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Structural Dynamics
Week 7: Module 03

Multi-Degree of Freedom System: Modal Orthogonality

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Welcome to structural dynamics class. So in this class we will modal orthogonality or orthogonality of motions. So let me give you a background of this modal orthogonality. In multi-degree of freedom system when we idealize the system into n single degree of freedom systems, so that means what modal in modal analysis n degree of freedom system we convert into n single degree of freedom systems.

So the process in which we convert this n degree into n single degree of freedom system we use a transformation matrix. So this transformation matrix is nothing but modal matrix, so mode shape or modal matrix are the property to convert or to change the coordinates from U to Q. So as you know we have used U=ϕ and Q term for denoting our response.

So this ϕ has a evaluated transform from U coordinates to Q coordinates. So how it is able to transform that, so that property is we are going to discuss today.

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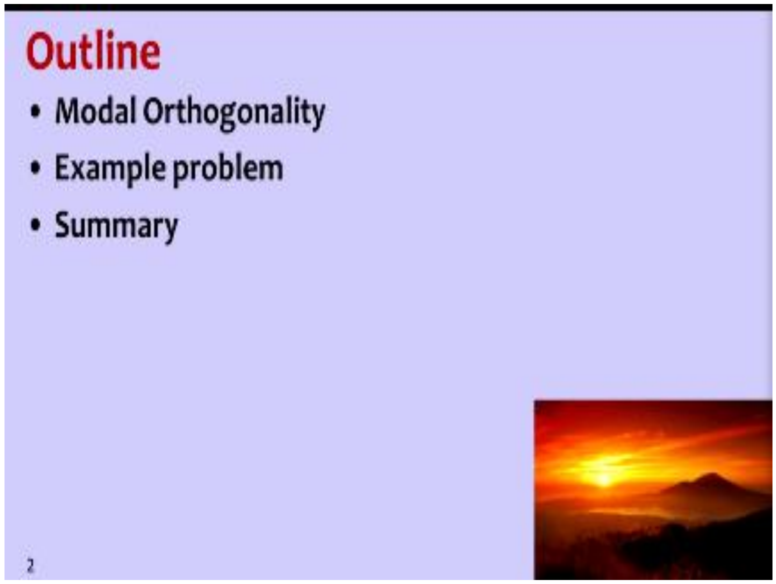

So modal orthogonality we will discuss and then one example problem also we will discuss to prove how mode shapes are orthogonal to each other.

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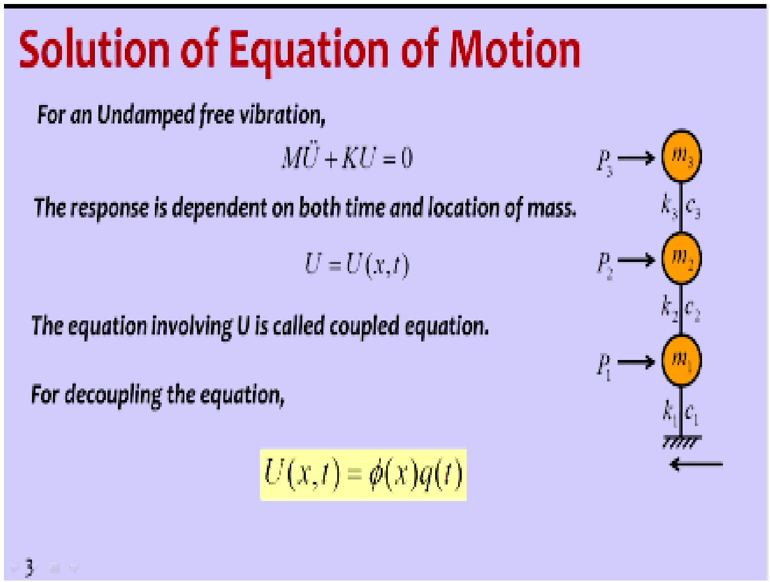

So this is a idealized mass, spring and damper system or lumped mass system, so where forces are acting at level 1, level 2, level 3 that is P1, P2, P3 the other forces, M1, M2, M3 are respective masses, K1, K2, K3 are stiffness terms towards stiffness and damping 1, C1, C2, C3 are damping terms.

