Finite Element Analysis
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Lecture No. # 06

In the last lecture, we have seen a boundary value problem, using the formal finite element procedure in conjunction with Rayleigh Ritz method and also Galerkin method. And the general steps involved in finite element procedure are: first, discretization over the problem domain and the derivation of element equations. And then, assembly of global equations. After that introduction of essential boundary conditions and then solution of global equations for the nodal values or nodal parameters, and then substituting these nodal values or nodal parameters into each of the element equations, you can calculate at element level, what the solutions at element level are.

So, this is a general finite element procedure and we have seen, whatever we have seen so far is all mathematical examples; we have taken a boundary value problem, the physical phenomena of which we are not aware of. So, it is time for us to start looking at some physical applications.

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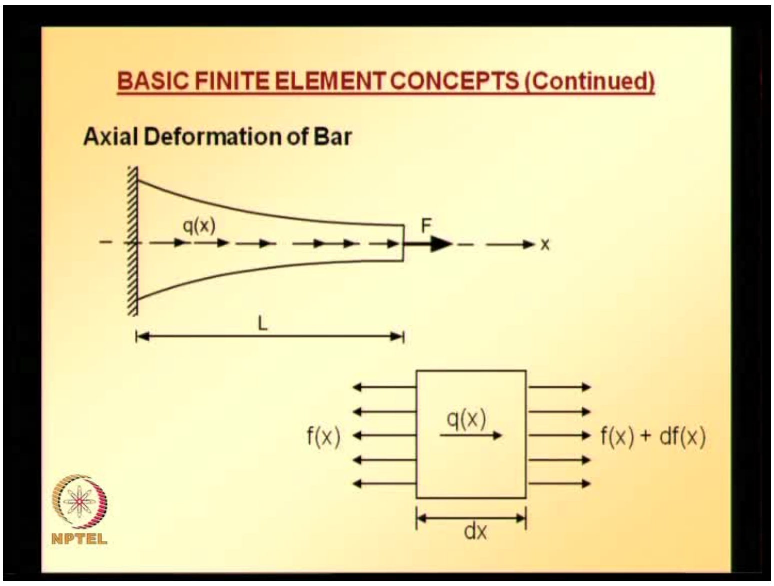

So, for that I have taken simple example, axial deformation of a bar. And here for generality, the cross sectional area of bar is taken to be vary and as indicated in the figure, the x axis is measured from the right side at x is equal to 0 will be equal to 0 , at x is equal to 0 will be 0 at left side and x is equal to l at the right side, at right side there is a point load applied f and a distributed load is over the entire length of the bar and the material property and cross sectional areas are also assumed to be a function of x. So, this is all for general case. If you have a specific case of a bar which is having uniform cross sectional area in material properties are constant over the entire length of the bar, then, you can easily incorporate those kind of things into the equations that we are going to derive. And we need to know, if you want to apply finite element method, what you need to do is, you need to explain this physical phenomena, that is, axial deformation of a bar in terms of a mathematical equation, that is, what we require to solve using finite element method in conjunction with variational method or Galerkin method.

So here, what we will do is to derive the governing differential equation, for explaining the physical phenomena of axial deformation of a bar. The first step is you just take a small differential element out of this bar and indicate all the forces that are acting on that differential element.

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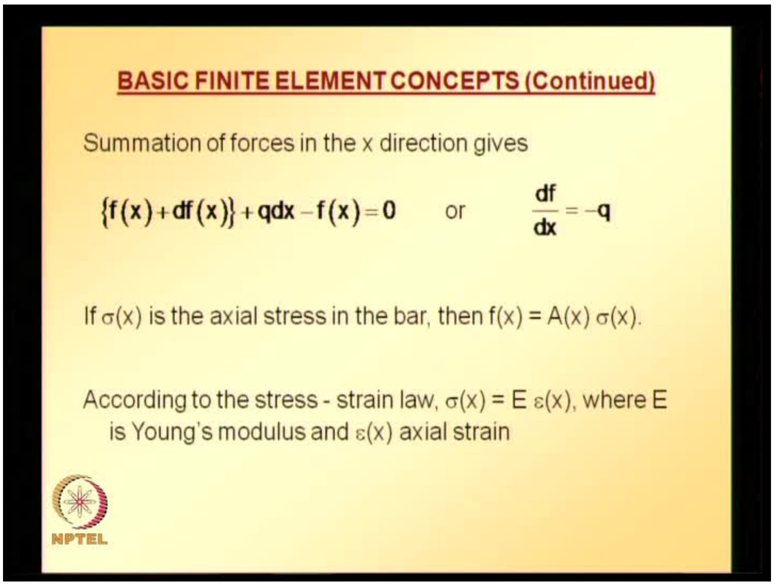

And then, sum up the forces in the x direction. You want this differential element d x to be in equilibrium, that may sum of all forces in x direction, because this is the one d problem we have only in the x direction, sum of all forces in the x direction equal to 0 to apply that condition, that gives us this equation, which we can rearrange that is, f and f as a function of x cancels with minus f as a function of x and will be left with this one after rearrangement. And we know that force is nothing but it is for one d problem it is a stress times cross sectional area and stress is equal to strain times Young's modulus and strain is equal to derivative of displacement this is all, what you have learnt from your mechanics of materials. So, we will apply all those conditions here. So, if sigma x is the axial stress in the bar then force effects is going to be sigma times cross sectional area everything is for generality everything is assumed to be function of x.

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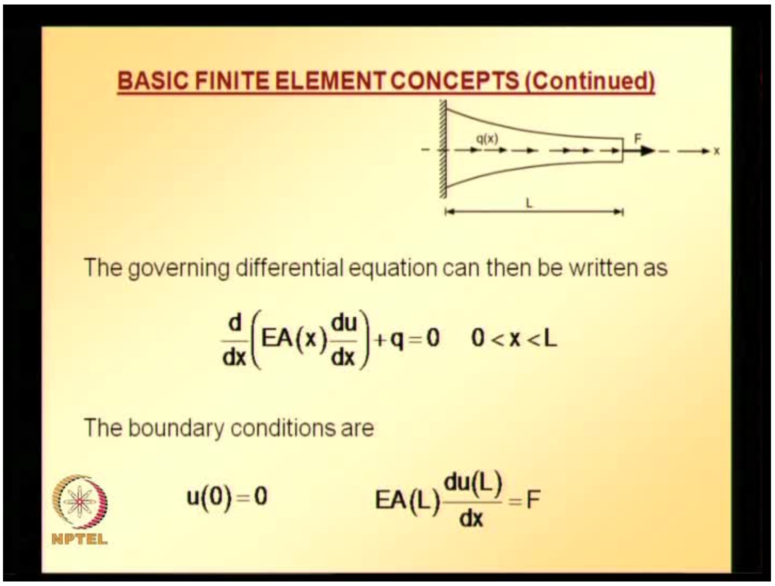

So, according to the stress strain law, sigma stress is equal to Young's modulus time strain and here capital is young's modulus and epsilon is axial strain again, all are functions of x. So, substituting all these quantities, the given are the whatever equation that we obtain earlier, that can be rewritten that is replacing f with e times area of cross section times derivative of u with respect to x.

So, that is, what is done here, so, this is the governing differential equation explaining the phenomena of axial deformation of a bar and this is valid over the length x going from 0 to L, because that is, the region we have taken for derivation of this differential equation. And now, if you see this is the second order differential equation and you require two boundary conditions, to solve this equation. So, the two boundary conditions, we need to look at the problem that is, the problem that we are already looking at and we need to write the corresponding boundary conditions.

So, the first boundary condition is, I mention to you the bar is fixed at the left side that is, at x is equal to 0, displacement is equal to 0 that is, the first boundary condition, at x is equal to 0 u is equal to u evaluated at x is equal to 0 is 0 and a point load is applied, a point load of magnitude f is apply at x is equal to L. You just learnt that f is nothing but, area of cross section times, young’s modulus times, derivative of u with respect to x, all these quantities we need to evaluate at x is equal to L in case, they are all functions of x.

So, when all these quantity that is, E times, A times d u d x all are evaluated at x is equal to L they must be equal to be f, the point load that is applied. So, this is the other boundary conditions. So, looking at the physics of the problem we have written the two boundary conditions. One is u evaluated at x is equal to 0 is 0, the other boundary condition is point load applied at x is equal to L is equal to the value f.

So, if you look at these two boundary conditions, you can easily recognize one of them is essential, the other one is natural boundary condition from the thumb role that I already mentioned to you in the earlier lectures that is, if you have a boundary value of problem of order 2 p, those boundary conditions of order 0 to p minus 1 are essential boundary conditions, those boundary conditions of order p 2 2 p minus 1 are natural boundary conditions. And also, if you see the physics of the problem u evaluated at x is equal to 0 is it is essential otherwise, if these boundary condition is not satisfied then in fact it is not a bar problem. So, in fact, it is essential that this boundary condition you need to satisfy whereas, the other boundary condition even if f forces f is not applied at the right tip then still a bar will be there. So, that is, why it is called natural boundary condition.

