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Structural Dynamics
Week 7: Module 02

Multi-Degree of Freedom System:
Solution of Equilibrium Equation

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Welcome to structural dynamics class. In this class we will discuss how to solve the equation of motion for multi-degree of freedom systems.

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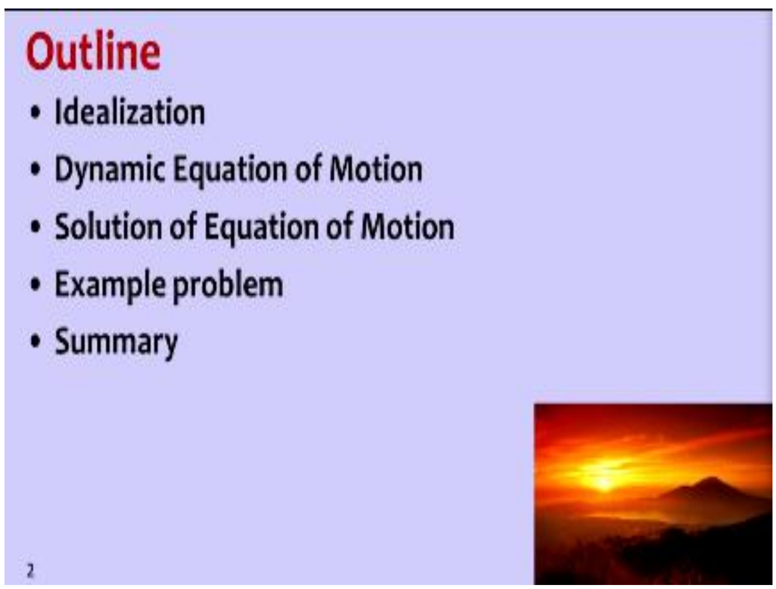

So the outline of this lecture is we will discuss about how the idealization has taken place and then dynamic equation of motion, solution of equation of motion and then we will solve one example problem in this.

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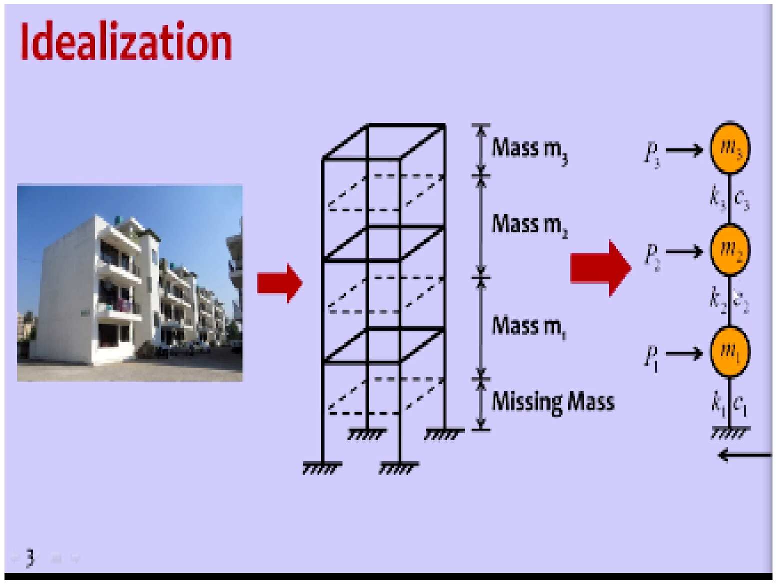

Now if you look at a building a complex system that can be idealized as number of degrees of freedom as many, as number of floors. So in this example you can see a building which is of three floors, so we are converting that into three lumped mass systems. So 3 degrees of freedom system, so this is idealization.

So we can calculate mass from this floor properties. So mass of each floor m1, m2, m3 and stiffness of each floor k1, k2, k3 damping from each floor c1, c2, c3.

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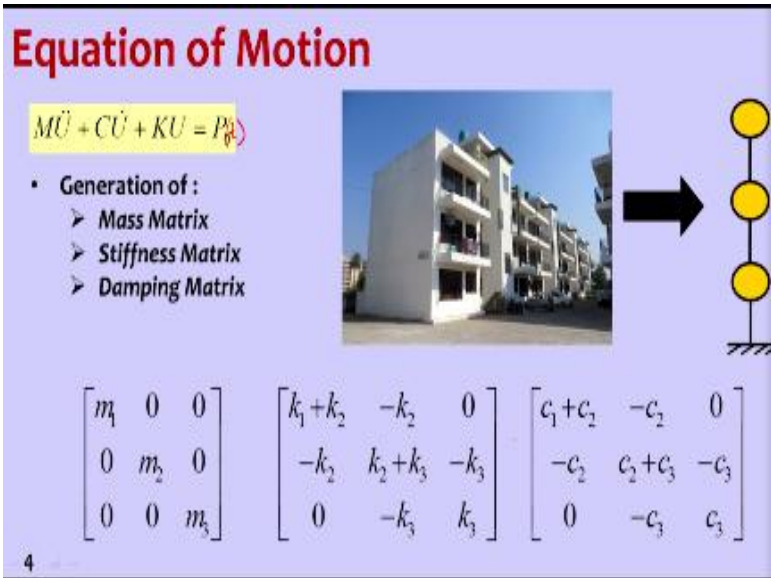