So for undamped free vibrations we know equation of motion that is in the form of MU..+KU.=0. So that means what all these forces are 0, so response is dependent on both time and location of the mass. So like that is because of which we describe displacement as U=U(x) and t. So in continuous systems this X is variable starting from 0 that is base up to the total length of the structure.

But in lumped mass system that is multi-degree of hidden system what we do is this X will have some discrete values, so in this case X will have if the floor height is 3m so this is 3, this is 6 and this is 9, and time is continuous. So time is continuous from 0 seconds up to which we want response in free vibration and in forced vibration in the time force is lasting. The equation involving U is called coupled equation. So for deep coupling that we need to use the properties of Ø and U = Ø (x) and Q(t).

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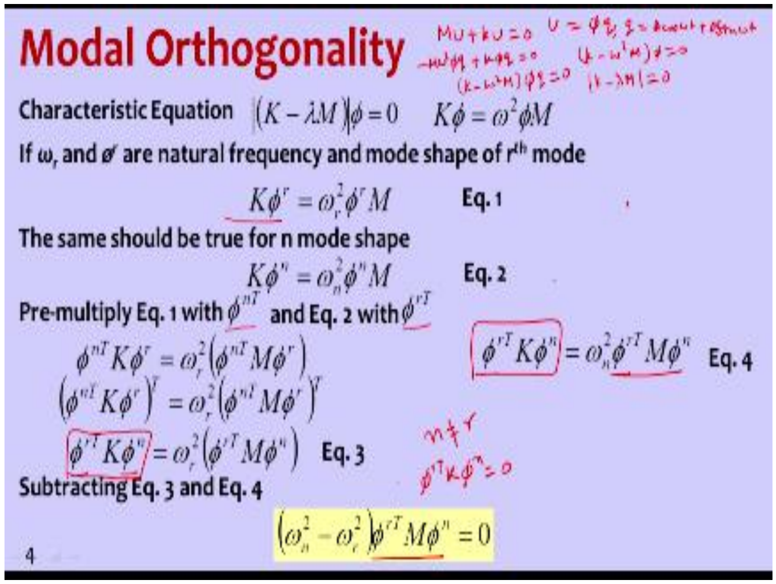

Now what is model orthogonality, so which we already discuss it that like hoe these characteristic equation can be derive from equation of motion, so let me give you a brief details of that Mu.. + Ku = 0 when we use U = Ø Q and Q is equal to some A cos ω t + B sin ω t and if we double differentiate that what we get is M ω^2 Ø Q + K Ø Q = 0, so from this equation we can write (K – ω^2 m) Ø and Q = 0 this we have already discussed, so Q cannot be 0 for non continuous solution we need Q1 as ω^2 m x Ø should be equal to 0.

So for Q cannot be 0 for non trivial solutions and then so Ø also cannot be 0 so m – λ k – λ m should be equal to 0 Ø cannot be 0 because Ø and Q cannot be 0 because if we there 0’s then u will become 0, so non trivial solution both cannot be 0, so we will be left with k – λ m is equal to so that is the equation here multiplied by Ø is equal to 0, Ø cannot be 0 so that is what we will find out now.

Then from this we can write k Ø = ω^2 Ø m if ω r is the natural frequency of r^{th} mode and Ø r is a mode shape of the r^{th} mode, so ω r and Ø r natural frequency and natural mode shape of r^{th} mode if we substitute that in it so it should be k time Ø r should be equal to ω r^2 Ø r . m so it should hold good for any mode shape so this is equation number 1 then the same should be 2 for nth mode so that means what.

We can wrote k Ø nth mode is equal to ω n^2 this nth mode Ø nth mode into m both are equal so it is equation 2 so now we will do some operations on this one, so let us pre multiply equation 1 with Ø n^{-1} and equation 2 with Ø r^{-1}, so let us now look at what will happen to first equation, by n transpose, so here I'm multiplying this equation with Ø n^{-1} k Ø r = ω r^2 Ø n^{-1} m Ø r, this is equation number 1.

