Finite Element Analysis

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Lecture No. # 05

We have seen in the last lecture, one approach of solving a problem using finite element technique that is long and approach, and which is almost similar to the approach that the technique that we adopted for classical approximation techniques. So, here the calculations are organized in way that it is made possible to obtain finite element solution in a manner similar to the classical approximate solution techniques. However, in anticipation of more complex in 3 dimensional problems it is necessary to integrals a more formal terminology, so that simple notation can be employed. And in the following we consider the same boundary value problem same boundary value as before, but this time using more general finite element solution procedure.

And this solution can be obtained using either the initial portion, that is you if you are adopting a Rayleigh-Ritz method or a variation method you need to get the equivalent functional, and if it is a weighted residual methods you need to apply the integration by parts and reduce the order of integration, order of differentiation, such kind of things are same so you can either use a weighted residual method or rely rage method in conjunction with this finite element method. So, now let us look at what are the key differences are not key differences sorry what are the general steps in finite element solution procedure.

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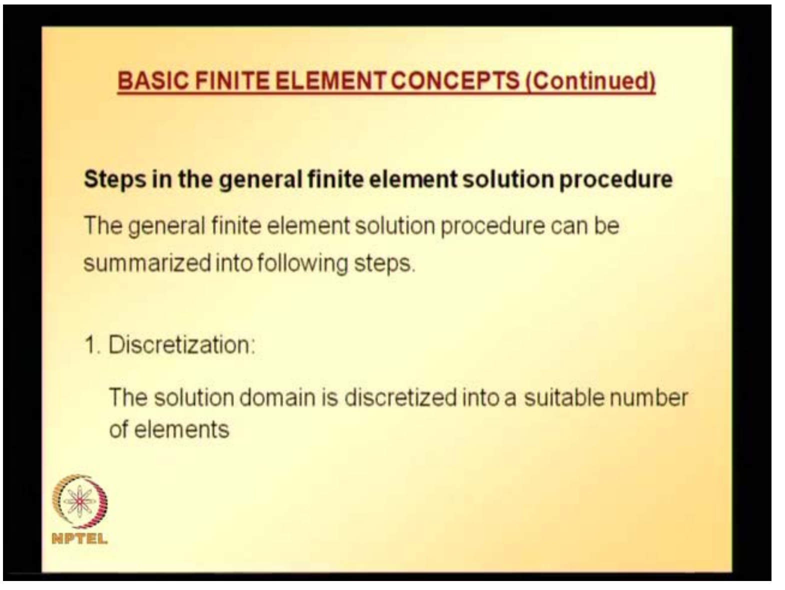

So, the first step is discritization solution domain is discretized into suitable number of elements,

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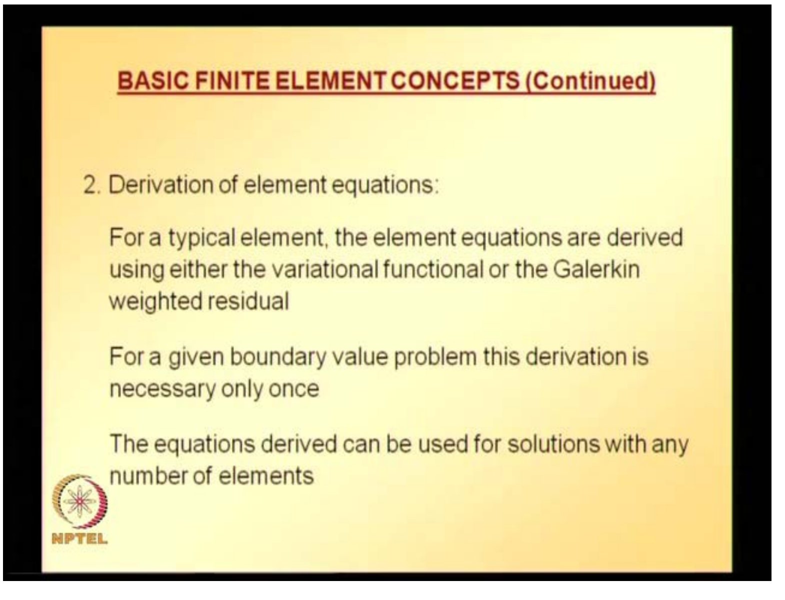

and derivation of element equations, we will do it for a typical element, element equations are derived again using either variational functional, or Galerkin weighted residual.

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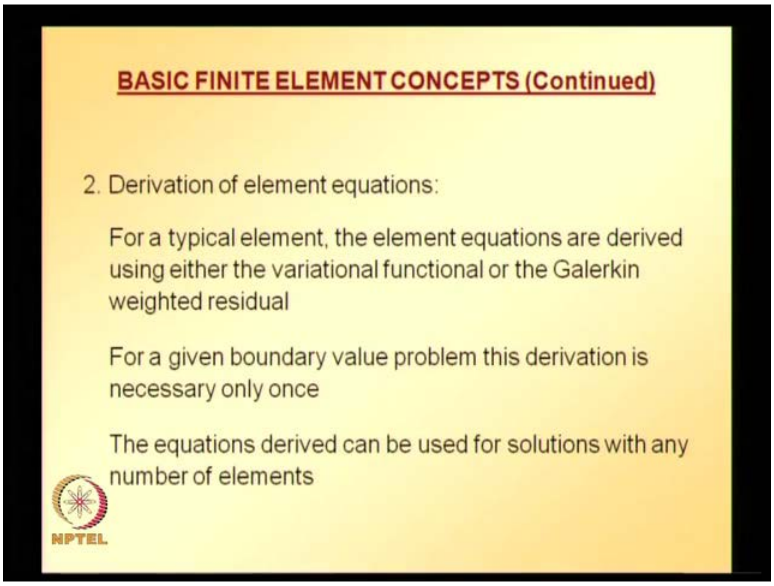

For a given boundary value problem this derivation is necessary only once, because you will see in a while, we if you do for a typical element that is enough, solution derived can be used for solution with any number of elements, whereas, this kind of thing is not there in the earlier the procedure that we followed, what we did in earlier procedure in the previous example, where I showed demonstrated or illustrated using finite element technique there we derived equations separately for each of the elements, element 1 and element 2, whereas, here we are going to derive for typical element, the equations derived for this typical element can be used for solutions with any number of elements.