So, now, we have the physics of the problem, the physical phenomena is already expressed in terms of differential equation and that is where we can start our knowledge that we gained already from the previous lectures that is, either you can use variational method or Galerkin method and we can substitute finite element approximations into it and solve for the unknown quantities.

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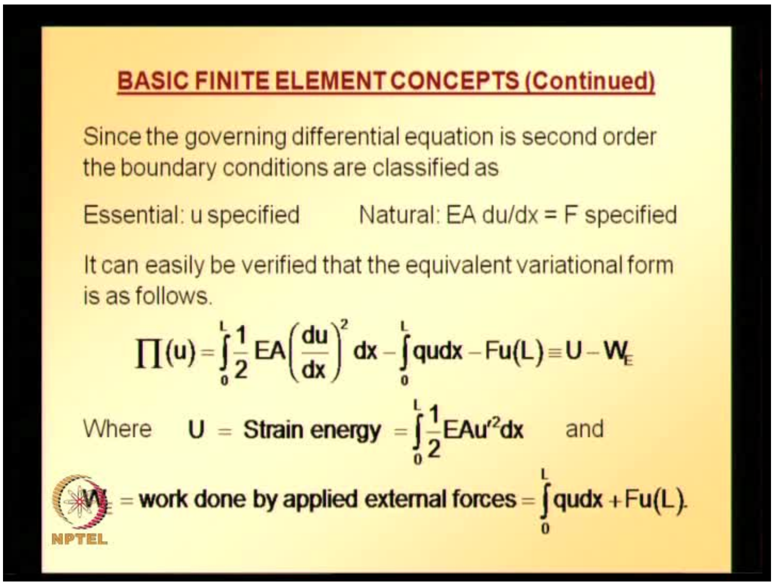

So, here, now we can write following, here what I mentioned u evaluated at x is equal to 0 is 0 that is, essential. And the other boundary condition is a natural boundary condition. And now we can use our knowledge in variational method Rayleigh Ritz method and we can write directly the equivalent variational functional for this problem. And that is you need to take the given differential equation. If you want to follow variational procedure and you just recall the steps involved take the differential given. The differential equation multiply with variational of the quantity that, we are looking for that is, variation of u and integrate over the problem domain equated to 0, and then, the next step is apply integration by parts reduce the higher order derivative terms and then you need to substitute the condition that, variation of u wherever essential boundary condition is prescribed at that point variation of u is equal to 0 and also, at this point you can also substitute any natural boundary condition that is given and then simplify it and bring it use the variational identities. And bring it into the form variation of some quantity is equal to 0 and whatever is there inside the bracket is what is called variational functional or equivalent variational functional and it turns out for this problem that is axial deformation of a bar this variational functional is nothing but potential energy.

And that is why here it is denoted using capital pi and if you are good in writing directly the potential energy equation for or potential energy expression for a structural system you can directly write it here, potential energy is nothing but strain energy minus work done by the applied forces.

And if you look at this potential energy equation, the first term is nothing but strain energy, if you recall from your mechanics of material for a bar, a problem bar under axial deformation strain energy is nothing but sigma square over 2 E times volume and because, if all are constant that is area of cross section is constant and if Young's modulus is constant if sigma is also constant, you can apply this condition but here all are assumed to be varying over the cross section entire length of the bar. So, it is given as here in the equation that is, strain energy is equal to integral 0 to L half E times A u prime square that is nothing but derivative of u with respect to x, if you can easily show whatever is there half E times A u prime square is nothing but sigma square over 2 E times volume A times d x is volume and half times E u prime square, what is sigma square over E? So, if you integrate over the volume, you will get the entire strain energy and if you recall what work is done by the applied forces, it is nothing but force times displacement since q is distributed load. So, it we need to use integral 0 to L, q is a function of x even though it is not explicitly written their q times u gives you the force and you need to integrate over the entire length of the bar, f is a point load and the displacement evaluated at x is equal to l f times displacement evaluated at x is equal to l gives you what is the work done by the due to the application of load f.

So, if you are good at writing a potential energy expression directly you can do it or you can go start with the given differential equation and apply whatever procedure we learnt for variational method or Rayleigh Ritz method follow that procedure and finally, you will arrive at this equivalent variational function. So, now, we are ready to substitute, if you know finite element approximation. Now, we can substitute finite element approximation of u and apply the stationarity conditions that we already did in the last class and get the element equations for this problem.

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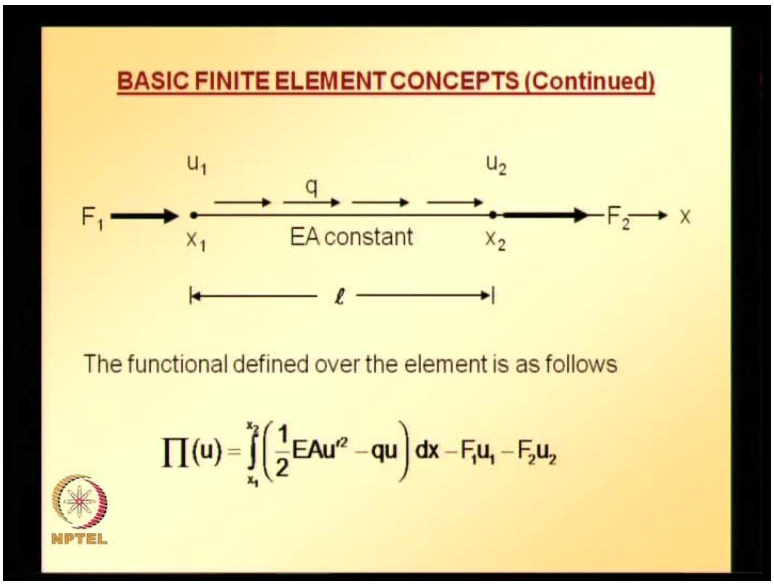

So, now, a finite element equations can be derived either you can use Galerkin method or variational method. And here a typical element is shown. And here, this element, for this element, at the nodes 1 and 2 point forces are F 1 and F 2 and also the sign convention. Please, see the forces are positive in the direction in which x is positive. And the nodal coordinate of node 1 is x 1 nodal coordinate of one node 2 is x 2 and q is the distributed load that is acting and E A is constant. The degree of freedom associated with node 1 is u 1, degree of freedom associated with node two is u 2 and length of this typical element is L. So, now, if you see this element actually you have 2 point loads applied. So, if you want to write potential energy functional or equivalent variational functional it looks like this, additional if you compare this equation with the previous equation that you got for a the bar which is fixed at one end, we have only one F times u there whereas, you have one F times u term there whereas, here you have F 1 times u 1, F 2 times u 2 because, you have point loads applied at the both ends, at 1and 2 because of that reason, we have F 1 u 1, F 2 u 2, two terms are there and the rest of the portion is similar to what you already have.

So, now, what we can do is we can this x 1 x 2, coordinate system ,we can map into s coordinate system or x coordinate system can be mapped into s coordinate system, such a way that, x 1 coincides with s is equal to minus 1, x 2 coincides with s is equal to 1.

So, in the last class, we have seen, what is the relation between such kind of mapping between x coordinate system and s coordinate system, which you can obtain easily using linear interpolation or that is, a similarity between that relation and equation of the straight line, I already explained all these things in the last lecture. So, now applying those things, you get the relation between s coordinate system and x coordinate system once again, s coordinate system is defined such a way that s is equal to minus one coincides with x one, s is equal to one coincides with x 2.

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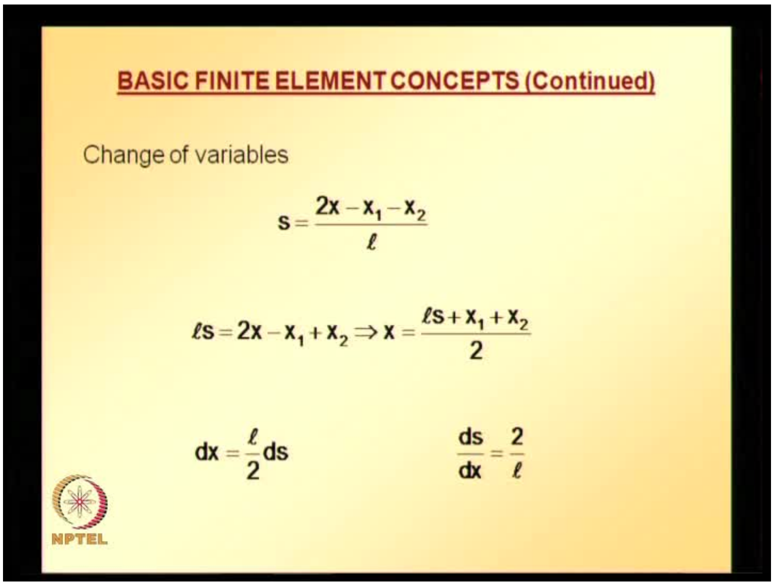

So, the relation between x coordinate system and s coordinate system is this. The advantage of doing this kind of mapping is the limits of integration, now becomes minus 1 to 1 instead of x 1to x 2. So, we can easily apply gaussian quadrature and now using this relation, we can manipulate and get the inverse relation between x coordinate system and s coordinate system and also we can take derivative on both sides, we get this relation. And now, we need to write trial solution in terms of s coordinate system.

