Mechanics of Material

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Constitutive relation, strain energy and potential
Lecture – 39
Young's Modulus and Poisson's Ratio

Welcome to lecture fourteen in mechanics of materials, the last lecture we saw the 2 remaining equation that connects the 4 concepts namely, the compatibility condition on the constitutive relation. We found that for a more dispersion field to be obtained from prescribe chain field. The chain field so, should satisfy certain restriction call as compatibility conditions, because we did not want surfaces to open up or penetrate each other.

So, that condition gave us in general curl of; curl of epsilon equal to 0 and for a plane state of strain.

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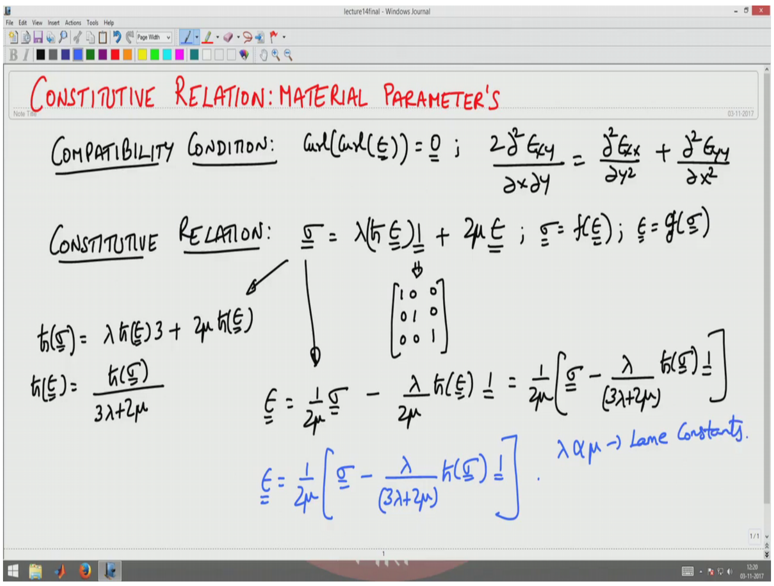

You got the equation has 2 times d square epsilon x y by d x d y equal to d square epsilon, x x by d y square plus d square epsilon y y by d x square. We also said that response where to be elastic, and if the material were to be isotropic, we got a constitutive relation for small definition is to be given by sigma to be given by lambda equal to stress epsilon identity plus 2 mu epsilon. These equations call as Hooks law or linearize relationship between stress and linearize strain. Now let us understand what does equation means. So, basically what we said was stress is some function of strain and we got this relationship between stress and strain because we said that the deformation small.

So, I r terms in the displacement gradient can be ignored. Now what this, this identity matrix mean. So, identity matrix means it is 1 0 0, 0 1 0, 0 0 1. Now we begin with the assumption that the stress of the function strain, but I can do the same thing there are same constitutive relation where I want write epsilon the strain as some function of the stress sigma. In particular I want to invert this relationship between stress and strain to get this strain as a function of sigma; to invert this first I have to take stress of this constitutive relation from here I find that stress of sigma is given by lambda stress epsilon, which is a scalar comes out of the stress expression. Stress of identity tensor is 1 plus plus 1 plus 1 which is 3 plus 2 mu stress epsilon right.

So, I get a relationship which says that stress of epsilon is, stress of sigma by 3 lambda plus 2 mu. Now I want to write epsilon in terms of sigma. So, from this equation I get epsilon to be 1 by 2 mu sigma minus lambda by 2 mu stress epsilon identity matrix. What I have done is I divided this equation by 2 mu, and I have taken a lambda stress epsilon to the other side of the equal to sign. So, I got this expression, now I substitute for stress of epsilon from this equation in your; from this in you are to get the final expression for epsilon has 1 by 2 mu sigma minus lambda by 3 lambda plus 2 mu, stress of sigma times and identity matrix ok.

So, the final expression for strain, which will carry forward is the following epsilon to be given by 1 by 2 mu times sigma minus lambda by 3 lambda plus 2 mu stress sigma identity tensor. This is the expression we will carry forward, now we have to device experiments or we have to device techniques to find this metal parameter lambda and mu where lambda and mu are called as lame constants or lame constants. It terms out that it is difficult to find this lame constant directly from the experiment.