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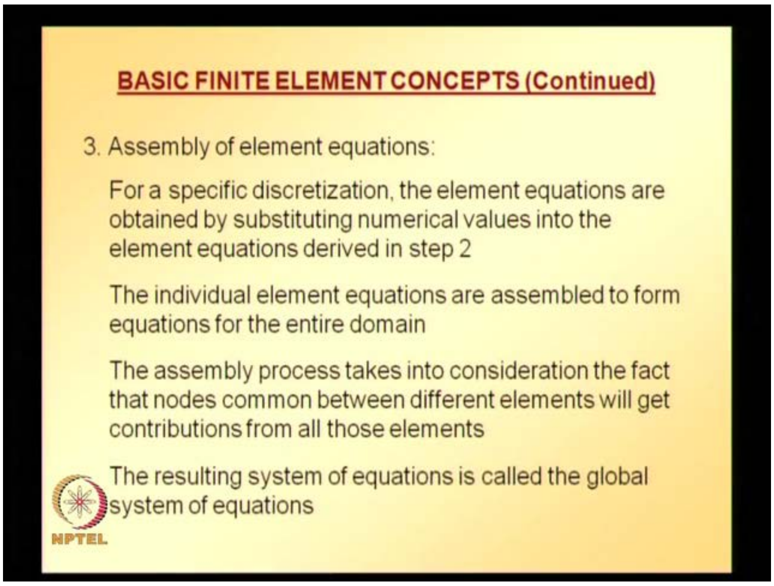

And then this third step is assembly of element equation, for a specific discretization element equations are obtained by substituting the numerical values into element equations derived in step 1. So, element equations will derive for a typical element, and if you want for a specific element what you need to do is you substitute the nodal values and the length of that particular are the geometrical properties of that particular element into the typical element equations and you get for that specific element. Individual element equations are assembled to form equations for the entire solution domain, and this assembly process and other things will be clearer to you once we solve a problem in a while, assembly process takes into consideration the fact that nodes common between different elements get contribution from all those elements, suppose a node is common to 3 elements so the contribution to that particular node in the global equation system comes from all the 3 elements.

If a node is common to 2 elements, the contribution to that particular node comes from the 2 elements. Resulting system of equations is called global system of equations.

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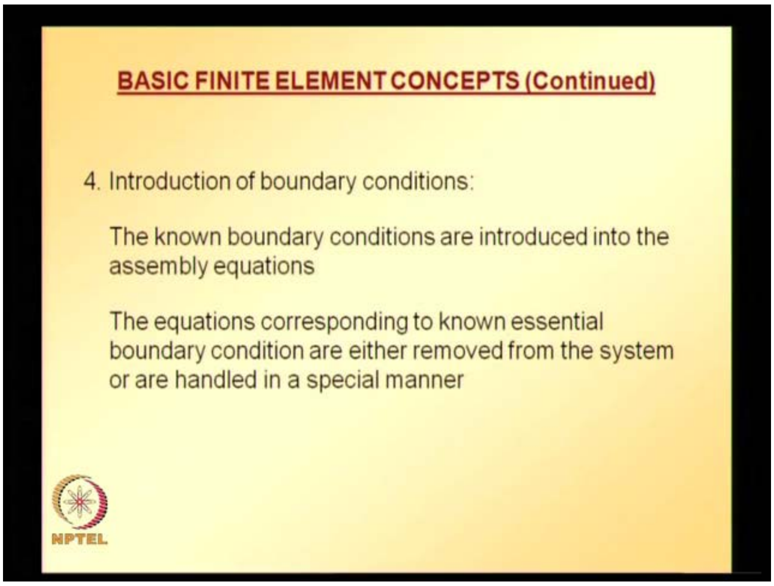

And in as I mentioned we are going to look at formal procedure of finite element technique, in this what we will be doing is after assembling the global equation system we are going to introduce the boundary condition, the known boundary conditions are introduced into the assembly equations. Equations corresponding to the essential boundary conditions are either removed from the system or handled in a special manner.

This point it will be clearer in a while, when we are solving this global equation system when we apply the boundary conditions we will be using partition of matrices to solve this global equation system, in that process we may remove some of the equations or we will handle them in a special manner which will be clear in an while.

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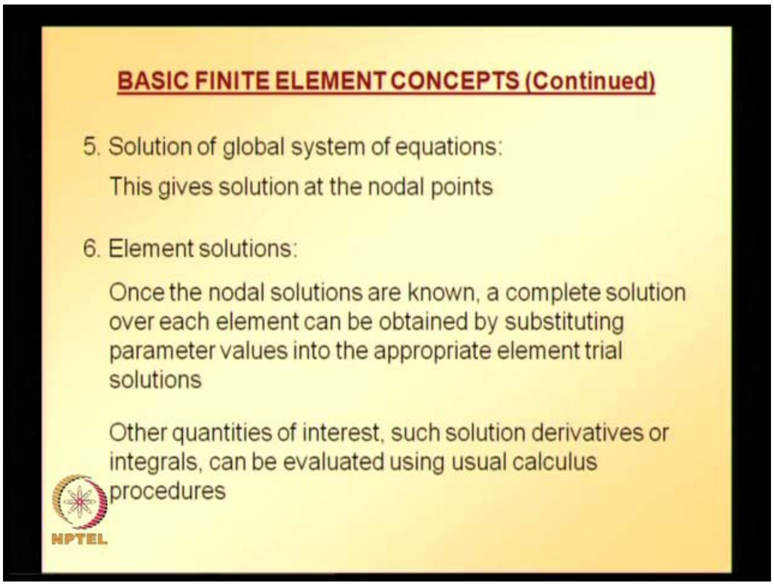

And once we have the global equation system after applying the boundary conditions, we can solve for the nodal values or the solution at nodal points. And a rest is similar to what you already experienced once you get the nodal values you go to each element, and the solution over each element can be obtained by substituting in the corresponding for that particular element may be all the nodal values may not be appropriate, The whatever values that are appropriate for that particular substitute into the trial solution for that particular element and you get the approximate solution and also the derivative of approximate solution. These points will be made clear to you other quantities of interest such as solution derivatives or integrals can be evaluated using usual calculus procedure which forms part of post processing. So, all these points will be clearer to you once you see this example.

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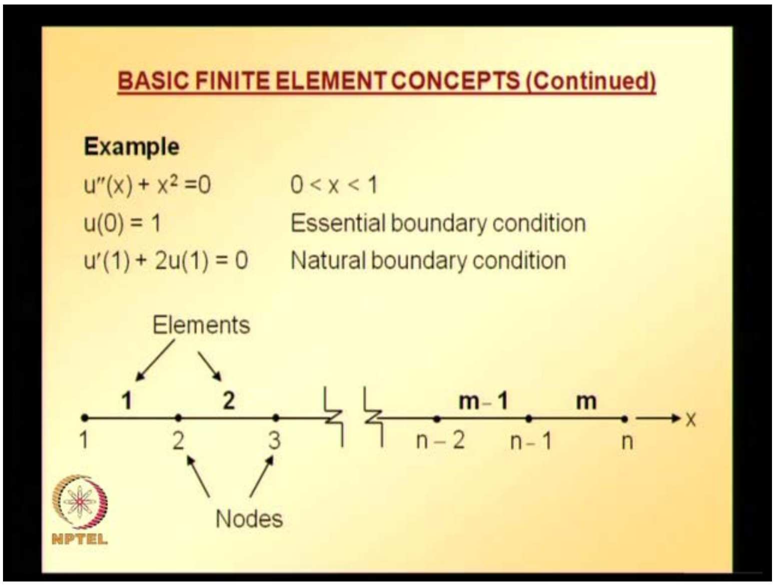

Is second order differential equation this problem we have solved earlier, you require 2 boundary conditions, the 2 boundary conditions given 1 is essential another is natural boundary condition, and we solve this problem using finite element method what we need to do is we need to discretize the problem domain, the problem domain is 0 to 1, 0 to 1 problem domain is divided into m number of elements and n number of nodes, and a typical element looks like this.