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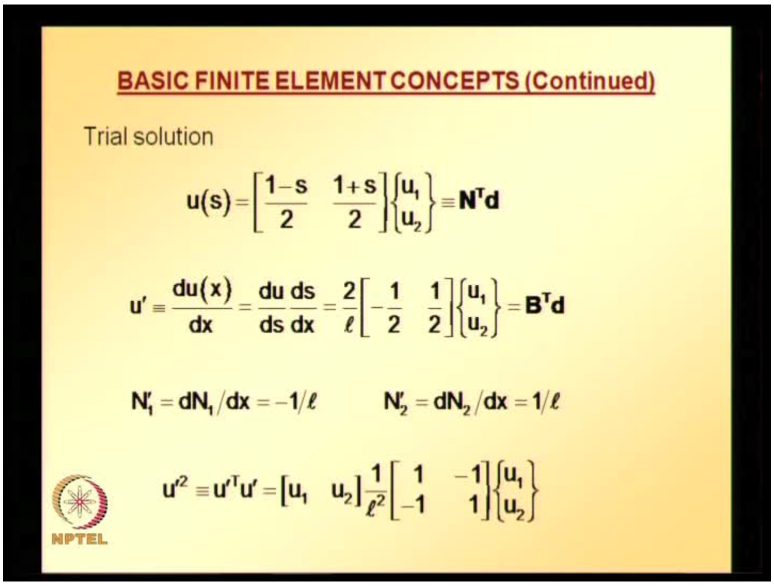

So, now, the entire problem we are going to work in s coordinate system. So, trial solution is u is equal to N 1 u 1 plus N 2 u 2 and the shear functions - N 1, N 2 are already explained these things in the last class, N 1 is equal to minus 1 1 minus s over 2 N 2 is equal to 1 plus s over 2 and these can be compactly written the trial solution can be compactly written as N transpose d; from there it follows: what is u prime u prime is nothing but derivative of u with respect to x using chain rule, you can write derivative of u with respect to s, derivative of s with respect to x. What is the relation between derivative of relation between s and s x coordinate systems?

So, you can write what is a ds over d x A d s over d x turns out to be 2 over L and du over d s you can use the first equation and get what is the du over ds. So, u prime turns out to be what is that b transpose d where, b transpose is defined as two over l minus half half. So, that is what is defined as b transpose and please note that this, notation is only applicable or is applicable for all the later lectures in this course and notation depends on other each, other will have different way of using different notations.

So, you should be careful when you are referring various text books. And now what is derivative of shape function with respect to x, what is derivative of shape function N 1 with respect x, shape function N 2 with respect x. That you can easily write it from the earlier and your knowledge you have seen N 1 is equal to x 2 minus x over L that is, N 1 and so you take derivative of it you will get minus 1 over L N 2 is nothing but minus x 1 plus x over L and if you take derivative of it you will get one over L. And if you sum up this derivatives of N 1 N 2 it turns out to be 0 and also shape functions that is N 1 plus N 2 is equal to 1 that is, some of shear functions is equal to 1 and derivative of shear functions is equal to 0, this is valid at any point in the domain and these are called consistency conditions. And now you got trial solution and what is u prime derivative of u with respect x.

So, now, if you see our potential energy equation, we have u prime square and please note that, u prime is a scalar quantity. So, u prime square, I can write it as u prime transpose u prime. If you have a scalar quantity, scalar square of a scalar quantity can be written as transpose of that scalar quantity times that scalar quantity.

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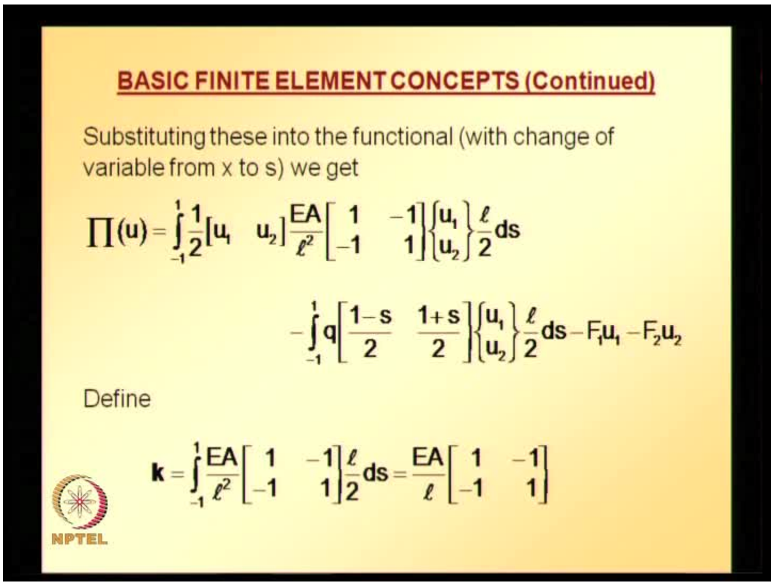

So, using that reason square is written as u prime transpose times u prime and you know what is u prime? It is B transpose d and substituting all those quantities, you get u prime square as given here and now you got u prime square. So, you substitute u prime square and also you can substitute what is u, u is n transpose d and finally, the given functional can be written in terms of finite element, approximation in this manner which you can further simplify.

So, the first integral inside whatever is there first integral, if you see the first integral term can be rewritten in this manner and please note that u 1, u 2 are nodal values or nodal parameters. So, they are not functions of the coordinate system, so you can pull them out of the integral and whatever is left inside the integral that is half is also constant you can pull out of that pull that also out of integral. Whatever is function of x E is a function of x or s is function of s L square 1 minus 1 minus 1 1 times L over 2 d s that is, what is defined as k and if you carry out integration, you get this one E A over L 1 minus 1 minus 1 1 and this E A while carrying out this integration it is assumed E A are constant. So, we can take them out of the integral, so whatever is given here k is equal to E A over L 1 minus 1 minus 1 1 that is, applicable only when Young's modulus and cross sectional area are constant and that is a specific case and if you have a general case you need to integrate them putting them inside the integral.

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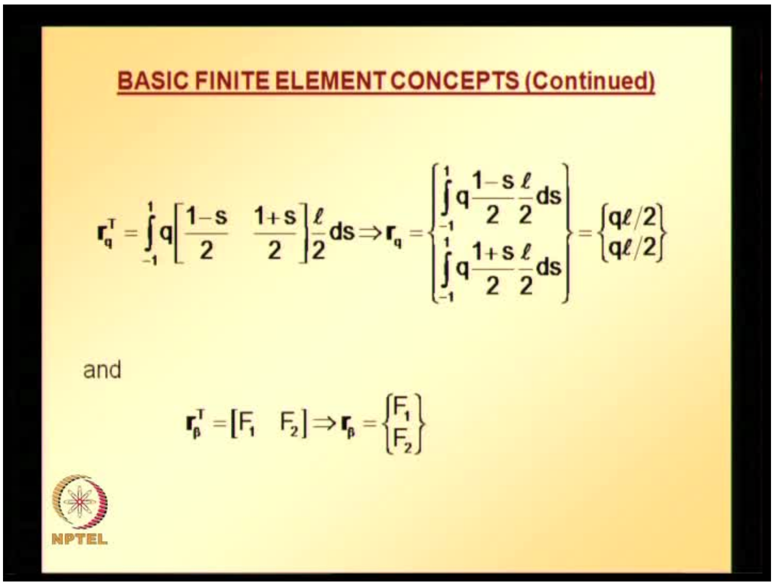

And k is defined like this r q, r q is a defined like this. The second term is defined as r q and here, if we see the second term actually it has n transpose d, d is taken out d consist of u 1, u 2 it is taken out of the integral. And whatever is remaining that is defined as r q and we can carry out the integration and finally, will get q l over 2 q l over 2 and if you see last two terms, you have F 1 u 1 times F 2 u 2, F 1 u 1 and F 2 u 2 these terms are there the last term and in that you can again put them in the form of this vector F1 F 2 times here d vector.

