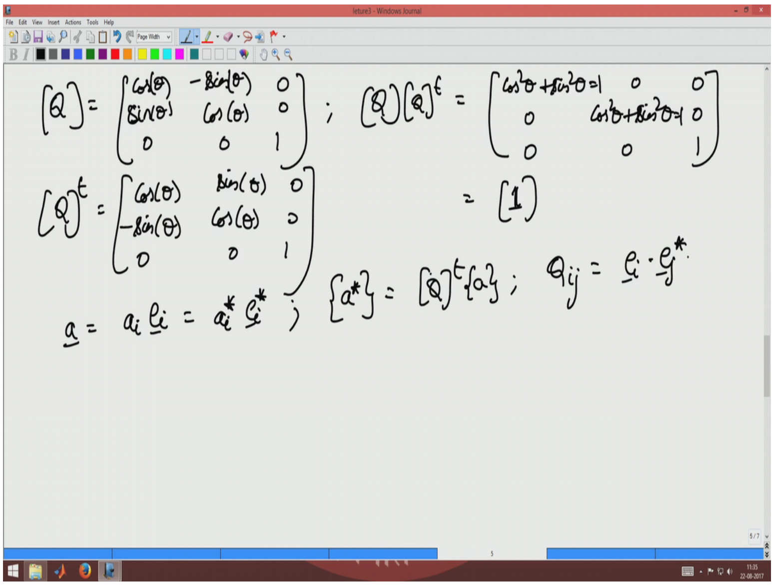

For our example Q is cos theta minus sin theta sin theta cos theta 0 0 0 0 1; I am interested in finding Q Q transpose what will it be. So, you can see that it is going to be if Q is that Q transpose would be cos theta minus sin theta sin theta cos theta 0 0 0 0 1 and if I multiply this is going to be cos square theta plus sin squared theta 0 0 0 again cos square theta plus sin square theta 0 0 0 1 from trigonometry you know that this is also equal to 1 and hence it is an identity matrix.

So that is why it was a, that is how it forms a transformation matrix. So, what you understood till now is I can the present a vector a as a i e i where a i are a i star e i star or a star would be related to a as Q transpose a where Q ij is e i dotted with e j star.