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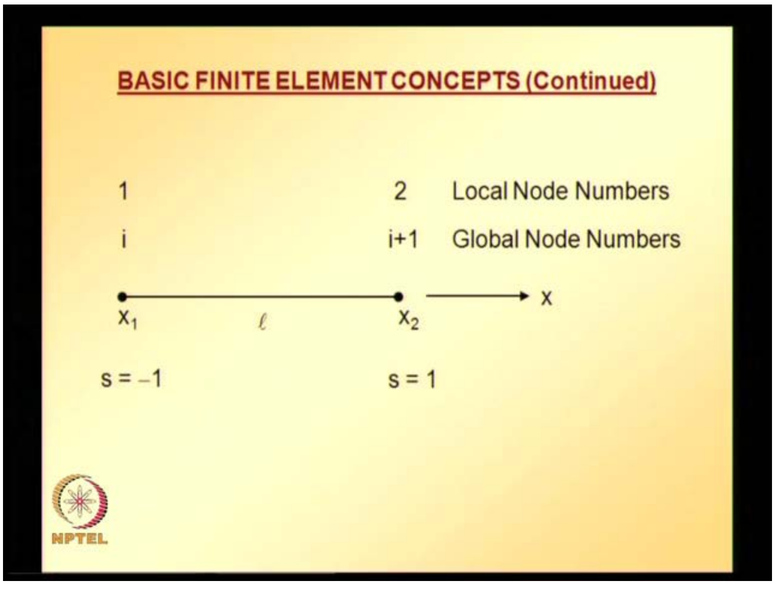
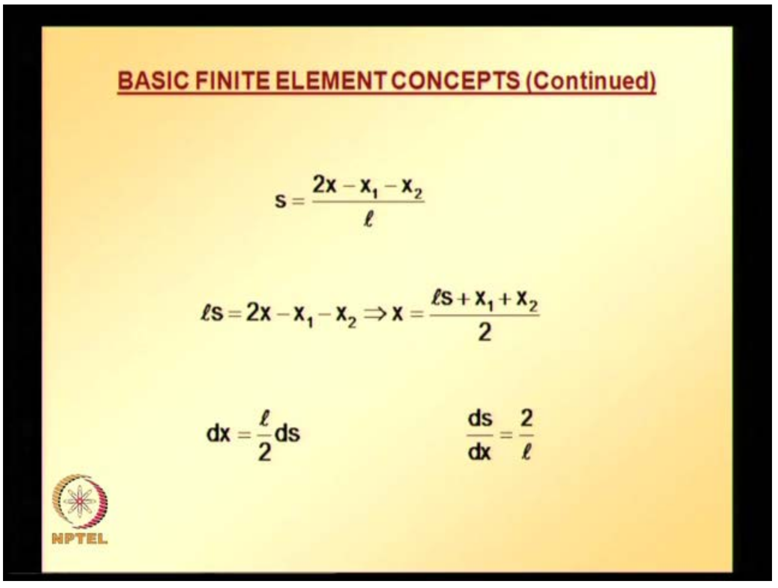


Here both global node numbers and local node numbers are indicated, for a particular element you may have at global level some particular node number, but if you take that element locally it may have node 1 and node 2. So, global node number I is denoted with I and I plus 1, local node number is denoted with 1 and two. And the first node spatial coordinate is x 1, second node spatial coordinate is x 2, length of this typical element is l, and here I am going to introduce you to a new concept that is, it is to later when we are actually assembling the stiffness matrix or the force vector we will be using numerical integrations, at this stage we will be using Gaussian quadrature, and the integration becomes simpler if your element ranges from minus 1 to one.

So, what I will introduce you is that, I will map this x 1 value to s is equal to minus, x 2 to s is equal to 1. So, this mapping is what is the advantage is it is makes integration simpler since the limits becomes from minus 1 to 1 instead of x 2 to 1. And what is the relation between this s coordinate system and x coordinate system that you can easily find using the formula for linear interpolation which is nothing, but equation of a straight line which you must already studied at your high school level. (Refer Slide Time: 10:16)



So, the relation looks like this, this is simple linear interpolation between x coordinate and s coordinate. And if you substitute in this equation if you substitute in this equation x is equal to x 1, x is equal to x 1 if you substitute you will get s is equal to minus 1, and x is equal to x 2 if you substitute you get s is equal to 1, that is 1 way of checking whether this relation is correct or not, because we are mapping x 1 to s is equal to minus 1, x 2 to s is equal to 1, and this equation can be rewritten in this manner, and which can be rearranged such a way that x is expressed into so s, and taking derivative of this on both sides results in d x is equal to l over 2 times d s, which can be rearranged in the manner d s over d x is equal to 2 over l.

The inverse relation that is d x over d s derivative of x with respect to s is called Jacobean, Jacobean has a physical meaning for a 2 node element like this, it is going to be half the length.

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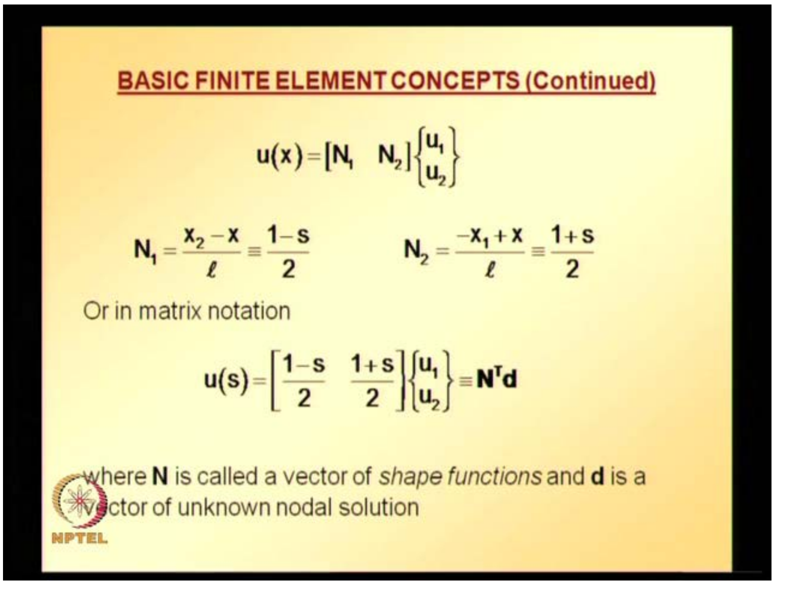