When we multiply with ϕ_n^T then if I take transpose of this one $(\phi^{nT}K\phi^r)^T=(\phi^{r2}\phi^{nT}M\phi^r)^2$ now in transpose if I take this transpose what will happen is in transpose this terms get reversed so it will become transpose value will become $\phi^{rT}K\phi^n$ so this got reversed and ω_r^2 exactly same and this one again gets reversed $\phi^{rT}M\phi^n$ so this is a matrix property, so if A,B,C if we take whole transpose what we will get is like C,B,A will get.

So say, C,B,A will get then similarly when we do equation number 2, in equation number 2 what we are doing is we are multiplying it with ϕ^{rT} so $\phi^{rT}K\phi^n=\omega_n^2\phi^{rT}M\phi^n$ so we are pre-multiplying it with ϕ^{rT} you can look at this one ϕ^{rT}. Now if we look at equation number 3 and equation number 4 and subtract 3 and 4 so that means what 3 from 4 if I subtract what will happen is this term the first term here and this term here both are same.

So if I subtract this one what I will get is we can see $\phi^{rT}M\phi^n$ this is common in both, so what I will get is so this left hand terms are same so that means answer is 0 there so $(\omega_n^2-\omega_r^2)\phi^{rT}M\phi^n=0$ so now what this tells us is if n=r then this is 0, the entire term is 0. But if n is not equal to r then this term should be equal to 0, so that means what $\phi^{rT}M$ ϕ^n should be equal tom 0. So similarly we will get another relationship something like this $\phi^{rT}K$ ϕ^n should be equal to 0, so that means what if we do pre-multiply and post multiply any matrix with modal matrix we will get all the diagonal terms only, so that is what is shown in this equation.

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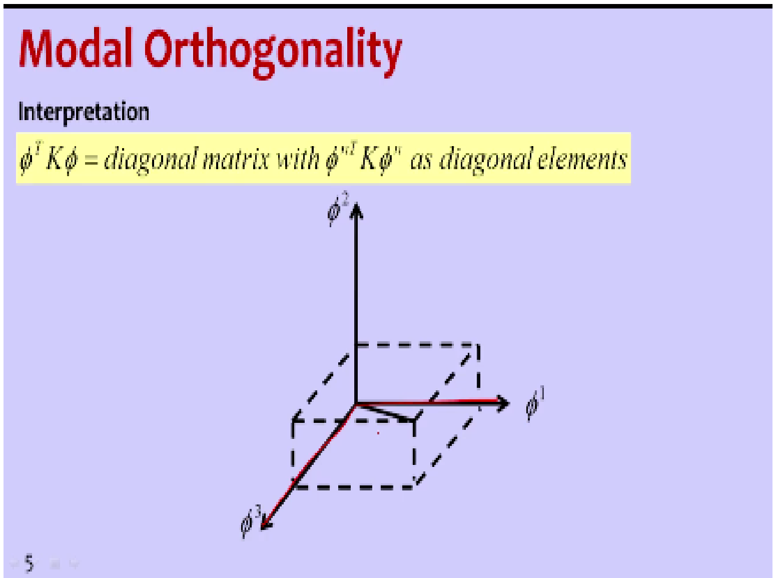

So how do we represent that, so if we right say ϕ1, ϕ2 and ϕ3 here and write this terms so this is something like x,y and z if we write and then represent modal matrix or some modal vector in the coordinate system so then what we will get is so any line what you draw for each mod shape in the coordinates these three lines will be perpendicular to each other, so that means what mod shape 1 as it is something like this so this mod shape 1, mod shape 2 and mod shape 3, so mod shape 1 is perpendicular to 2 and 3 and mod shape 2 is perpendicular to 1 and 3, mod shape 3 is perpendicular to 1 and 2, so that is what is modal orthogonality.