Then we wrote equation of motion that is MU..+CU.+KU=P(t) which is a function of time force. Now we have discussed already that how to generate mass matrix, so how to generate stiffness matrix and how to generate damping matrix. So mass matrix is m1, m2, and m3 with of diagonal terms as 0, k1+k2, -k2, 0, -k2, k2+k3, -k3, -k3, and k3 this is stiffness matrix and then damping matrix in the similar manner as stiffness matrix.

So these are three matrices mass matrix, stiffness matrix and damping matrix. So we are already discussed that these matrices are positive definite. So positive definite means if you take the determinant of these matrices this will be more than zero, so that means positive value will come. And then look at this one, in this one half diagonal terms are zero on mass matrix.

So the reason for that is, so we are putting degrees of freedom here only, the centre of mass and then mass of first mode is not vibrating in the mass of the, in the direction of the second mode. So this we will discuss in the next class about how mode shapes are helping us to understand the complex vibration of the whole structure into simpler manner. So this k1+k2 here you can see there are half diagonal term.

So this is, we can call this as coupled in the form of in stiffness and coupled along damping. So it is uncoupled along mass matrix. So in simple words we can say that when we write this three equations of motion, so three equations of motion for each point mass, we cannot solve these equations independently.

Why because, when we have here mass 1 is associated with acceleration 1, but here where we come to stiffness, stiffness of this first term is associated with the displacement of first mass, second term is associated with the displacement of the second mass. So that means what, in one equation we have U1 terms and also U2 terms, and in another equation if you have U2 terms and also U3 terms.

So that is how we cannot, that is the reason where we cannot solve, so we need to have employ special techniques for solving it.

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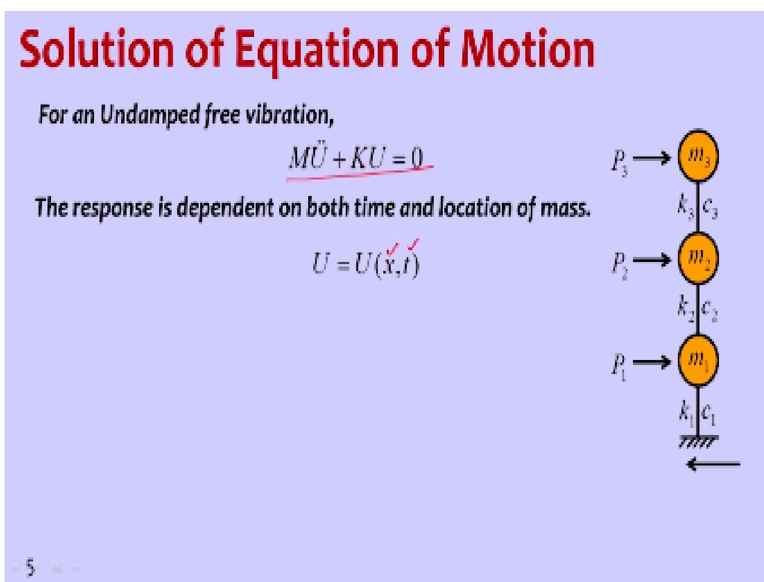

Now how do we solve. So if we remove say damping and then take it as a free undamped vibration so we are left with this equation that is MU..+KU., as you can see here, now the response is dependent on both time and location of the mass. So that means what, U is a function of time as well as location. And it is something like as you can see this is a multi-degree of freedom system building idealized system.

So if this building is vibrating, so if we say displacement U we have to first specify where we are talking about. So at different, different floor levels and there are different, different times. Definitely the structure like this, now when we say displacement where are we talking displacement, which location and which point of time. So at what time, so suppose this is floor 1, floor 2 floor 3 and 4 floor here.

So we are seeing that in lumped mass system we are seeing that at this location and also at this point. So that is the reason why our displacement function has two items in it one is location and the second one is time, so now if you look at the displacement function.