And now the trial solution N 1 u 1 plus n 2 u 2 can be put in this matrix and vector form like this, you know from previous lecture you know N 1 is nothing but u 2 minus x over l for a 2 node element, now we want to replace this x coordinate system with s coordinate system where s goes from minus 1 to 1, and we already decided or we already have an understanding that x 2 corresponds to s is equal to 1, and x say in the s coordinate system a x corresponds to s, so N 1 can be rewritten in this manner, x 2 you replace it with 1, x you replace with s, and length of this element is goes it is going from minus 1 to 1 in the s coordinate system so length is equal to 2.

Similarly, you can write the expression for n 2 in terms of s coordinate system, again you can crosscheck whatever that you learnt earlier, that is N 1 plus n 2 is equal to 1 and derivative of N 1 with respect to s, if you add with to the derivative of n 2 with respect to s it is going to be equal to 0. So, some of all shape function should be equal to 1, derivative of sum some of the derivative should be equal to 0. So, in a matrix notation substituting N 1 and n 2 values, we can write in this manner, and in a compact notation this vector 1 minus s over 2, 1 plus s over 2 is denoted using n transpose, that is n transpose comprises of N 1 n 2, and d vector is nothing but, all the nodal values u 1 u 2, n is called vector of shape functions, d is vector of unknown nodal solution, and this kind of notation is it is all individual thing, so if you see different text books you will have different kind of notations. So, whenever you are trying to read a text book a particular text book, you better make sure that are you understand that notation that is been followed in the text book clearly before you sometimes instead of n transpose in some text book you may have n, so do not confuse between n transpose and n. So, you need to look at the notation carefully.

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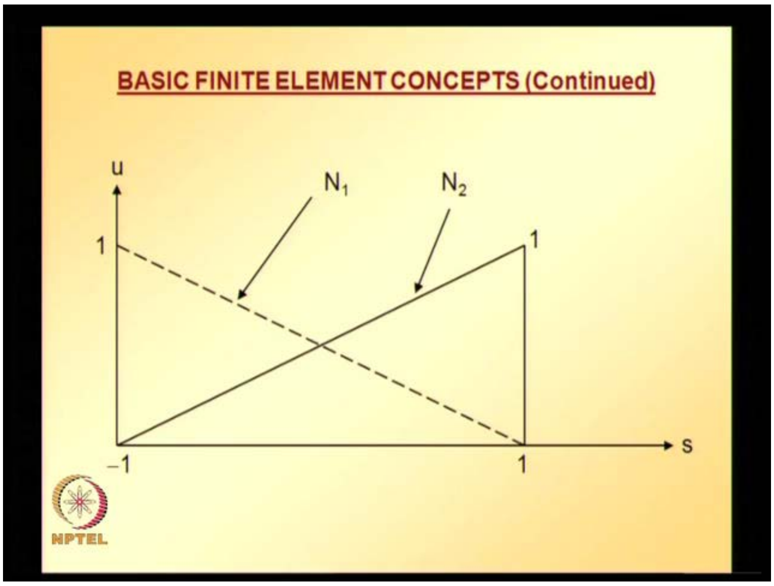

And this N 1 n 2 if you plot on the s coordinate system, the variation of N 1 n 2 looks like this, N 1 is going to have a value equal to 1, at its own position that is N 1 is going to be 1 at s is equal to minus 1, N 2 is going to be 1 at s is equal to 1, and N 1 will be equal to 0 at s is equal to 1, N 2 will be equal to 0 at s is equal to minus 1, and this is what is called Corniceer delta property. That is N 1 evaluated at N 2 location is equal to 0, N 2 evaluated at the node corresponding to N 1 is equal to 0.

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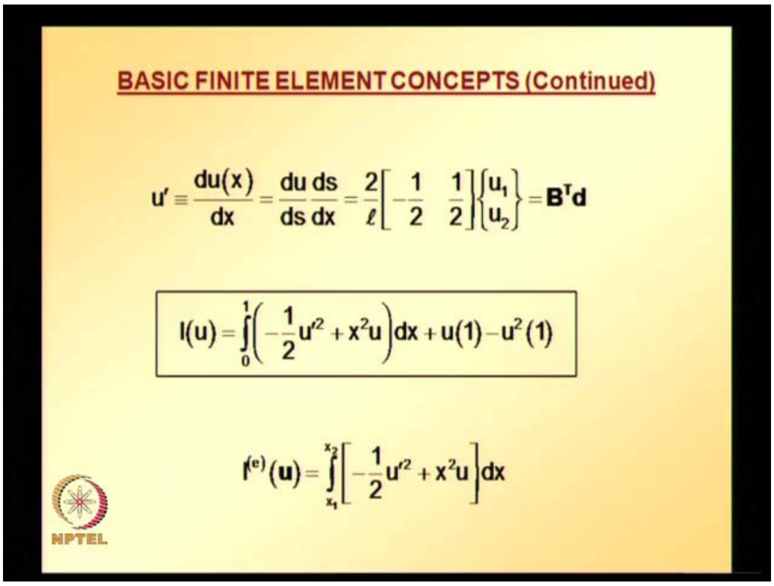

And now if you look at our the equivalent functional, or the equation that we get using weighted residual method we may require this kind of quantity derivative of u with respect to x, derivative of u with respect to x can be in a using chain rule we can write it has derivative of u with respect to s, derivative of s with respect to x, and you just learn derivative of s with respect to x is nothing but 2 over l. And derivative of u with respect to s is nothing but shape function matrix derivative times the nodal values vector that is you need to take derivative of n transpose times d that gives you derivative of u with respect to s. So, plug in all this information you get 2 over l times a vector consisting of minus half half times, a vector consisting of u 1 u 2, and this u 1 u 2 is denoted with d and 2 times minus half half is denoted with b transpose, that b transpose if you see, it relates strain like quantity, derivative of u is strain like quantity on 1 side, the other side you have d, d is nodal parameter.

So, it relates strain like quantity to the nodal values it is called strain displacement relationship, this equation is called strain displacement relation. b relates strain like quantity, I cannot say whether u is strain or not because still we do not know what is the what physical quantity u represents, but if at all if it represents displacement then derivative of u with respect x represents stress, sorry strain, so strain like quantity is related to displacement via b matrix. And now you can plug in all this information, you can plug in all this information into the equivalent functional, and if you look at equivalent functional this is how it looks, what we will do is as I mentioned a in the points that I shown you related to finite element method.