So, will do a uniaxial experiment first, which is a common experiment that is done on material to find the metal parameters, and late this Young’s modulation Poisson’s poissons ratio to the lame constants, where Young’s modulation Poisson’s ratio or parameter that you obtain from a uniaxial experiment ok.

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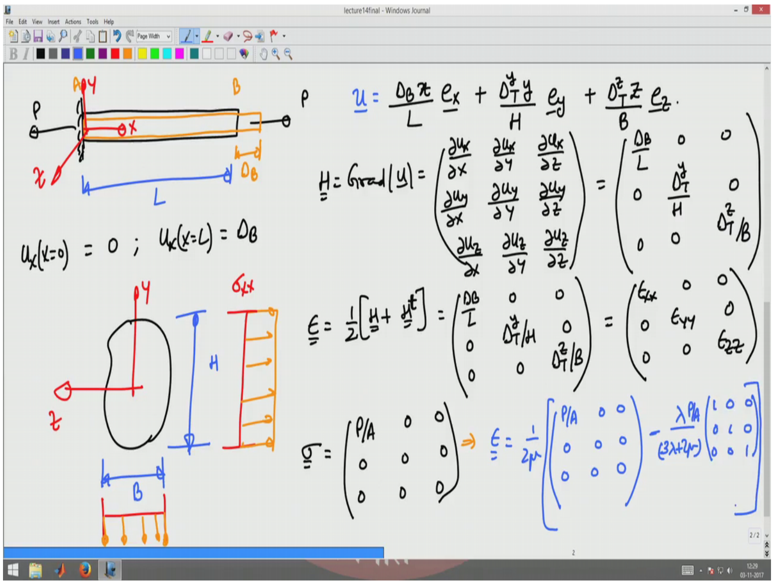

So, what is this uniaxial experiment? Now uniaxial experiment you apply a force along the axis of the member alone. Say this is a member, this is a rod or this is a square cross section bar and you pull this bar along a particular direction with the force p, you fix this end. So, that be reaction force p also coming in there. So, this end this fixed, there will be a reaction for p coming in there. So, what happens when I pull this bar? It is going to string and it is going to expand. Going to string in the (Refer Time: 05:54) direction and it is going to expand the other direction. Now let us say this displacement was delta B this is A and this is this end is B, and that deflection was delta B. Now I have to assume a displacement field to start solving bound value problem right ok.

So, displacement field I assume is assume a displacement field u, which is I want this displacement to be delta B at the sense L from the end a say the length of the bar initially was L and let us assume a coordinate system 2, let us assume this is x this is y and this is z, then it is getting displace by among delta B along the x direction, at x equal to L. And x equal to 0, I know that the x comfortable displacement field is 0 that is I know that the x component of displacement at x equals to 0 is 0, and u x component of to the displacement at x equal to L is delta B. Since I know this 2 values I linearly interpolate for the intermediate displacement as to get u x component to be delta B x by L. You see that x equal to 0, this component is 0 and when x equal to L this component is delta B like what we want ok.

Similarly, let us assume that u y displacement is given by sum delta transfers y by the depth along the direction H. Delta transpose along y direction where in the cross section is where my have some arbitrary cross section with y and z like this, and I am assuming that this distance is H, and I am assuming that this with this going to be B. So, then I can write the displacement field as e y plus, delta z T Z divided by B into e z ok. Now I have assume the displacement field, somewhat appropriated to the problem based on experimental observation that the strain that I measure is not wearing along the axis of the B axis of the member. So, that is displacement field.

So, now I have to compute the gradient of displacement field e H, which is gradient of u which you know is dou u x by dou x, dou u x by dou y, dou u x by dou z, dou u y by dou x, dou u y by dou y, dou u y by dou z, dou u z by dou x, dou u z by dou y, dou u z by dou z this will evaluate to be delta B by L, 0 0 0 delta transpose y by H 0 0 0 delta T Z by B. Now I want to find this strain, the strain linearly strain is define as 1 half H plus H transpose. So, I will get a essentially it is only a diagonal matrix, I will get a same diagonal matrix again it will be delta B by L 0 0 0 delta T y by H 0 0 0 delta T Z by B in here. Now what is the stress corresponding to this or I can rewrite this strain as some epsilon x x 0 0 0 epsilon y y, 0 0 0 epsilon z z component of the strain ok.