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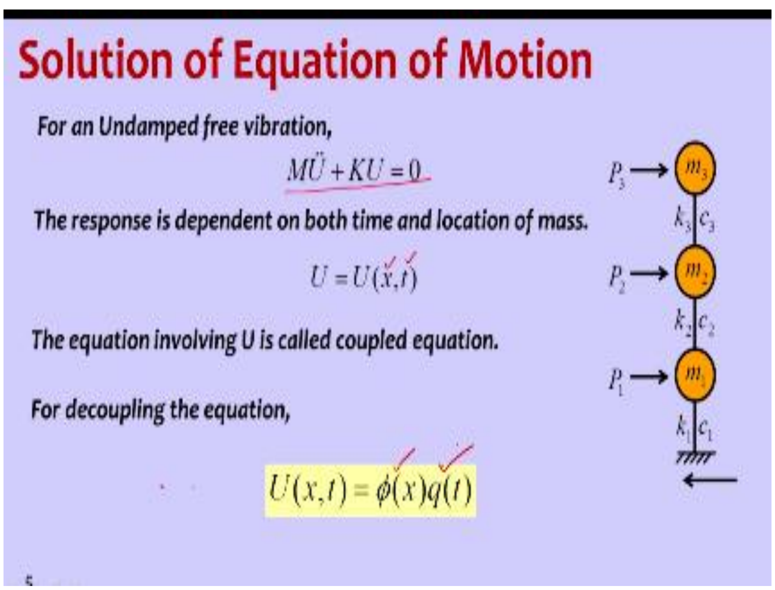

That is U, U is a function of space that is location and it is also a function of time so the equation of motion u is called couple equation why it is coupled equation, because in one equation we are having say u1term and u2 terms also u3 terms so that is why there is a coupling so we need to decouple it, so for decoupling the equation so we are using say u x(t) so x and u(t) that we are converting that into say Ø(x) and Q(t). Is a space variation function and Q is a time variation function.

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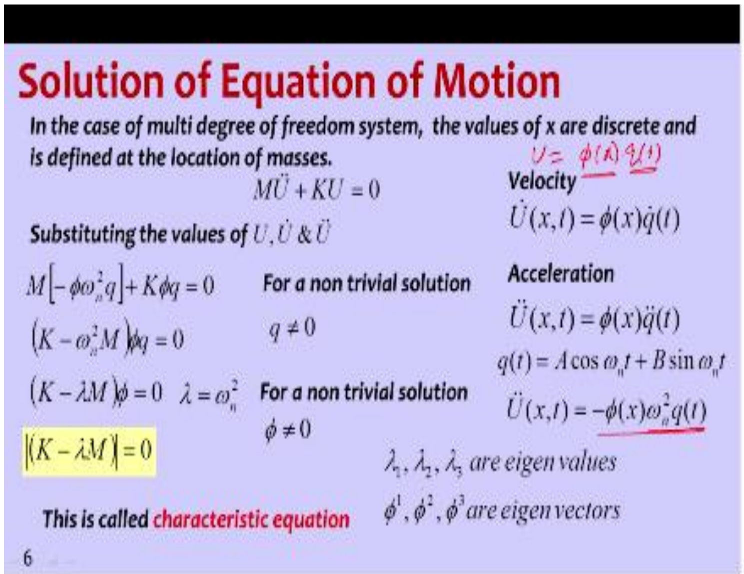

Now in case of multi degree of freedom system the value of x are discrete and but in case of continuous system this value of x can be any value from 0 to the total height of the building or the structure so this discrete and it defined at the location of mass, so now how do I if say u = Ø(x) and Q(t) okay, how do I define velocity, so we differentiate it once so if we differentiate it once this is a space quantity we cannot differentiate it will be there then Q is q.(t).

Then acceleration u.. x(x, t) = Ø(x) Q..t, now what is Q, Q. so Q. is nothing but so if a structure is vibrating so it is vibrating means simple harmonic motion so we can write that simple harmonic motion as Qt = A cos ω nt + B sin ωnt as in the case of single degree of redeem system so we know that this represents simple harmonic motion, now if we double differentiate this one so what we get is Q.. = - ω^2 A cos ω nt and then - ω n^2 B sin ω nt.

So that means what, ω – ωn^2 Q we get so that is how is written here that is what is written here so that means what u..t = - Ø(x) ωn^2 Q(t), so that is written, then if we take this and substitute the values in this equation basic equation so what we get is Mu.. so u.. is replaced with this - Ø ωn^2 Q + K so Øq, then if we rearrange the terms what we get is k - ωn^2M multiplied by Ø and Q it should be equal to 0.

So for non trivial solution Q cannot be 0 so if Q is 0 there is no solution if Q is 0 then there Is no solution, so that will give us k – λ M into Ø = 0 so λ here they representing as – omega n^2 so – λ is – omega n^2 , so λ is omega n^2 the for non trivial solution Ø also cannot be 0 so if Ø is 0 solution is 0 if Q is 0 solution is 0, so what we want is, Q is non 0 term and Ø is also non 0 term, so in that case so what we need is k- λ M should be 0.

So determinant of this Q – λ M should be 0 so this is called characteristic equation, so we need to solve this characteristic equation to get characteristic roots characteristic vectors so that is λ we get say as many number of degrees of freedom it has so if it is 3 x 3 matrix we get 3 λ values and corresponding λ values what we =can put is λ 1 here and solve for Ø so we get Ø 1.