We will develop equation system for a typical element, so what we will do is in this equivalent functional let us assume an element, if you see the last 2 terms they corresponds to x is equal to 1, let us take an element which is for which x is equal to 1 is not a part of that element, so for that particular element you do not have the last 2 terms in this functional equivalent functional, that is u evaluated at x is equal to 1, u evaluate u square evaluated at x is equal to 1, they form a part of that particular element which has 1 of the nodes at x is equal to 1. So, now let us take an element which is not having that, so that is denoted with this one, u superscript e some element understand typical element in which the boundary terms are not part of that element, so for that particular elements the previous equation can be written in this manner.

So, now what we will do is, if you see this equation you have u prime and u, and now you can substitute u prime in terms of b transpose d, and also u in terms of n transpose d, and we can simplify these terms, first let us see what is u prime square before we plug in u prime square value we need to know want is u prime square, please remember u prime is a scalar quantity, u prime is a scalar quantity scalar means it has only a single value single component, if you take a vector it has 2 components, whereas, scalar has only a single component, so u prime square can be written as for a… if you have a scalar u prime square can be written as u prime transpose u prime, you will understand why I am writing like that, u prime square I will write it as u prime transpose u prime, whereas, u prime transpose is what? b transpose d so that is what is written here.

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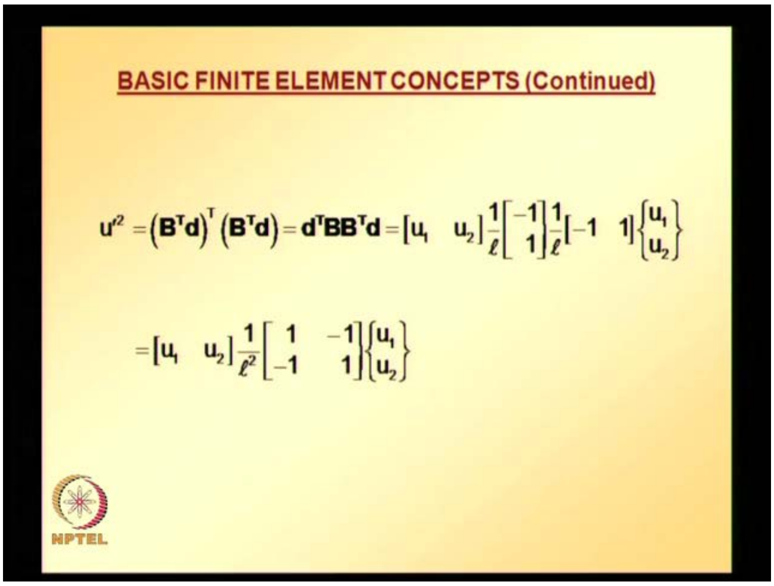

u prime square is written as u prime transpose u prime, u prime is b transpose d, so b transpose d transpose b transpose d, and if you apply your matrix knowledge you can simplify it into d transpose b, b transpose d. And you know what is d, and you know what is b, you plug in all those things you get this one. And you can do all the matrix and vector multiplications and you can compactly write it.

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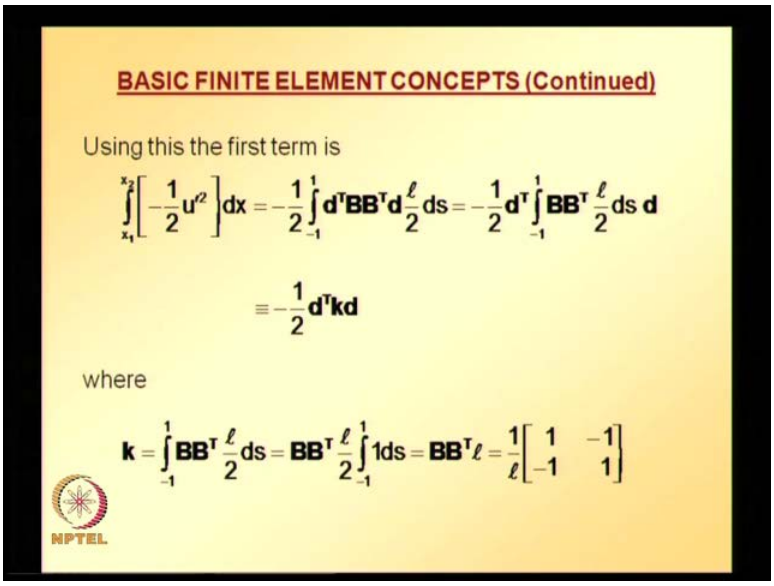

Now, you go to the first term, in the first term you substitute what is u prime square, and then please note that d consists of nodal parameters or nodal values, so it is not a function of s so you can pull that out of the integral. d is not function of x, s it is you can pull out of the integral, and whatever is there in the integral… and please here one more thing I want to point out d s and d x, if you see the first term limits of integration are from x is equal to 1, limits of integration are from x 1 to x 2, they are replaced in the second term from minus 1 to 1, because in the s coordinate system x 1 corresponds to s is equal to minus 1, x 2 corresponds to s is equal to1, and d x is replaced with you know the relation between d x and d s, d x is replaced with l over 2 d s, so that is you obtain the second term, and the third 1 by pulling out the d which is not function of s out of the integral you get a third one; third term.

And this can be compactly written like this, where k is defined as integral minus 1 to 1, b transpose b, l over 2 d s, and if you see b, b is also not a function of s for this particular element, for a 2 node element b is not a function of s, for this particular element b is not a function of s, so for a 2 node element so you can pull b, b transpose out of the integral and this entire quantity simplifies to 1 over l 1 minus 1 minus 1 1, and if you check this matrix; k matrix, it is non invertible means it is a singular matrix, determinant of this is equal to 0, What is determinant? a d minus b c if you have a matrix a, b, c, d, a d minus b c is… you see a times d is 1 b times c is 1, 1 minus 1 is 0, determinant of this matrix is 0, so we will see the significance of this later.