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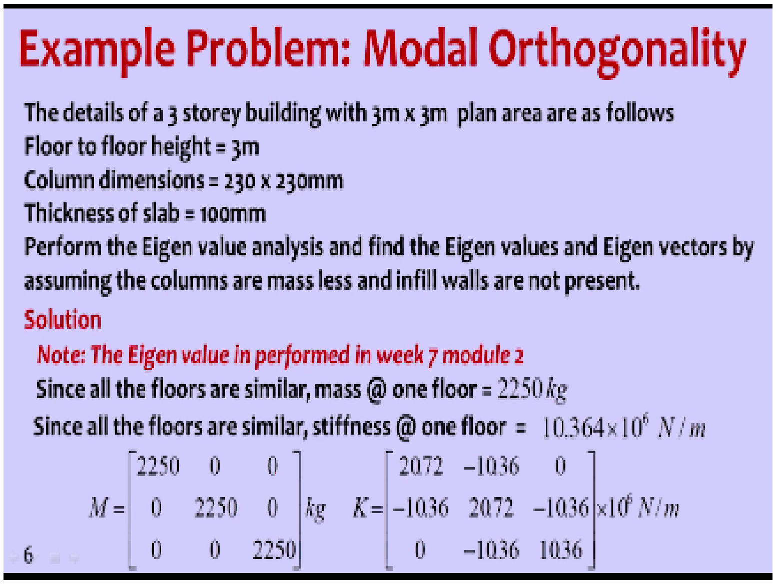

Now let us look at the problem, and try to understand by using numerical values, so the structure the same example we have solved in the earlier class the same problem I am taking so where Eigen value is performed so we know that in week 7 module 2 we performed already this analysis, so we get stiffness matrix and then we have mass matrix so you can clearly see that mass matrix is already diagonal so off diagonal terms are 0 but when it comes to stiffness matrix some off diagonal terms are there.

So that means what if we write equation of motion what we will get is we will get coupled equation coupled equation means in the same equation we will have u 1 terms u2 terms or u1u2 u3 terms together or u2u3 terms together three different equations. So if we have one equation if one ware known is there we can find out so one unknown it is derivatives are okay so that in differential equation say if u1 is there u1. U1.. is alright but in one differential equation if we have u1 u2 and u3 or it is derivative we cannot solve that.

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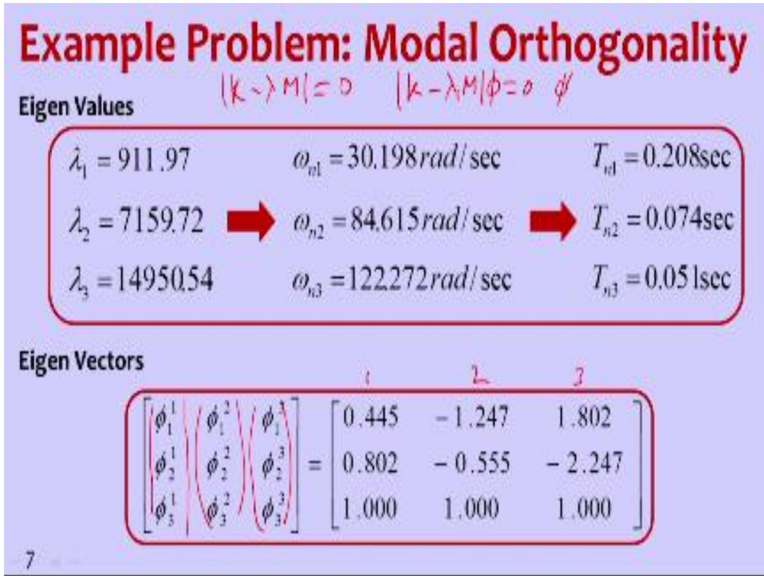

So let us go further, so these are Eigen value so if we solve this matrix k – λm / o when we solve this one we get λ1 λ2 λ3 upon substitute of each λ1 we get λ 1 means corresponding the natural frequency corresponding natural frequencies so λ1 λ2 λ3 and ω1ω2ω3 values are there corresponding natural period are given now if we substitute λ 1 in the equations say this equation k – λ 1m x 5 = 0.