So, I will rewrite it as this now I would not write the stress matrix I know I am applying a force only along the x direction. So, form there I in far the component of the stress in the prescribe x y z coordinate system will be given by P by A 0 0 0 0 0 0 0. This is because I am assuming that the variation of stress along the depth of the cross section. The variation of sigma x x which is component stress due to the applied force is uniform along the depth. So, uniform along the depth and it is uniform along the width of the cross section or it is uniform over the entire cross section area that is the assumption that I am making, it is uniform along the width of the cross that is assumption I am making in here ok.

Now, I know this stress because I know what force I have applied. So, from here I used a constitutive relation that was prescribed here that we obtain here in terms of stress; I use this constitutive relation to get the strain from here. I get the strain to be in terms of the stress I have 1 by 2 mu P by A 0 0 0 0 0 0 0 0 minus lambda by 3 lambda plus 2 mu p by a into 1 0 0 0 1 0 0 0 1 right. So, I equate both this strain I equate the strain obtain here to the strain matrix here, you can see only the dymal terms are non 0 as we obtain from the assumed dispersion field.

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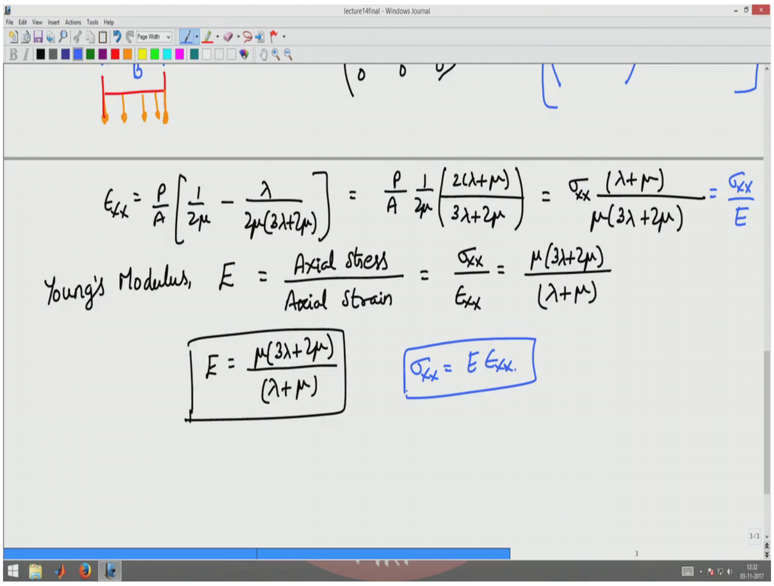

So, basically now, from here I get to know that epsilon x x would be given by P by A into 1 by 2 mu minus lambda by 2 mu 3 lambda plus 2 mu ok.

So, this on simplification it will give me P by A into 1 by 2 mu into 2 times lambda plus mu divided by 3 lambda plus 2 mu, which simplify is to let me call it as sigma x x stress into lambda plus mu divided by mu times 3 lambda plus 2 mu. Now I defined a first moduli Young's modulus, which denoted by E for a purpose e r is defined as the ratio of axial stress by the axial strain, that is the axis along which stress is applied divided by the strain dollar along the same axis, which in our case would be sigma x x by epsilon x x which from the above equation will boiled down to mu times 3 lambda plus 2 mu by lambda plus mu, which will boiled on that expression in here. Thus Young's modulus is given by mu times 3 lambda plus 2 mu by lambda plus mu.

Now, if I defines Young's modulus as this, then the equation here then this equation can be written as equal to sigma x x by E right, this equation become sigma x x by E. So, from there I get sigma x x equal to E times epsilon x x, that is the constitutive relation. Now how do I find E I do an experiment I measure the force that I am applying and I am measure the actual same epsilon x x.