So λ 2 corresponds to Ø 2 λ 3 corresponds to Ø 3 so these are Eigen values and Ø1, Ø 2, Ø 3 are called Eigen vectors so we will discuss about the Eigen vectors in detail in the next class.

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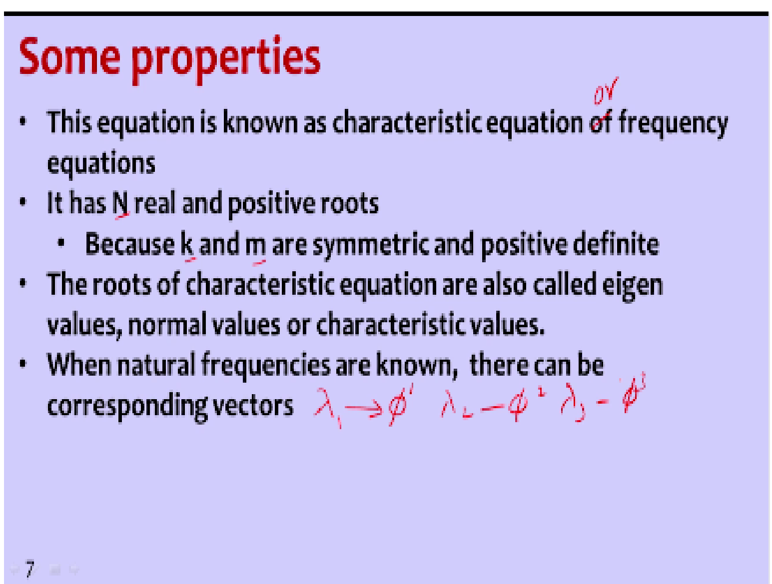

Now some properties of characteristic equation so this can also be called as frequency equation, or frequency equation so this frequency equation it has n real roots so if the matrix size is say n/n it will have n real and positive roots now why because k and m are symmetric and positive definite so for that reason we will have N real roots.

So roots of characteristic equation are also called eigenvalues they can also be called as normal values or characteristic values. So when natural frequencies are known there can be corresponding vectors, as I told you so for λ_1 natural frequency or Eigen value there is one vector that is ϕ_1 so for $\lambda_2\phi_2$ we can get for $\lambda_2\phi_3$ we can get.

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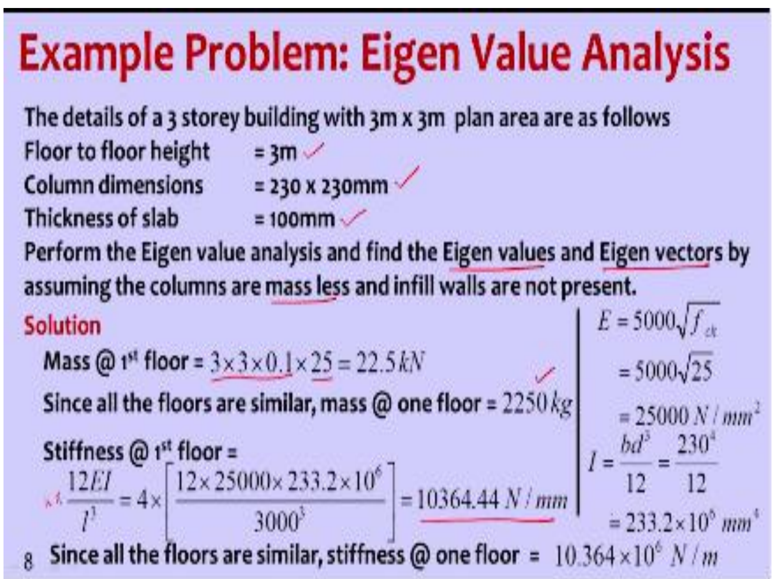

Now let us solve that du is this one and let us solve one example problem using for understanding multi degree of freedom systems in a proper manner. So let us take a structure, okay is a real regular structure so details of the structure are say three storey building with 3mx3m plan area okay, the properties are as follows. So floor to floor height is given as 3 meters column dimensions 230x230mm and then thickness of slab 100mm and perform Eigen value analysis and find out Eigen values and Eigen vectors.

By assuming columns are mass less and then infill walls are not present, so we are taking these assumptions for simplifying the problem. Now let us solve it, first let us find out mass so mass at first floor so that is 3mx3m is given so and multiply by 0.1 so this will give volume of mass and then this is weight density, so if you get weight density so 22.5 kN weight that we can convert as 2250 kg mass in all floors, so multiply by 1000 and divided by 9.81 so we get 2250 kg all masses same.