So when we solve this one this how to solve this one w have already discussed in the previous lecture so when we solve this one we get 5 value which is corresponding to λ one so that is 51 we get. So like that first mode 51 first level of second level third level so 51 51 51 52 52 52 and 53 53 53 this is mode one this is mode 2 and this is mode 3, so values of this mode 1 mode 21 mode 3 are known so this is first mode this is second mode and this is third mode. So what we are trying to discuss in this class is how these things are orthogonal to each other.

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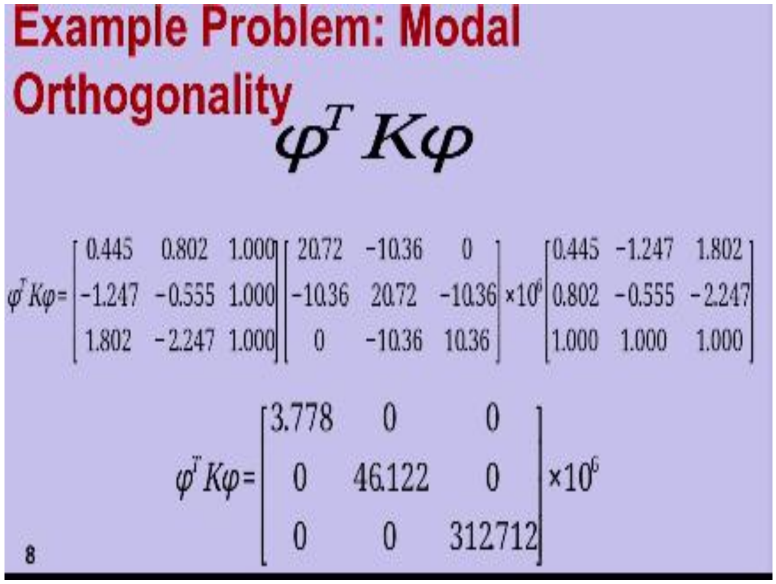

Like matrix wise if we solve five transpose k5 so this is 5 transpose that is model matrix this is k and this is 5 trans pose k 5 if you solve this one we are getting this diagonal terms so how do we multiply first this and this we have multiply and the product of the together we will multiply with this one so we will get this matrix.

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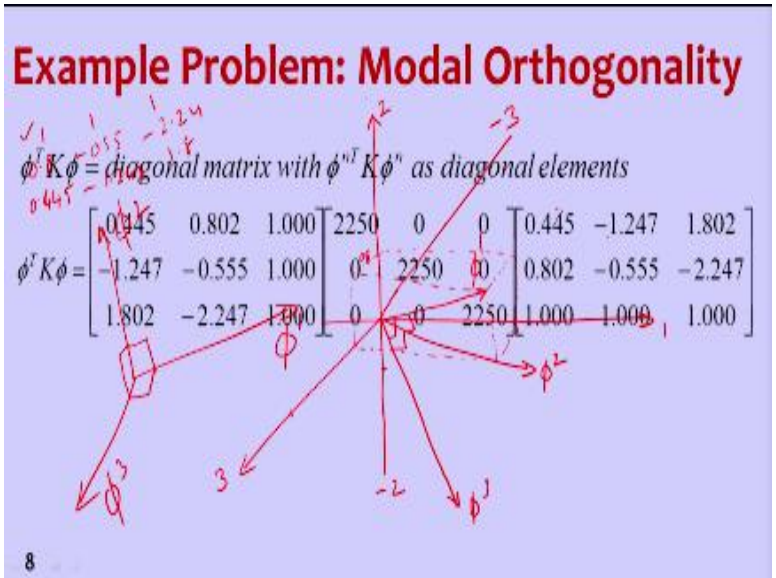

Okay but let us discus how this as come now let me take say three axis system so this is one this is two this is three this is degree of freedom okay so this is positive one axis positive two axis positive three axis, now if we look at the values of let us say first mode one pointed 0.45 so first mode is 1.8 0.445 second mode is 1 – 0.55 – 1.247, let us take third mode also 1-2.24 + 1.8 yeah now let us plot the first mode.