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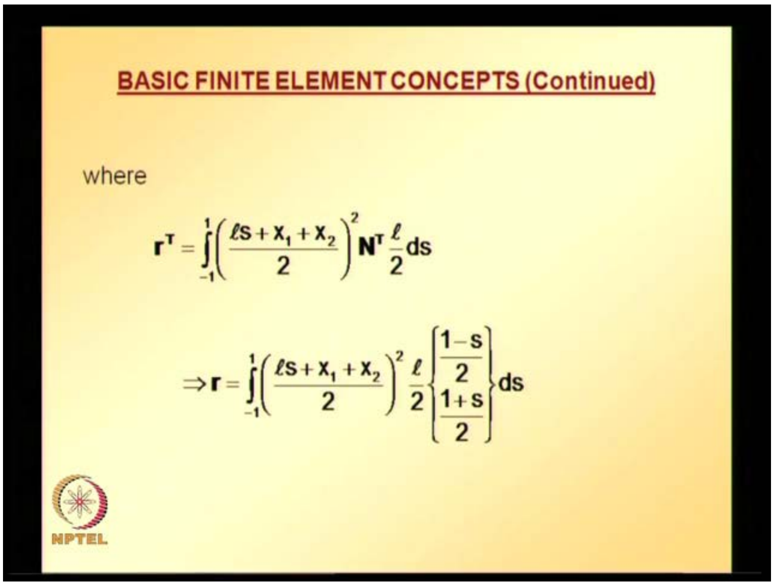

And the second term, we need to substitute what is u value, and substituting x in terms of s and u in terms of n transpose d and d x is replaced with l over 2 d s and the limits of integration are changed from minus 1 to1, x 1 is replaced with minus 1, and x 2 is replaced with 1, and you can simplify these to this one, and again here d is pulled out of the integral, because it is no longer a function of s, d is nothing but nodal parameters or nodal values, so it has nothing to do with the integral you can pull out of the integral, and r transpose is defined whatever is there in the integral, and you can simplify this r further by substituting what is n, n is nothing but vector of shape functions we substitute that, now everything inside the integral is function of s so you can do integration.

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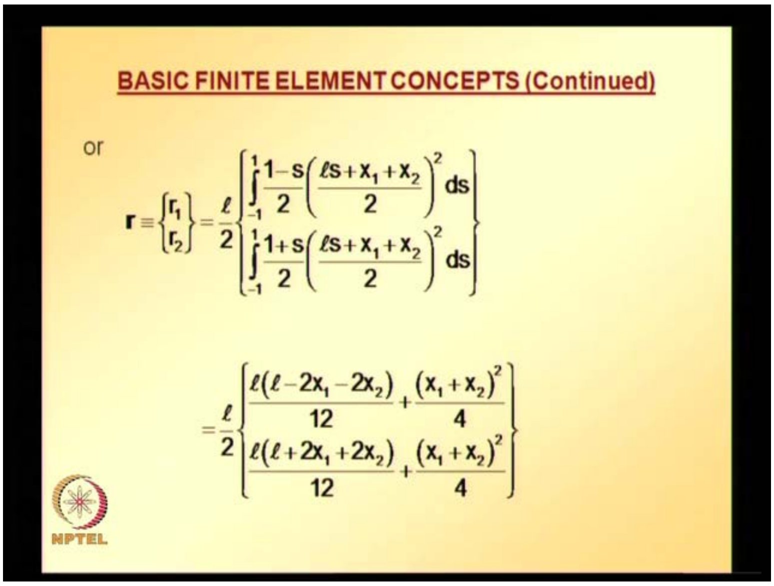

And that result in r is equal to this, so you can easily assemble r once you know the nodal coordinates of a particular element, you plug in local node 1 as x 1, local node 2 as x 2, and a length of the element you plug in you can find what r.

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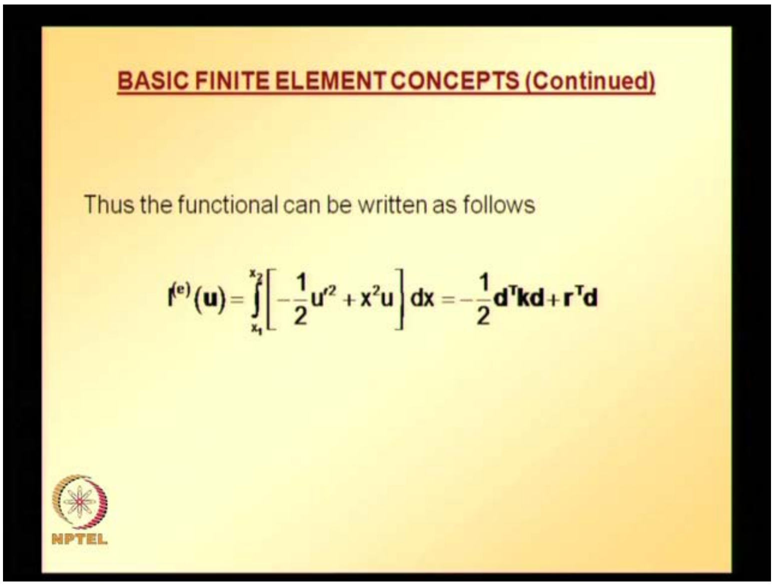

So, you got the first term, you got the second term, you put them together this is a equivalent functional for a typical element for which boundary terms that is u at x is equal to 1 is not a part of it for that kind of particular that that kind of element this is the equivalent functional.

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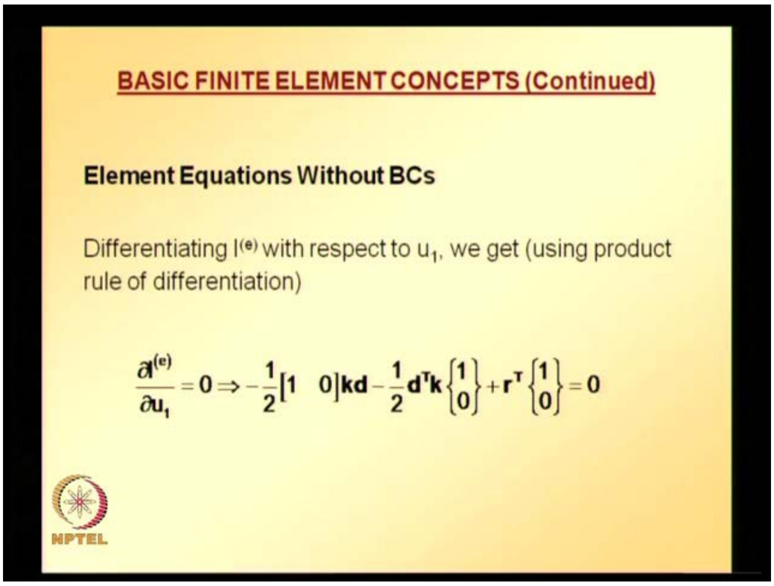

And now you apply the stationarty conditions, element equations without b c's, without boundary conditions. Stationarity condition is what? If you see the previous equivalent functional it is function of d nodal values, d is comprising of u 1 u 2, so apply this stationarity condition that is differentiate I superscript e with respect to u 1, you get 1 equation, and please note here that stiffness matrix is symmetric, if you recall stiffness matrix is 1 over l 1 minus 1 minus 1 1 which is symmetric. So, this equation what I can do is, I can actually last 2 terms I can transpose them once again, and finally this everything is a scalar quantity, so if I transpose them if I transpose a scalar quantity I will get a scalar quantity, so what I will do is I will transpose the last 2 terms, that is half d transpose k times 1, 0 that vector plus r transpose 1 0 that 1, first and second term I will transpose and add first 2 terms.