And then we want stiffness so for that we want modulus of elasticity so it is assumed that the material grade is m25 grade concrete so 5000√fck so we get 25000 N/mm^2 as modulus of elasticity and then moment of inertia of the column cross section so since it is square cross section so both sides are both so we are taking $bd^3/12$ so that is 230 power force so this is also 230 this is also 230, so that is why $230^4/12$ so $233.2x10^6mm^4$ is moment of inertia.

So now we need to calculate stiffness so stiffness of one column is $12EI/l^3$ so since four columns are present on each floor so we take 4 multiplied by 12xE is 25000 so we are substituting it here I we are substituting from this divide by h^3 so h is or l^3, l is 3 meters height, so total stiffness value is this stiffness value of floor, floor stiffness this is. Since all the floors are similar same so this value is same in all the floors.

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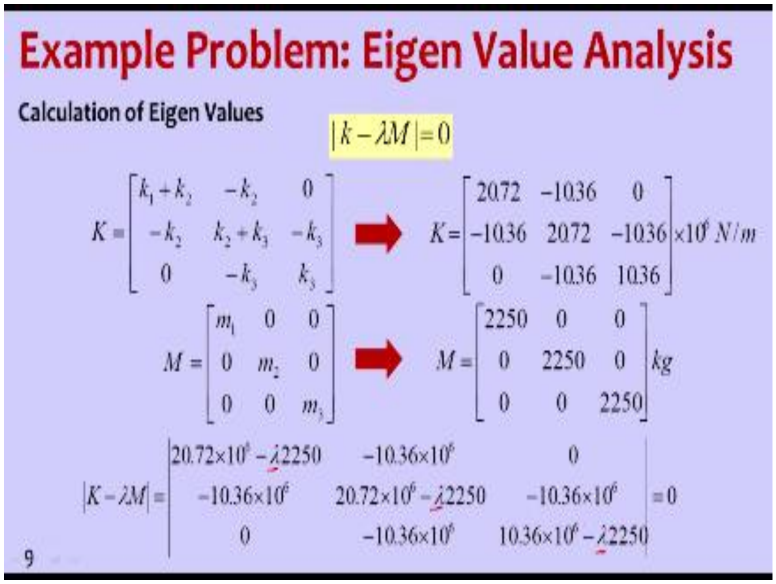

Here you can observe that we have converted this into N/m. then we calculate Eigen values, so how do we calculate Eigen values by using characteristic equation, so we have stiffness matrix as we have generated from the earlier class so $k=k_1+k_2$ $-k_1$ 0, $-k_2$ k_2+k_3 $-k_3$, 0-k_3 k_3 and then mass matrix is also there so we have this stiffness matrix so if we plug in the values here so we get stiffness matrix, and then mass matrix so mass same all masses are same so we have mass matrix characteristic equation can be written like K-λM is equal to like this so here you can see λ values are there, λ values. Now we need to solve this matrix that is take the determent of this.

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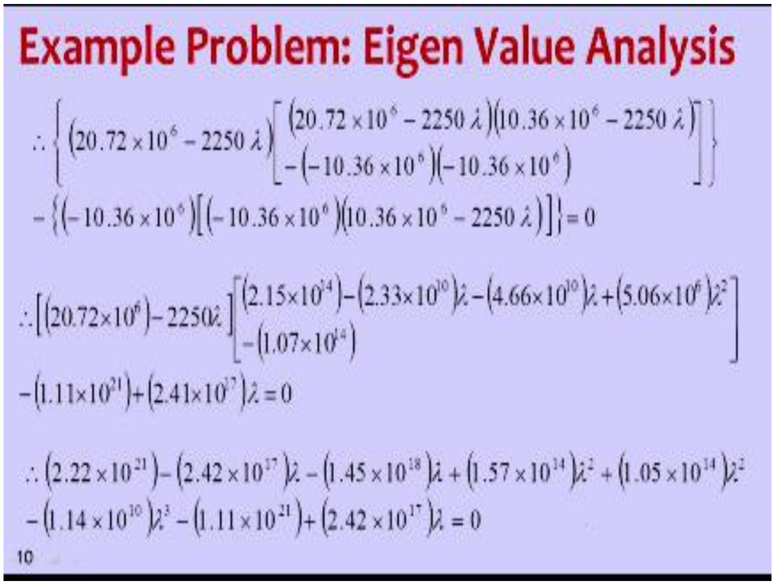

Okay, so if you expand it, it will look like this and then further expansion and then so if you solve it then what we get is this quadratic equation.