So first mode this first coordinate value so that is a top one it is one so let us put one the second coordinate is 0.8 so in the second coordinate let us go to 0.8 so this is 0.8 okay so let us take and call it this as one this is third coordinate 0.445, all positive values so this 0.445 so what we need to do is we need to draw a line.

So this is the first mode this is first mode so we can see on x 1 x coordinate or one coordinate we went up to one on the second coordinate we went up to 0.8 and third coordinate we went up to 0.445 then the second mode second is in the first coordinate is one only but in the second coordinate is -0.55 so it is this value and then the third coordinate is -1.2.

So that means what one in that direction so that means what if we look at this coordinate system so what will come is so all positive is one side so this has two negatives and one positive so two negative means is it will come on this plane below so negative this one and the negative so what happens is we will get something like this on the on 1 -3 line and -2 line so we take is that coordinate below so in that we are writing 1 -0.5 – this one and the third one is again 1 and this one second axis is negative so negative means down third axis is again positive that means what positive 1.8.

So what we get is in this so 1 then we go down then we come on this side so it is this so this is 5 1 5 2 and this is 5 3 so what I’m mean to say is this 5 1 will be perpendicular to this and this perpendicular to that so that means what 5 1 5 2 5 3 are perpendicular so in ever if we transform and write this one something like this, this will be 5 1 this will be 5 2 this will be 5 3 so 5 1 is perpendicular to 5 2 line and 51 is perpendicular to 5 3 line similarly 5 2 line is perpendicular to 5 3 line and all these lines are perpendicular to 5 1 line 5 3 is perpendicular to5 1 and 5 2 like this. So this is model orthogonality.

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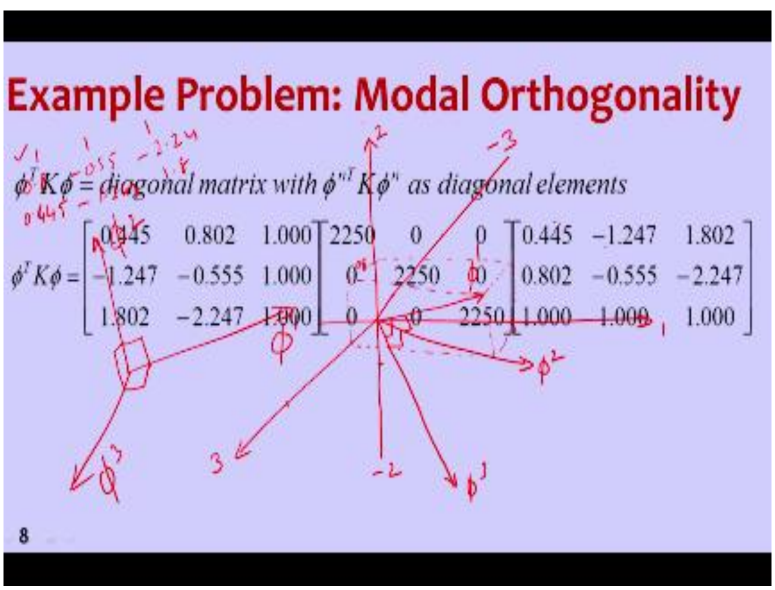

The same thing we have seen in the form of numerical problem.

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So in summary what we have discussed in this class is how mode shapes are orthogonal so this property of model orthogonal we will use in converting a coupled equation in to a decoupled equation so if so usually mass matrix is always diagonal matrix of diagonal terms are zero so us is already un coupled but when it comes to damping matrix and stiffness matrix these are coupled matrices so half diagonals terms are all not 0.

So what we can do is if we pre multiply and post multiply with model matrix then all things will become diagonal terms so we are converting one coordinate system that is U U1 U2 U3 into a system of Q1 Q2 Q3. So that means n degrees of freedom system we are converting into single degree of freedom system so for that we use model orthogonality principle of model matrices.

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