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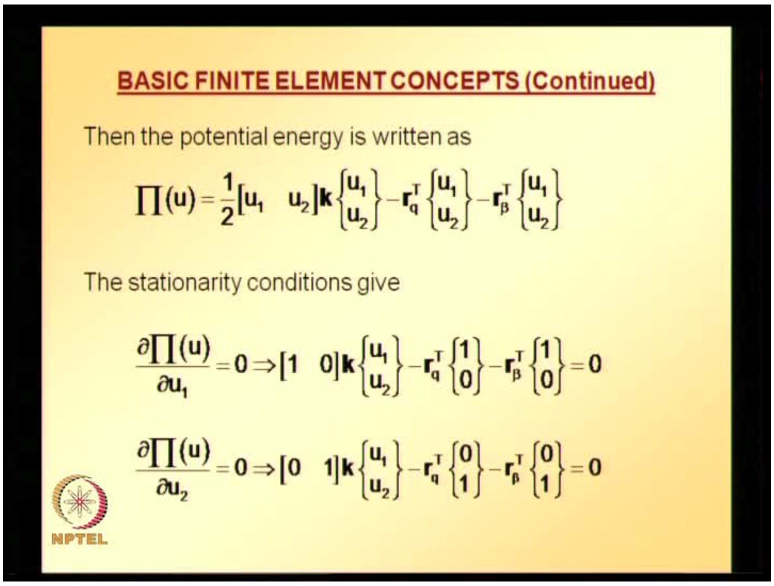

So, r beta is defined like this. So, once we define k or q or beta, we can write the potential energy in a compact form like this. And now you can see potential energy is a function of u 1, u 2 and the requirement is variation of potential energy should be equal to 0 or the other condition is we can variation of potential energy is equal to 0 is satisfied, when partial derivative of potential energy with respect each of the nodal unknowns or nodal values is equal to 0 that is, partial derivative of pi with respect to u 1 is equal to 0.

You get two equations. There is a first equation, applying the first condition and the second condition. As you can see here whatever the procedures you have learnt we are now applying to a physical problem, you can see the applications of those things that is, a stationarity conditions.

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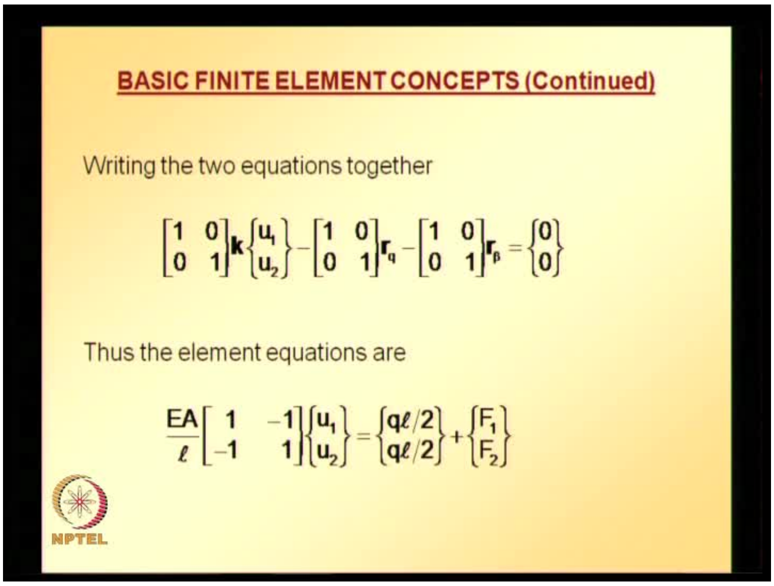

And now you got two equations, these two equations can be are looking similar so we can put these two equations in a matrix form, in this manner and if you see 1 0, 0 1 is identity matrix, so, whatever you multiply with identity matrix you got a same thing. So, this can be compactly written k value is substituted, k is equal to E A over L 1 minus 1 minus 1 1 that is worth k and r beta and r q values r q is nothing but q l over 2 q l over 2 r beta is F 1 F 2.

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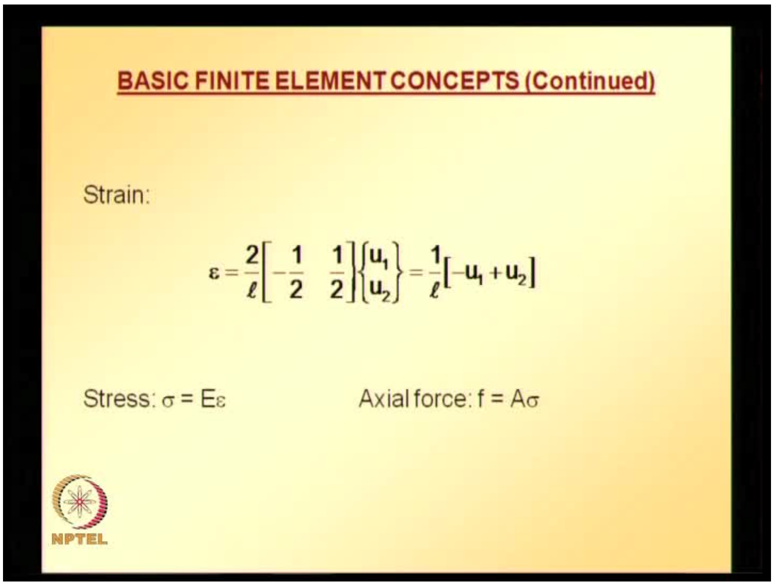

And if you see this equation and the first matrix is what is called stiffness matrix u 1, u 2 is displacement vector and whatever you have on the right hand side, they are all related forces and if there is no body force or distributed force to you than that worked is going to be 0 and if you have only point forces then you will have only F 1, F 2. And you solving that equation system you can get what is u 1, u 2. Once you got that is, for one element, you need to assemble such kind of equations for all elements in the system and assemble the global equation system and apply the boundary condition essential boundary conditions, solve for the global equation system. Once you get the nodal values you get, you can go back to each element and then find strain in each element. Strain is given by B transpose d, strain is nothing but derivative of u with respect x you already know what is derivative of u with respect x. B transpose d, B transpose is defined as 2 over L minus half times d, d is nothing but u 1 u 2.

So, strain you can obtain using this equation, once you solve for u 1, u 2; once you know the values of u 1, u 2 you can calculate the strain in the element using this relation and once you know the strain, you can find what is stress and axial force. And now will take an example and apply whatever we learnt till now related to the axial deformation of a bar.

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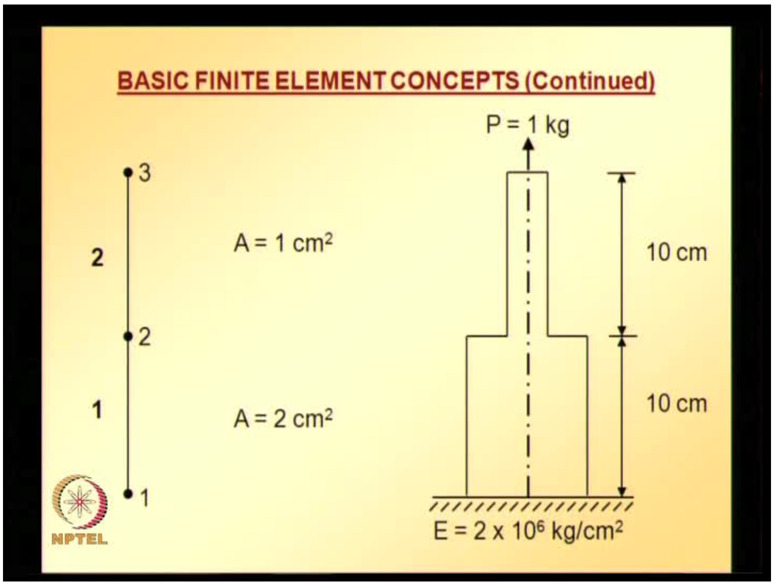

So, let us take a step bar like this, which is having different cross sectional areas over its length and the cross sectional areas, the value of cross sectional areas and also the lengths over, which those cross sectional areas are valid and also the material property and the load applied everything is given here and since, bar is having different cross sectionals at a particular point, we will take a element 1 or node 1 coinciding with the lower end, node 2 coinciding at the point, at the location, at which the cross sectional areas are changing and node 3 is at the point where load is applied. And lets call the element connecting nodes 1 and 2 as one element first element and element connecting nodes 2 and 3 are second element, which is indicated to s 2 there in the figure and also, the cross sectional areas are indicated for clarity there. So, now, you have element 1 and element 2, you know what is Young's modulus cross sectional area length of element 1. So, you can easily assemble what is the stiffness matrix of element 1 or elemental equation system, you can assemble and element 1 if you look, no load is applied in the region of element 1; if you look at element 2, all the material properties and cross sectional areas, all the geometrical details and material property details are known to you E A L and also the load that is applied is known.