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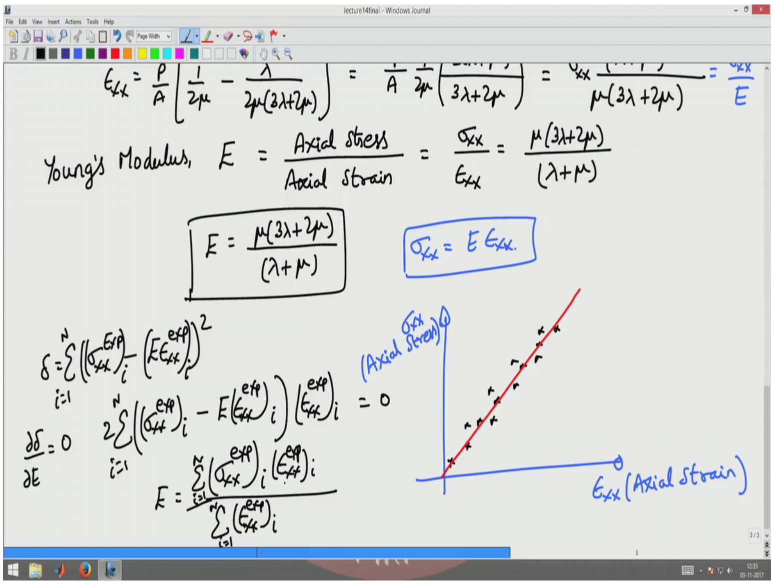

And then I plot epsilon x x versus sigma x x actual stress versus the actual strain or this is the actual strain and this is the actual stress. Now I will get some data points like this I will get some data points like that from the experimental data. So, now, how do I find what is this value of E? I would do a to find the value of e I would do a curve fitting, basically I want to find a relationship between sigma x x and epsilon x x which is linear which is linear and which is close to the data point that are prescribed here ok.

So, I want to find a curve which will pass through this points something like this, such that the error between the experiment and predicated equation predicated sigma x is the list. For this we use the list square method to find the error and to find Young's modulus assuming this relationship, that is you define and error quantity called delta which is sigma x x experimental minus E times epsilon x x experimental. This E times experimental will give you the theoretical estimate of the stress, this I want to square and some for each ith value of this stress and ith value of the strain that I am estimating. For each of the i, n number of experimental data points that I have. So, I equal to 1 to n and define this as the error measure then to find E, I want to minimize this error.

So, I say dou delta by dou E as to be equal to 0, which will give me essentially 2 times summation i equal to 1 to n, sigma x x e x p, i minus E times epsilon x x e x p, i into epsilon x x e x p i is equal to 0. From here this is the linear equation E, I will solve this linear equation to get E as epsilon x x e x p i into a epsilon sigma x x e x p i divided by epsilon x x e x p i both n 1 to N, i equal to 1 to N. So, basically I take a product of all the stress quantities that I have obtain, with the strain corresponding strain quantities and divided by the square of the strain quantity that I obtain to get the Young's modulus value. Once I get the Young's modulus value it will be pass to the origin that is the constant that we have and sigma 0, the strain as to be 0 that is constant that we have and hence we get this expression for the Young's modulus.

Now, this is basically relating axial stress to the axial strain. So, let us go back to the equation we had before, which connects epsilon x x a which connects which will tell what epsilon y y and epsilon z z are in terms of the apply sigma x x. You will find that both are the same expression, epsilon y y equal to epsilon z z which is given by the same expression which is this term alone ok.

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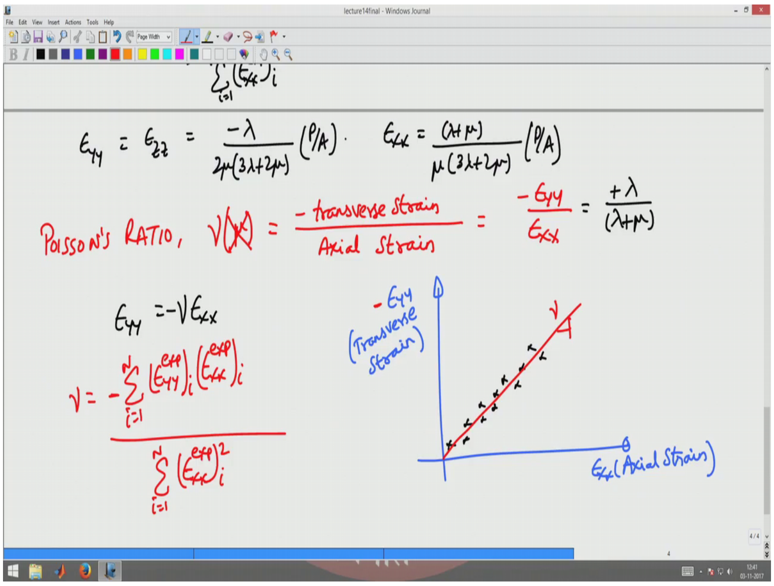

So, let us go add and do that you will find that epsilon y y is equal to epsilon z z is equal to minus lambda by 2, mu 3 lambda plus 2 mu into P by A. In comparison to this epsilon x x was lambda plus mu divided by 2 mu divided by mu, 3 lambda plus 2 mu times P by A ok.