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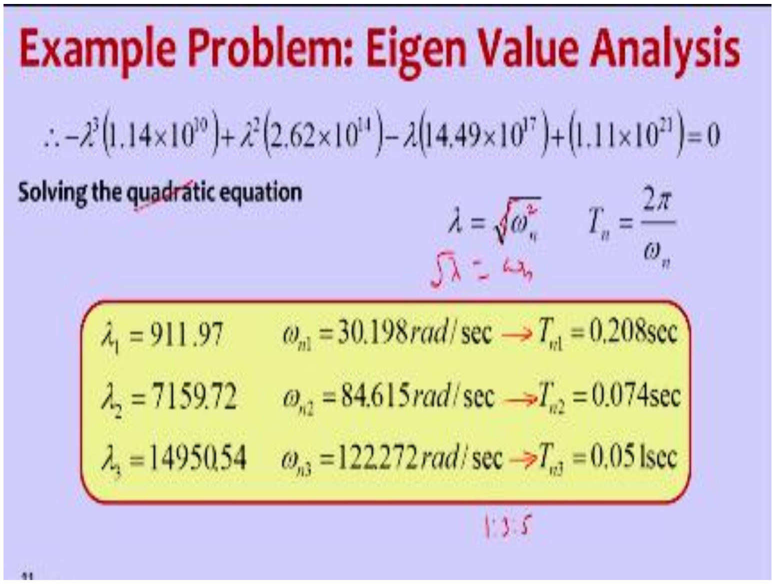

Okay, cubic equation invade not quadratic equation cubic equation λ_1 we have so and we get T_n so natural period of say correspondent to each λ so that is $2\pi/\omega_n$ so $\lambda=\sqrt{\omega^2}$ so λ_1, λ_2, λ_3 oh, in fact λ is square of this λ is square of ω, okay so n factor √λ is ω, so we get ω values, so we get ω1ω2ω3 so this is Eigen value one correspondingly we get natural frequency that is 30.198 and corresponding natural period is 0. 2 second so λ 2 so we will get ω2 corresponding natural period is 0.074 seconds λ 3 third Eigen value.

So we get this natural frequency and we get third natural time period. So we can clearly see that these natural frequencies are spaced so usually in regular buildings these values will be in the order of 1:3:5 orders. So one time 1:2:5 order if it is regular building we will have a value like this ω1ω2ω3 will be in this order, so ω1ω2ω3 will be in the order of 1: 3: 5 then not be exactly ion this order but almost around that values.

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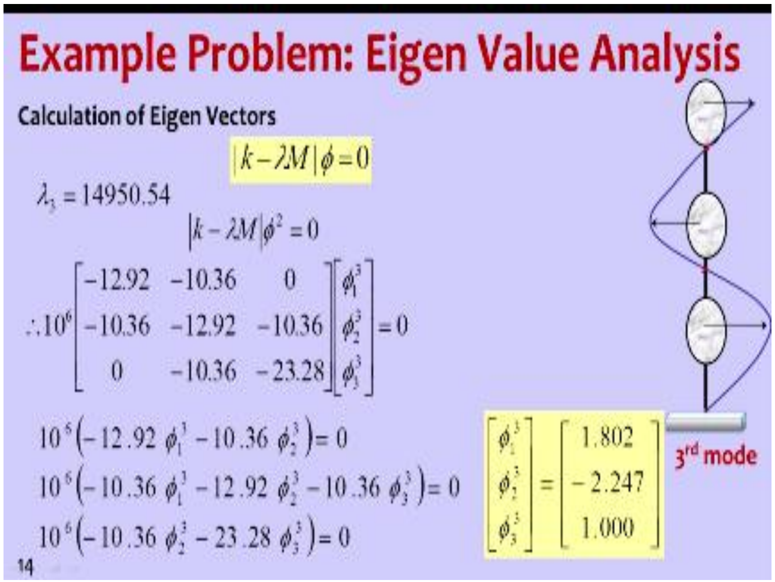

Now if we calculate Eigen vectors corresponding to $\lambda 1$ then we get so this is corresponds to $\lambda 1$ so you can see this 1.802 .445 value so if you solve this one as you can see let me explain you that k – λ m x 51 = 0nso this is k – λ, so this λ m value is already substituted here so we got this so in this one so first one 18.67 x 51 – 10.36 x 52 so this will have 51 52 53 three term this will have 52 and 53 terms so what we can do is from one equation one so we have to assume some one value ion this one so in equation three since it is representing the roof of the structure at the roof level of the structure in the first mode.

So we are assuming this value as one so if you assume this value as one so relatively we get 52 of first mode so we get that value that is 0.82 again the we take 53 and 52 values and substitute in the second equation we get 51value and put 51 and 52 value to check whether they are matching or not so that is how we get so 51 52 53 of first mode, and we get mode shape if we plot like this so this is one this is 0.8 and this is 0.445.

So usually first mode will touch the axis this is the axis of the building at this location only at this base only it will not touch anywhere.