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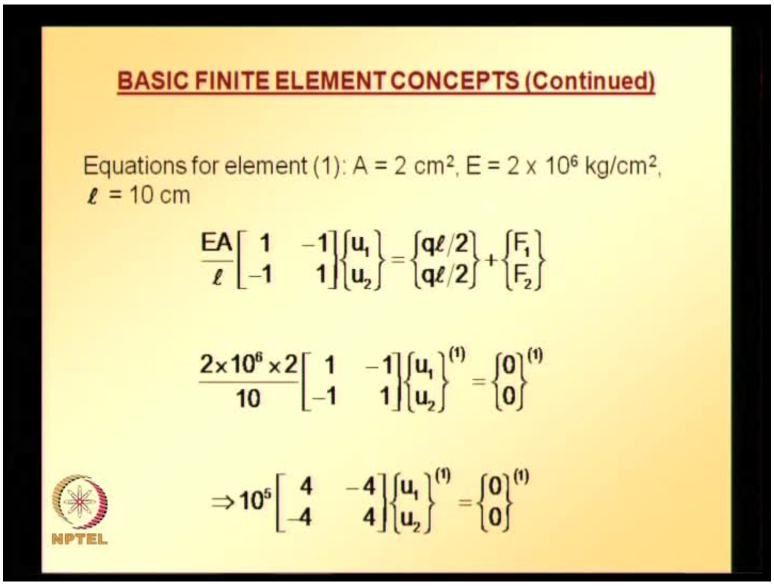

So, you can easily assemble elemental equations for element one; noting down the geometrical properties, material properties, length of that element we can substitute into this equation that we just derived and please note that the equation system that we derived is applicable only when cross sectional area Young's modulus are constant throughout the entire cross section area of cross section and young's modulus are constant over the entire length of the element. Since, the element that we are looking at is having same Young's modulus and cross sectional area we can apply these equations element equations, substituting the corresponding values for this element we get these equation element. Superscript here is denoting the element for which it is this equation system is for.

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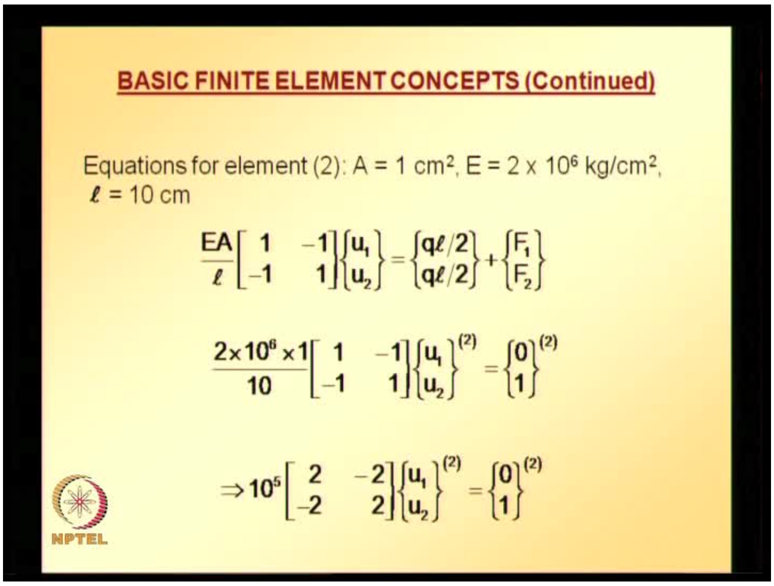

So now, we can also write equations for element 2 and simplification of this gives you this. Equations for element 2, again note down cross sectional area Young's modulus length, substitute into the element equations, again superscript is denoting the element number for which this equation system is applicable. And if see the difference please note that F is applied F 2 here is taken to be equal to1, because force 1 of 1 kilogram not load not force, load of one kilogram is applied at node 3 and the positive x direction, our positive x direction is going from the fixed end to the top. So, the load is acting in the same direction as the positive direction so, this taken as 1 and we get the equation system for element 2 which upon simplification gives you this.

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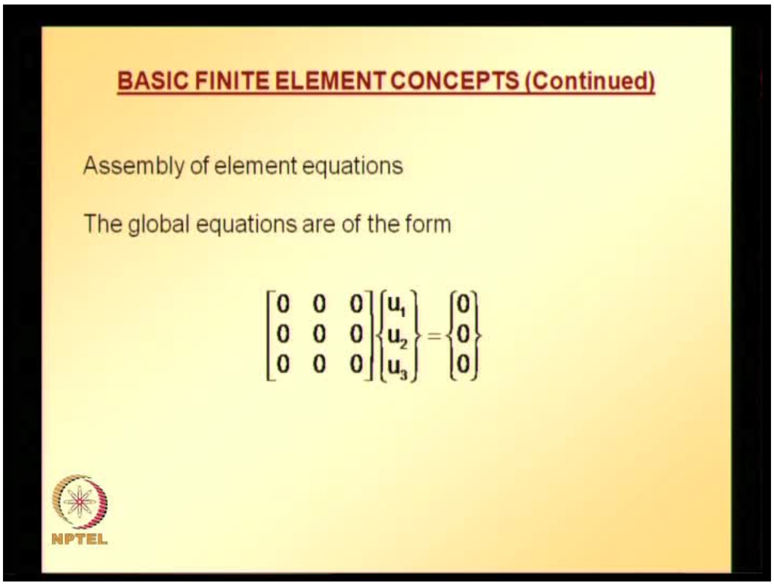

So, now you got elemental equations for element 1, elemental equations for element 2. So, now we are ready to assemble the global equation system assembly of element equations and please note that, this problem is a 1 dimensional problem and there are 3 degrees of freedom one at each node.

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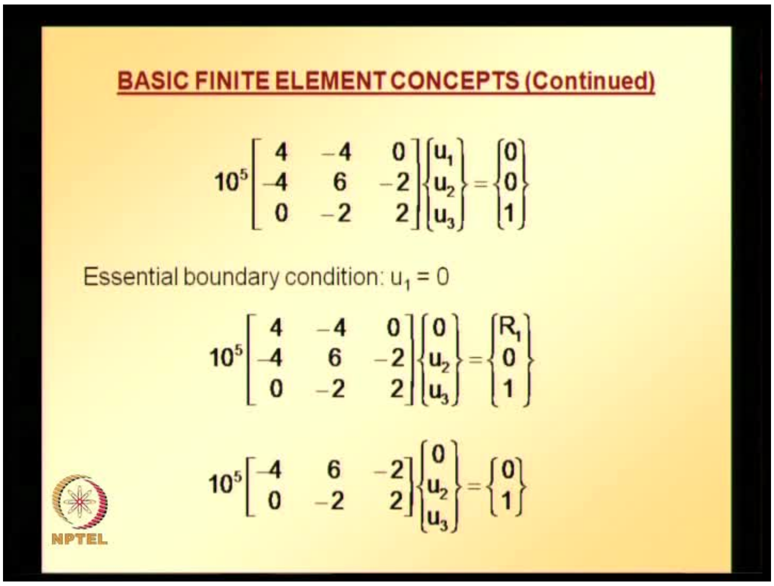

So, you will have global equation system of size 3 by 3 and the element 1 contribution goes into the location 1 and 2, because element 1 is connecting nodes 1 and 2, element 2 contribution goes into the locations 2 and 3, because element two is connecting nodes 2 and 3. So that is what is substituted and this is the global equation system. Similarly, the force vector from element 1 goes into locations of the a global force vector and also from element 2 the contribution goes into 2 and 3 locations of the global force vector. And now after substituting all these, we get this global equation system. Now, we are ready to apply the essential boundary condition, essential boundary condition is u evaluated at x is equal to 0 is 0 that is, bottom it is fixed that is u 1 is equal to 0.

So, that is substituted u 1 is equal to 0 is substituted and please note, that wherever you apply displacement boundary condition wherever you apply displacement boundary condition, reaction will be develop. So, that is the one of the important thing that you need to note down. It is not that because u is 0 that is here, u 1 is 0 it is not because of that reaction is develop, u 1 you can take any value reaction will be developed.

So, since, displacement is equal to 0 at node 1 reaction R 1 is assumed to be developed at node 1 and this equation system, what you can do is you can use partitioning of matrices and solve for unknowns u 2 u 3 and then back substitute those unknowns into the first equation and get back calculate what is reaction R 1 or you can actually multiply or you can actually eliminate the first row of the equation system and you can get this.

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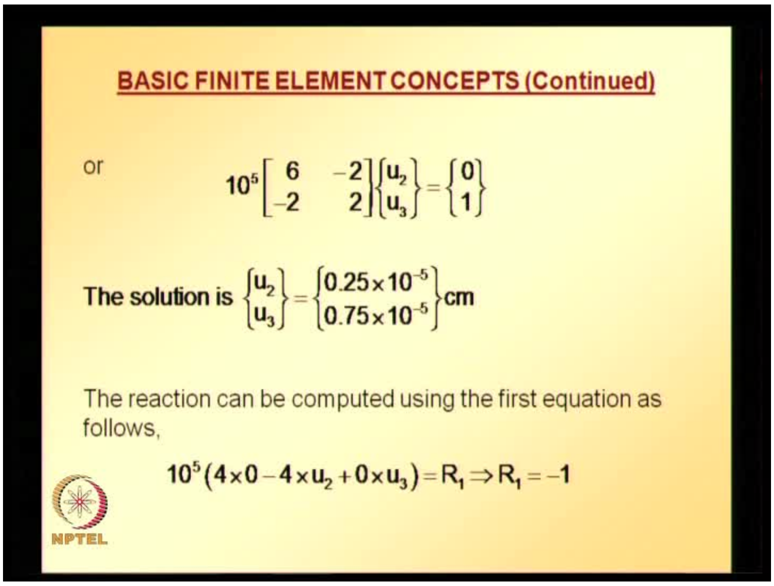

And if you do multiplication of this, on the left hand side it reduces to this one. If you compare this, what is called reduced equation system with the equation system that you have without applying the essential boundary conditions, what we basically did is wherever essential boundary condition is applied remove that particular row and column in the global equation system then we can get the reduced equation system directly.