Now, I define the second parameter that I can that I want to find from the uniaxial experiment that is called as if Poisson's ratio, which will denote by a symbol nu some use the symbol mu also for this, but this we are already use for lame constants. So, we are not following this notation in here. So, this is defined as the negative of transverse strain to the actual strain, which in our notation is nothing, but minus epsilon y y by epsilon x x. Now substituting for epsilon y y and epsilon x x from this equation, you find that this ratio is given by plus lambda by lambda plus mu, is the Poisson's ratio ok.

So, you have now the relationship that epsilon x x or epsilon y y is equal to minus mu times epsilon x x. So, now, how do you estimate the Poisson's ratio from a uniaxial experiment you do the same thing you plot the actual stress epsilon x x or the actual strain versus epsilon y y or the transpose strain, we will get some data points here. What you want is you want a best (Refer Time: 23:00) straight line passing to those data point, the slop of which would be you are Poisson's ratio. So, similar to the expression we got for Young's modulus we will find that Poisson's ratio will given by summation epsilon y y e x p, i into epsilon x x e x p i, i equal to 1 to n divided by summation epsilon x x, e x p i squared i equal to 1 to n, there will be you expression for the Poisson's ratio ok negative slop here, though because this will be in third coordinate actually this is plotting negative of epsilon y y.

So, that is going to give epsilon for the Poisson's ratio. Now there are 2 way where which can measure the strain epsilon x x and epsilon y y, one is you can use this expression find the elongation in the axial direction the contraction in the lateral direction to find the strains, you can use this expression to find the strain from over all displacement are you can use a strain gage and measure the axial and transverse strain. We saw what is strain gage is in a previous lecture. So, what I will do is, I will stick a strain gage here like this, this will give me epsilon x x and I will stick a strain gage like this to get me the strain epsilon x x, and I will stick a strain gage perpendicular to that to get epsilon z z. So, I can directly use the strain gage to get this strain and I can measure it from the displacement that I see in the body ok.

So, now we are form the expression for how to expression for Young's modulus and Poisson's ratio in terms of the lame constants. So, we have form the relationship between Poisson's ratio, Young's modulus express in terms of the lame constants ok.

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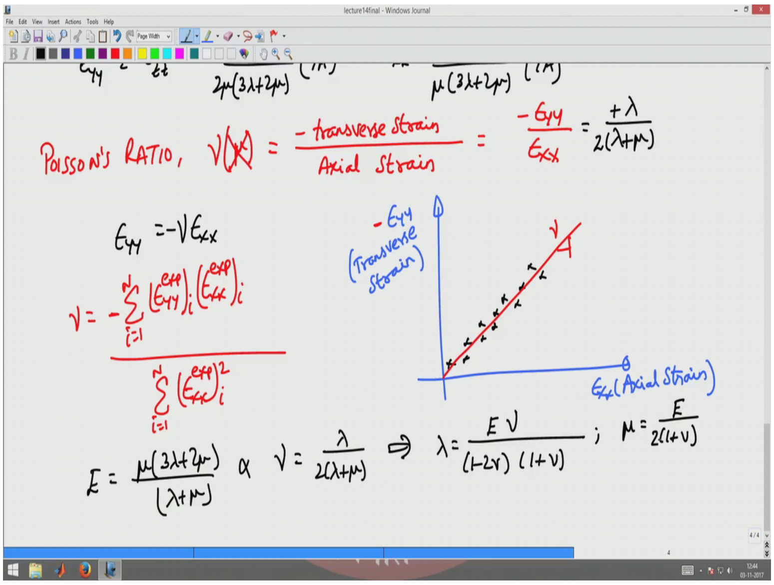

So, basically we found that the Young's modulus is given by mu times 3 lambda plus 2 mu by lambda plus mu and Poisson's ratio is given by lambda by 2 times lambda plus mu. From here you will find that I can express lambda as solving this equations in terms of the Poisson's ratio as 1 minus 2 mu 1 plus mu, and this lame constants would be e by 1 plus mu 2 times 1 plus mu. So, you will get this expression for lame constant and Poisson's ratio solving these equations.