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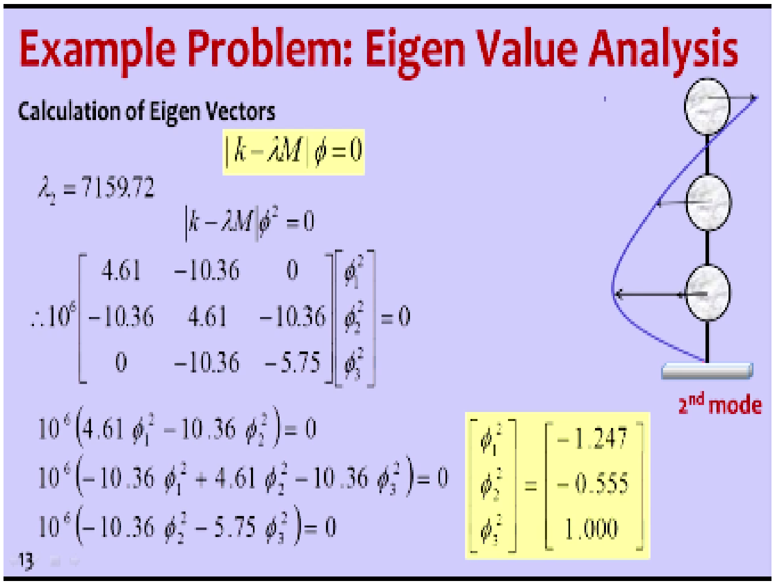

Now when we check the λ2 that is the second Eigen value so we get second Eigen vector so second Eigen vector is you can see this one so it is crossing the axis at one point this is the beginning and then it is crossing at only second location so this two okay.

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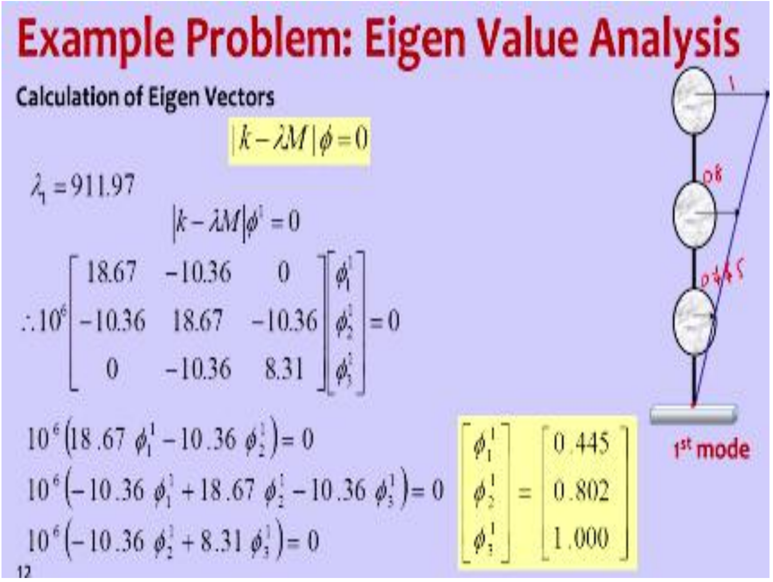

In first one Eigen vector it touches axis only at one point and at no other place it touches and the second one it cross at this location and it touches the axis this location so when you substitute Eigen value three in this equation and get Eigen vector values so 515253 of mode 3. So we get it like this so you can see this one is crosses the axis at one point second point and this is beginning so three places.

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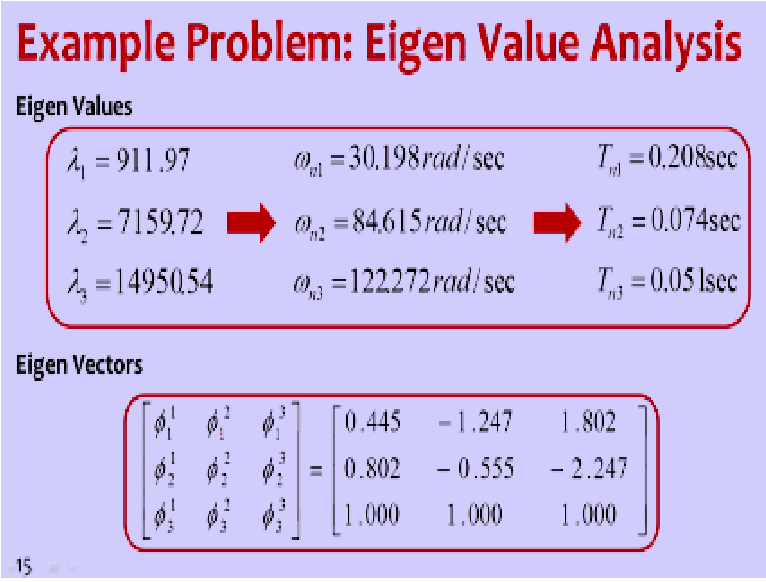

So if we look at all three at a time so this is 5 matrix that is in values matrix okay so Eigen 3 Eigen vectors are there. So it is called mode shaped matrix so first mode second mode and third mode so first mode vector second mode vector third mode vector so if we represent all these.