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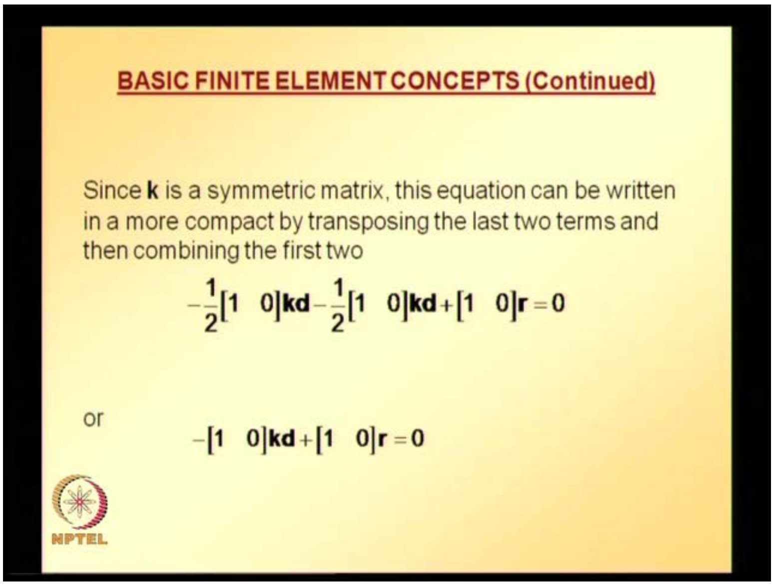

Since k is symmetric this equation can be written in a more compact by transposing last 2 terms, then combining the first 2, this is possible only because k is symmetric matrix. So, then I get this equation and which can be simplified further, because first 2 terms are looking similar, so this is the equation that I got by applying stationarity condition with respect to u 1, that is partial derivative of I with respect to u 1 is equal to 0, applying that i got this equation.

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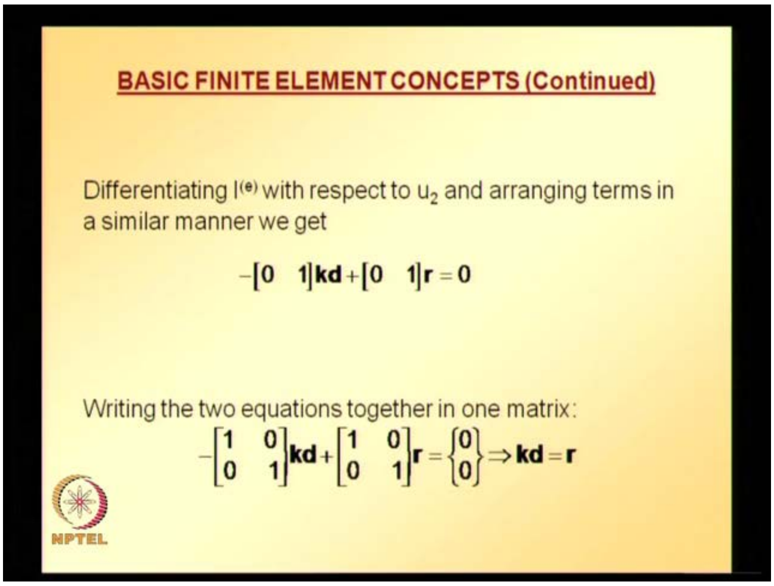

And we need 1 more equation that we get by differentiating i with respect to u 2, we have another nodal parameter is there u 2, and follow the same procedure you get this equation. So, you got 2 equations, you can put these 2 equations together in 1 matrix, and that can be written in a compact manner as k d equal to r because if you see 1 0 0 1 is identity matrix, so anything you multiply with the identity matrix you get the identical quantity, now plugging in what is k what is d what is r?

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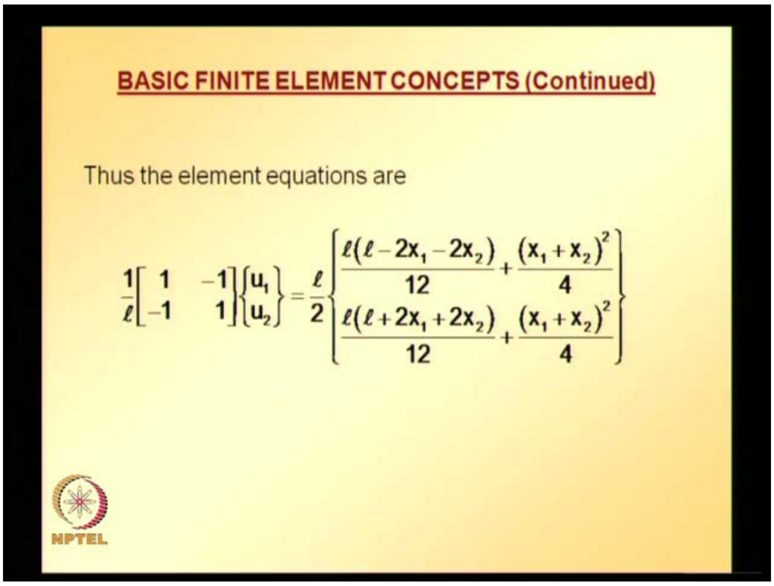

These are the elemental equations or element equations for a typical element for which x is equal to 1 is not a part of that element. That is why in the beginning here we have written element equations for an element without applying the boundary conditions. So, why I am doing this it will be clearer to you in a while, now let us develop equation system for an element for which x is equal 1 is a part of that element.