So, now we can solve these two 2 by 2 equation system for u 2 u 3 we get this and once we get this u 2 u 3 what we can do is how do you check whether this u 2 u 3 values are correct or not. What you can do is, you can go back and calculate what is R1 reaction and you know that for a body to be in equilibrium the sum of all forces should be equal to 0.

So, substitute this reaction in the free body diagram and see whether forces balances are not actually in fact here, the reaction is having same magnitude as the load that is applied one kilogram is applied.

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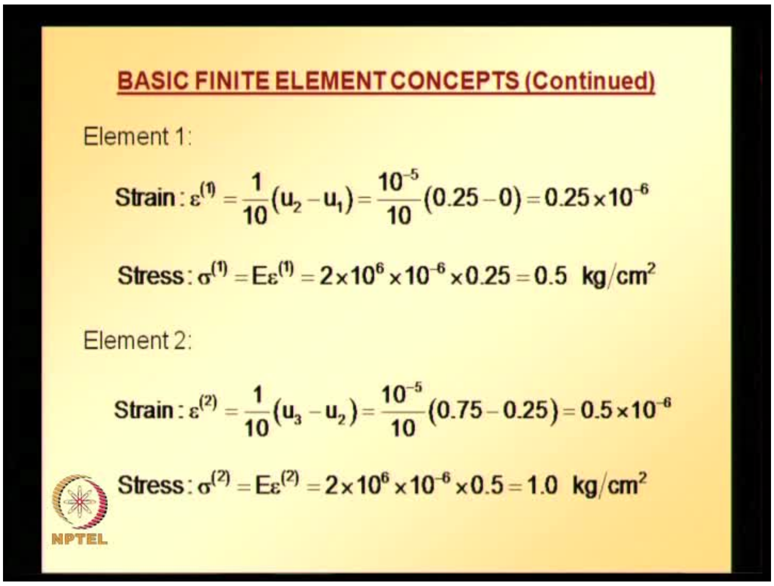

So, R 1 came out to be minus 1, because it has to balance the positive 1. So, it is negative 1. So, we got what is u 2 u 3. So, what we can do is and u 1 is equal to 0 using this nodal displacement values, we can go to each element, element 1: we can calculate, what strain is, stress again element 2: strain and stress. If you further want what is the load or what is the load in each axial force in each of the elements you multiply the stress value with cross sectional area of each element, you get what is the force carried by each part of that bar.

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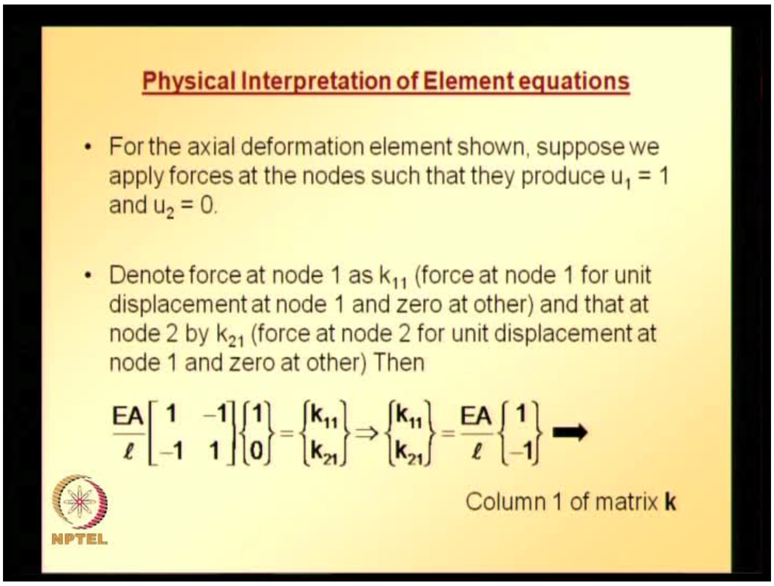

So, now we solved a problem, we have seen a physical application of the techniques that you learnt. So, now we have seen a structural mechanics problem that is, axial deformation of a bar. And now let us see, what is the physical interpretation of this element equation? For axial deformation element shown, suppose we apply forces at nodes such that they produce u 1 is equal to 1 u 2 is equal one.

You have the axial deformation of bar, the equations system is there with you that is E A over L 1 minus 1 minus 11 times u 1, u 2 vector is equal to q L over 2 plus F 1, F 2 vector. You have this equation system with you, so in that equation system lets denote the force at node 1 as k 1 1 - Force at node 1, for unit displacement at node 1 and 0, at the other and at the other - node 2 by k 2 1 - force at node 2 for unit displacement at node 1 and 0 at the other. So, what is basically done is in the equation system that you already derive u 1 is taken to be 1 u 2 is taken to be 0 and instead of F 1 F 2 ignoring distributed load the force vector consists of only F 1 F2. F 1 is denoted as k 11 F 2 is denoted as k 2 1.

So, this is the equation system you have and now you multiply on the left hand side. If you multiply, if you carry out multiplication on the left hand side that is, E A over L 1 minus 1 minus 1 1 times vector 1 0. It turns out that k 1 1 is equal to E A over l, k 2 1 is equal to minus E A over L. If you see k 1 1 value, k 2 1 value that is nothing but first column of k matrix.

So, now what I will do is I will take the same equation system except that, u 1 will substitute u 1 is equal to 0, u 2 is equal to 1 and let say corresponding forces let them be denoted using k 2 1 and k 2 2.

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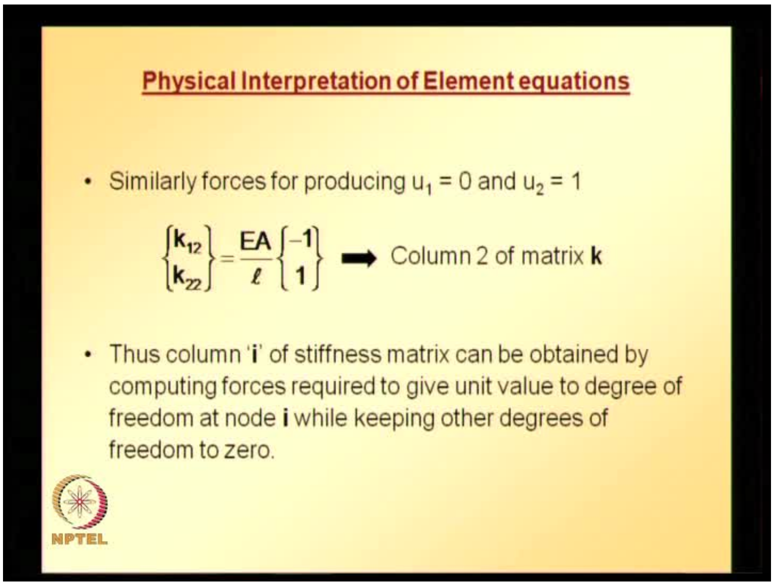

Similarly, forces for producing u 1 is equal to 0, u 2 is equal to 1 and if you carry out the multiplication, it turns out that k 1 2 k 2 2 are nothing but whatever values you get are nothing but second column of matrix k. So, what we can learn from here, column i of stiffness matrix can be obtained by computing forces required to give unit value to degree of freedom at node i while keeping other degrees of freedom is equal to 0.

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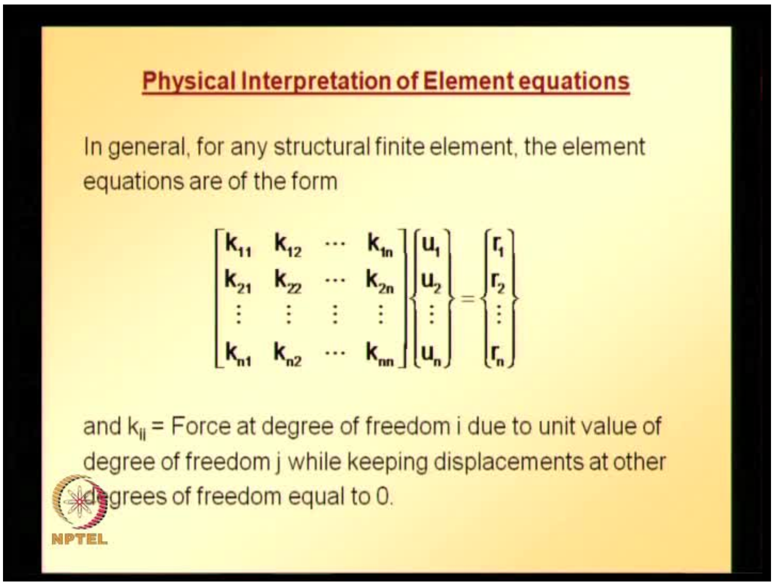

So, and we can generalize this to A N by N equation system where, i j component of stiffness matrix is nothing but force at degree of freedom i due to unit value of degree of freedom j or unit value of displacement at j while keeping other displacements equal to 0. So, that is what each component of stiffness matrix is.