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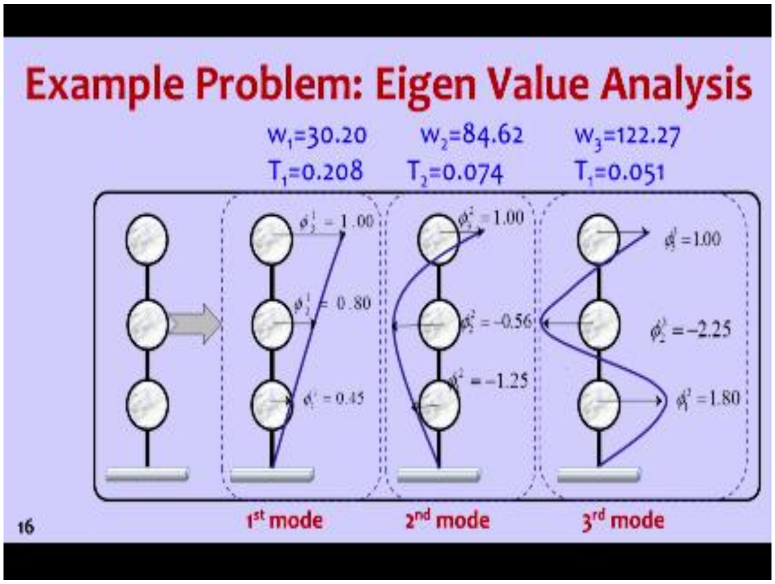

Three in a location it's the same place so this is the total idealized mass of the building or the idealized building so first mode is represented like this second mode is represented and third mode is represented so usually if the top value as say one if we normalize it to one it has its advantages so each mode shape will carry some mass of the building in its vibration along the when it is vibrating it mass participates in this one so much how much mass is participating in mode one how much mass is participating in mode two how much mass is participating in mode three we discuss in the next class.

So first mode second mode and third mode so these modes give the decomposition of a complex displacement behavior of the structure into simple idealized structure.

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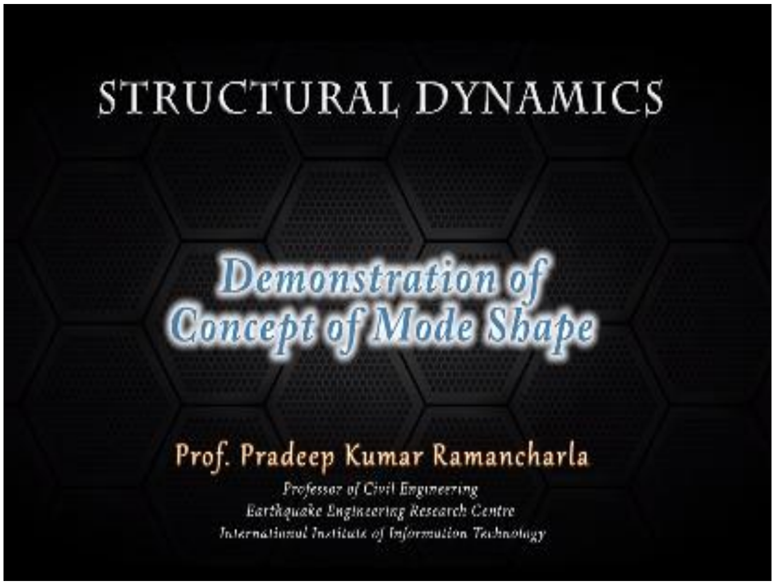

Multi degree of freedom system or three storied building system is oscillating in the first mode so all three so this slap, this slap and this slap are all three in the positive direction one direction on the left side direction and then if it is oscillating in the second mode then this will be in the right direction and this will be in towards left of the axis and if it is vibrating in the third mode then this will be in the right direction this will be in the left direction this will be in the right direction.

So this is a concept of mode shaped in the building so if I oscillate it slowly then it will get exacted in the first mode if I increase the frequency then it will excise in second mode then further increase its frequency of exaction it will increase it will move in the third mode in summary what we have studied in this class so we have studied how to formulate mass matrix and stiffness matrix and how to solve the characteristic equation and then how to the solution of the characterized equation is represented in the form of Eigen values so Eigen values is nothing but the frequency of the structure in equal mode.

So first mode second mode third mode so omega Eigen value 1 represents the frequency of the structure in first along first mode and second mode is Eigen value and third mode is third Eigen value so this is represented by representing the frequency and then the mode shape is their so mode shape is representing the, the deflection profile of the structure defile profile of the structure in first mode defile profile of the structure in the second mode deflection profile of structure in third mode so these mode shapes have a some important properties so with which we can understand the behavior of the structure under the action of dynamic load in much better manner so we will discuss about mode shapes and the properties of the mode shape in the next class.

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