So, this is a physical interpretation of element equations and now, let me ask you we have taken a bar under axial deformation and here we have derived the element equations, but if I replace that bar with a spring having spring constant, how the equation system looks like.

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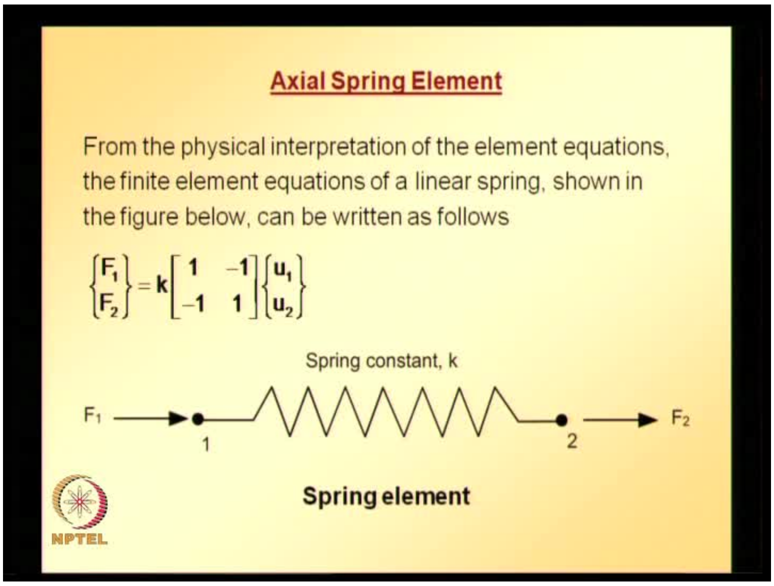

Please note that, if you see the equation system of the elemental equation system for a bar under axial deformation that we just derived, k is E A over L, stiffness matrix is E A over L 1 minus 1 minus 1 1. In that E A over L is what is the stiffness is like a stiffness of a spring, stiff stiffness constant of a spring.

So, what we can do is if you want finite element equation of a spring we can easily write it as k times 1 minus 1 minus 1 1 times u 1, u 2 vector is equal to F 1, F 2 and how the spring looks this is the spring nodes 1 and 2 are defined and spring constants we need to know spring constant. So, if you want to replace a bar element with a spring element what is you need to do you need to find a spring, having spring constant is equal to E times A over L of bar element so this is a spring element.

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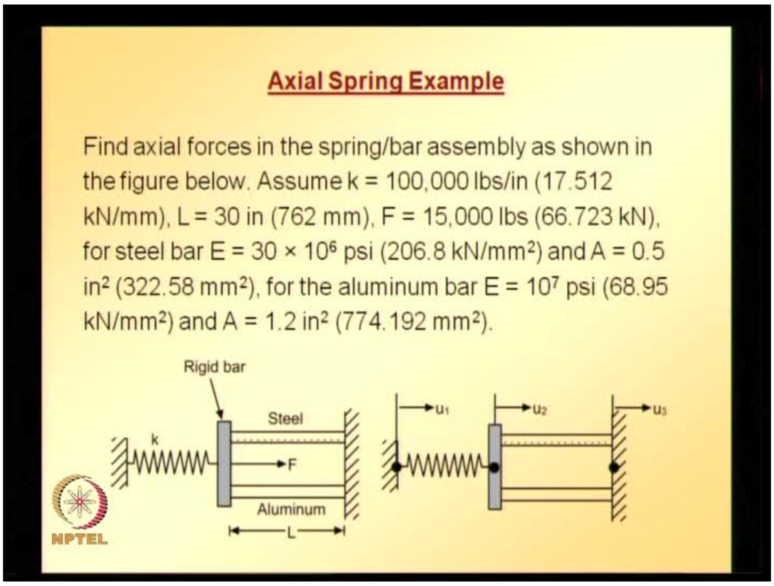

So, now you learnt what is a bar element and what is a spring element, now let us see this example. Find axial forces in spring bar assembly shown in the figure and here units are given in FPS units and corresponding S I units are given in the brackets and since, FPS units are looking in a more without any decimals much decimal points.

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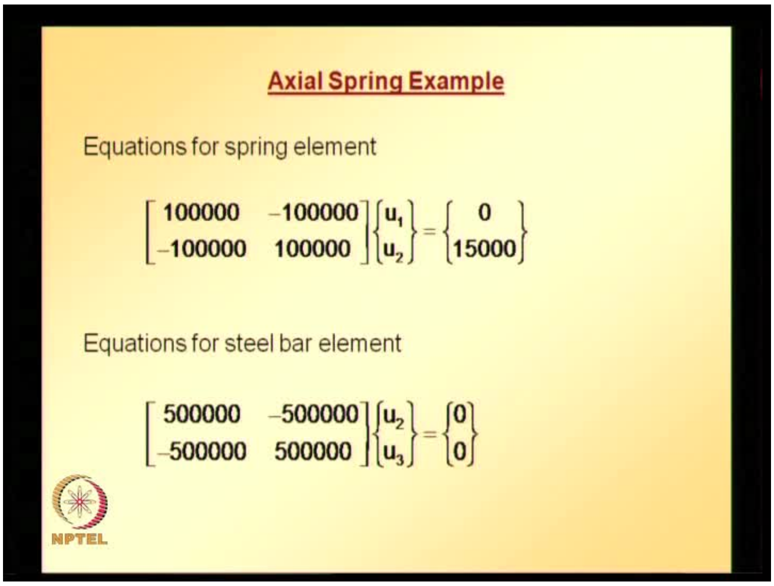

So, decimal places will work in FPS units. So, all the values, I have given here spring constant, length of the bar elements, the steel and aluminum and force that is applied Young's modulus cross sectional areas all are given here and a figure is also shown with all the degrees of freedom u 1, u 2, u 3. Please note that u 1 and u 3 are fixed that is, u 1 u 3 values will be equal to 0 they are the essential boundary condition and the natural boundary condition is force F applied on the rigid bar is equal to 15000 pounds and here the force F is applied in the middle of the rigid bar, when you are assembling the elemental equation system, this F when you are assembling this spring element, you can actually put it for contribution for spring element or steel or aluminum bar, but it is acting at this node 2. So,k that is what you are going to see here, equations for spring element spring element stiffness matrix is k times 1 minus 1 minus 1 1.

So, you find what is the spring a constant for spring element substitute that you get this equation system here, if you see spring element at node 2 this force 15000 pounds is acting that is why this is shown as a contribution to the spring element and for steel bar element equation system, you know what is Young's modulus of steel, it is given area of cross section of steel bar, length of steel bar.

So, using that information you can find stiffness matrix and the force here, this is what I mentioned the force is actually the force contribution is added to the spring element that is why it is not shown here for steel bar element.

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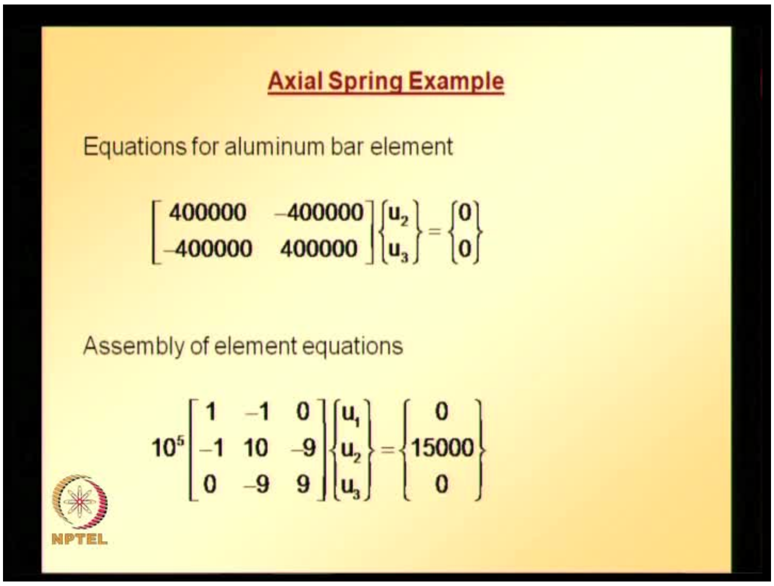

Similarly, equations for aluminum bar element can also be assembled; taking the value of cross sectional area of aluminum bar, Young's modulus of aluminum bar and length of aluminum bar.

So, now you have the elemental equations for all the three kinds of components here. So, now, we are ready to assemble the global equation system, spring contribution goes into rows 1 and 2 rows and columns 1 and 2; steel bar element contribution goes into the location 2 and 3 and aluminum bar contribution also go into location 2 and 3 and the force is applied at node 2.

So, that is what is shown there and u 1 is equal to 0 u 2, u 1 is equal to 0 and u 3 is equal to 0. So, what you can do is you can eliminate rows and columns corresponding to the degrees of freedom which are equal to 0 that is, eliminate rows and columns 1 and 3.

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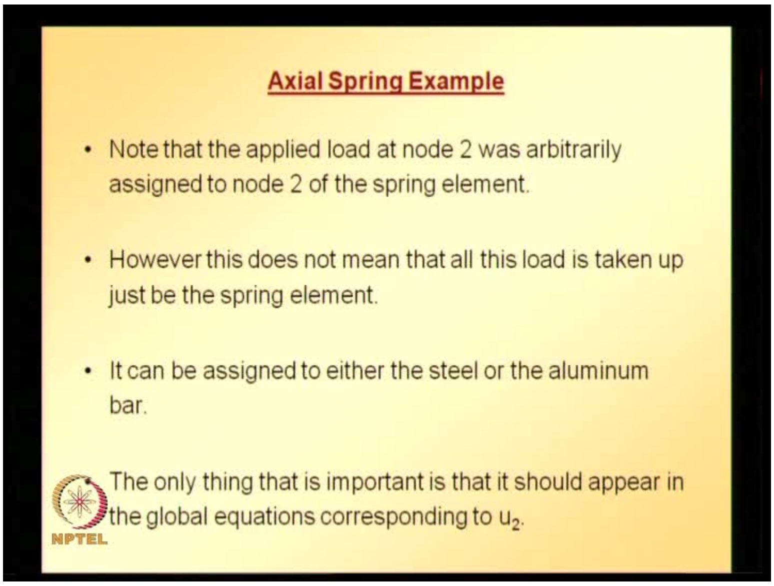

So, you will be left with 1 equation, 1 unknown and you can solve for u 2 and while we did all this what we did is, note that applied load at node 2 was arbitrarily assigned to node two of spring element this is what I mentioned however, this does not mean that all this load is taken up by spring element.

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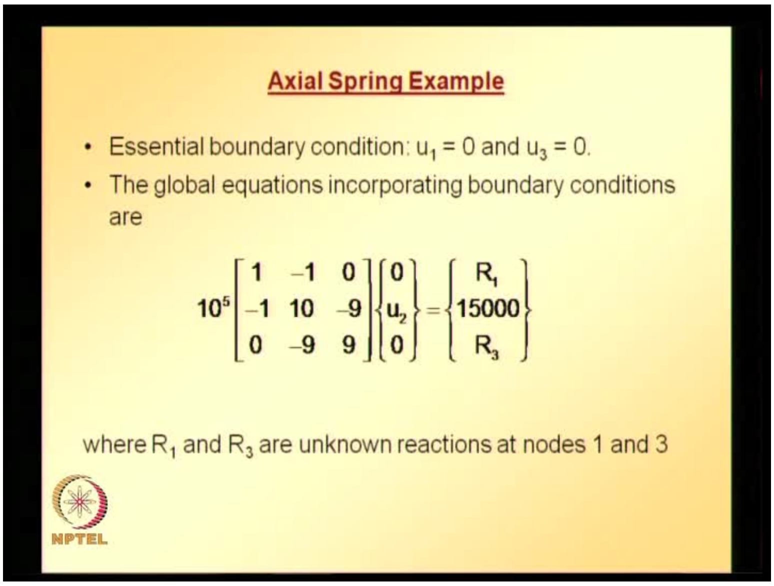

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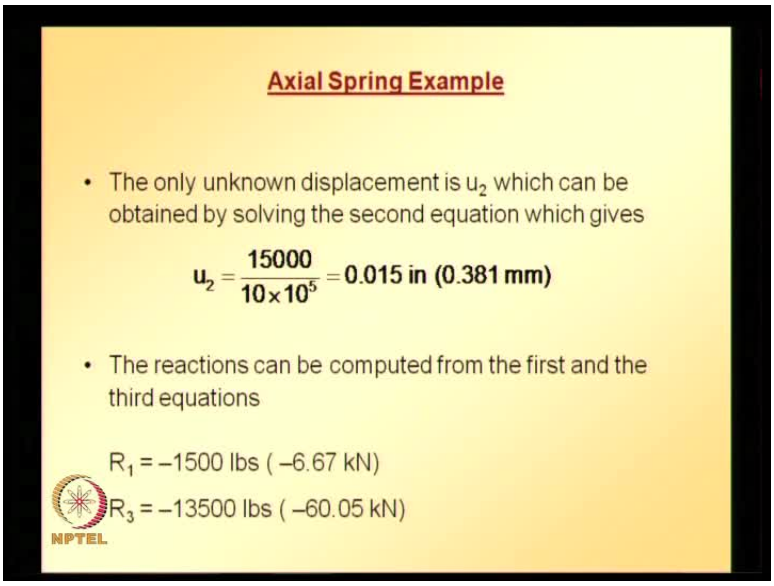

It can be assigned to either steel or aluminum bar the only thing that is important is it should appear in the global equations corresponding to u 2, so, this is what we have done. So applying the essential boundary conditions u 1 is equal to 0, u 3 is equal to 0 the global equations incorporating these boundary conditions becomes this and reduced equation system. You will have only 1 unknown u 2 which can be obtained solving the second equation which gives u 2 is equal to this. And now you know what is u 2, you can calculate the reactions R 1 and R 3.

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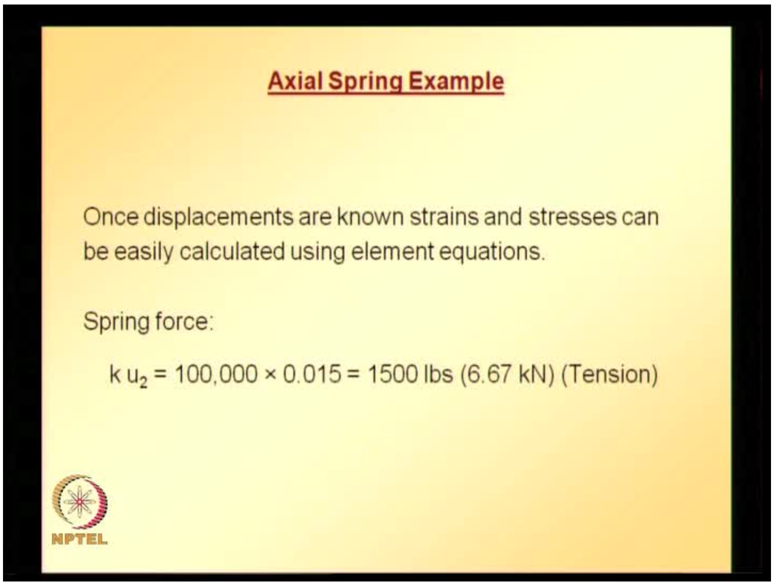

Please note that if you sum up all the forces that is, force applied 15000 pounds and plus R 1 plus R 3 should be equal to 0 for force equilibrium. So, now you got all the nodal displacement values you can calculate the forces. Once displacements are known strains and stresses can be calculated using element equations. First one is spring force, spring force is what u 1 is already 0, u 2 is what you just found so spring force is going to be stiffness times u 2, it gives you the spring force both F PS units corresponding SI units are shown and also as expected, it turns out this value is positive means tension, negative means compression that is our sign convention.

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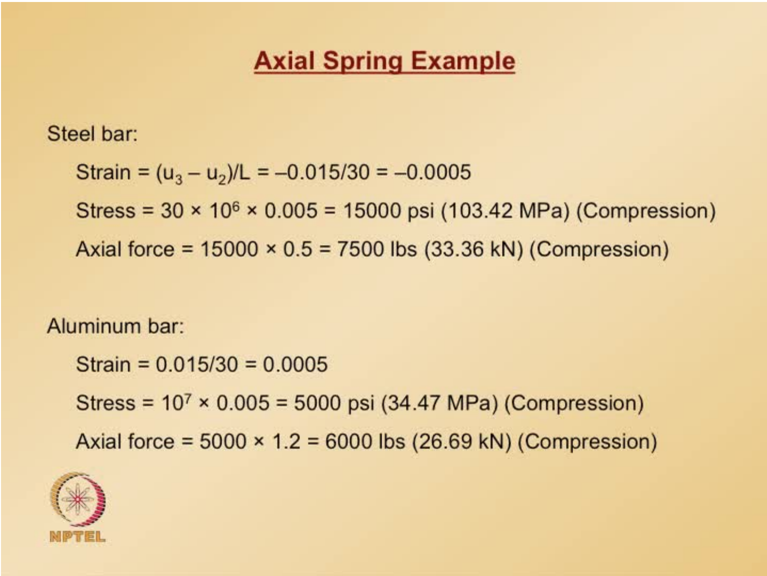

So positive means tension, so it actually make sense because the force is trying to pull the spring and what is the strain in steel bar, stress in steel bar and here strain is negative means stress whatever value is calculated that is compression axial force is compression.

Similarly aluminum bar, aluminum bar strain, stress here negative sign is missing and this is expected that is steel bar and aluminum bar will be in compression and spring will be in tension, because force is acting from the left side towards right side. Since, spring is towards right and the aluminum and steel bars are towards left, aluminum bar and steel bar will be in compression whereas, spring will be in tension. We will continue in the next class.