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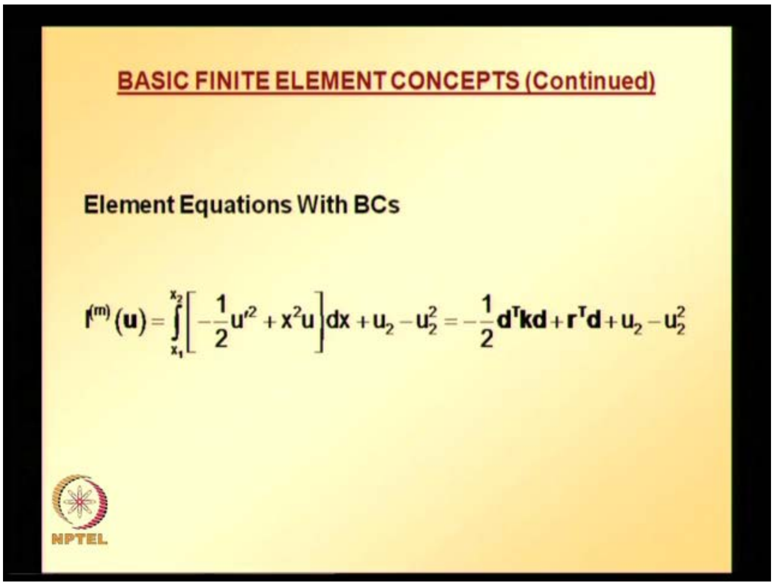

Element equations with boundary conditions, so I am saying let that equivalent a equivalent functional be I superscript m, again same thing this is looking similar except the last 2 terms that is u 2, other than that you have everything same, so now you substitute the first integral whatever the simplified quantities that you already obtained you substitute those here, that is minus half d transpose k d, plus r transpose d that is similar to what you just derived for element equations without boundary conditions, in addition to that you get 2 more additional terms.

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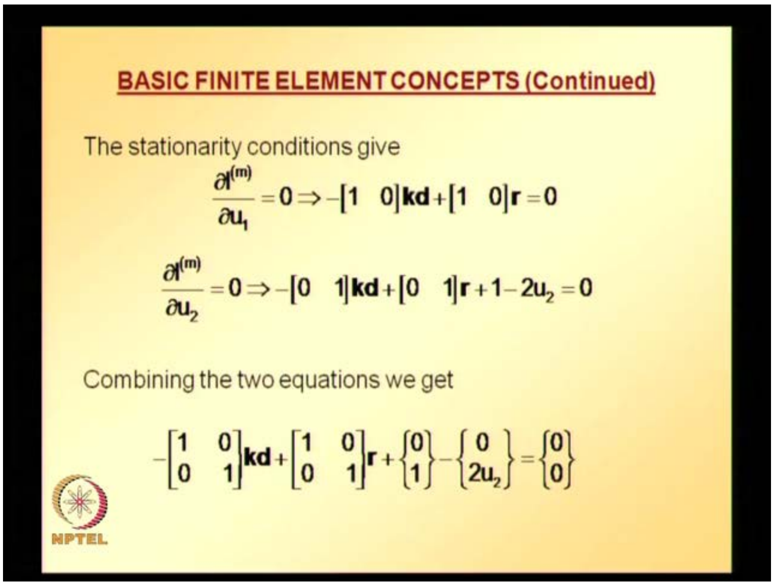

And the rest of the story is same, you need to apply the stationarity conditions, now for this equivalent integral that is partial derivative of I with respect to u is u 1 is equal to 0, you get 1 equation, this is similar to what you already obtained for element equation without boundary condition the first equation you obtained is similar to this one. And the second equation partial derivative of I with respect to u 2 is equal to 0, you get the second equation, again same thing similar to what you had done earlier, you put these 2 equations in a matrix form you get this equation system, and rearrange this 1, because 1 0 0 1 is identity matrix, so whatever I multiply with identity matrix I get same quantity.

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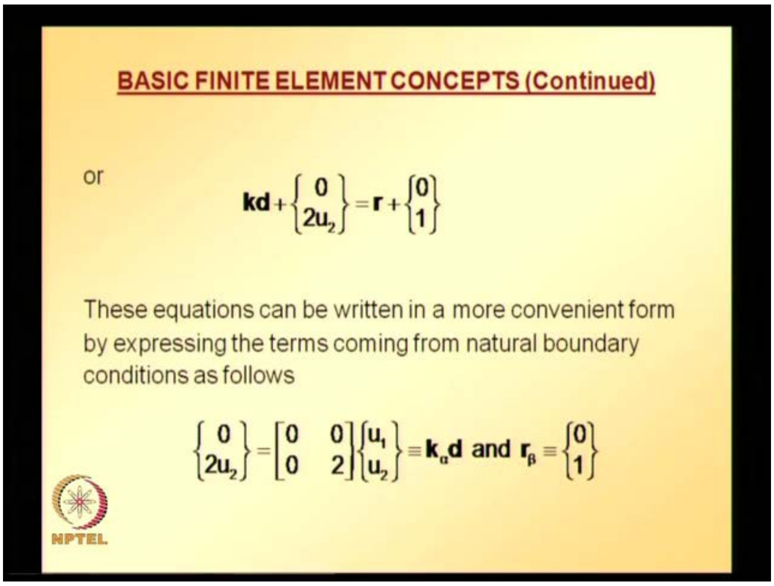

This equation can be rewritten in this manner, and you can define a quantity called r beta consisting of 1 and 0 as a vector, and k alpha times d this 0 2 u 2 can be written as in a matrix and a vector form like 0 0 0 2 times the vector consisting of u 1 u 2, and the matrix consisting of element 0 0 0 2 that is defined as k alpha. So, these equations can be written in a more convenient form by expressing in this manner.

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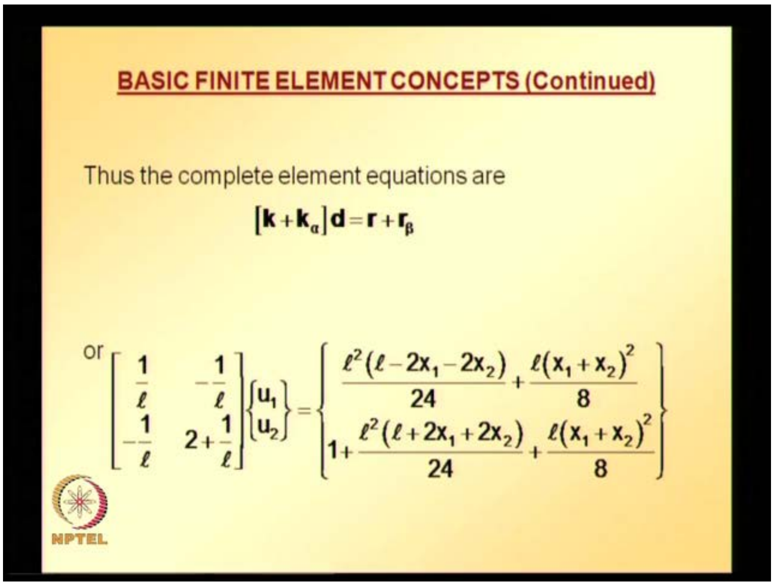

So, the final equation system for element with boundary condition looks like this, now you substitute what is k, k alpha d r, r beta? You get equation system. So, what we did is we develop element equation system with boundary condition without boundary, condition that means we developed a equation system for an element for which x is equal to 1 is part of that element, and x is equal to 1 is not part of that element. These 2 situations will come across in a while please wait.

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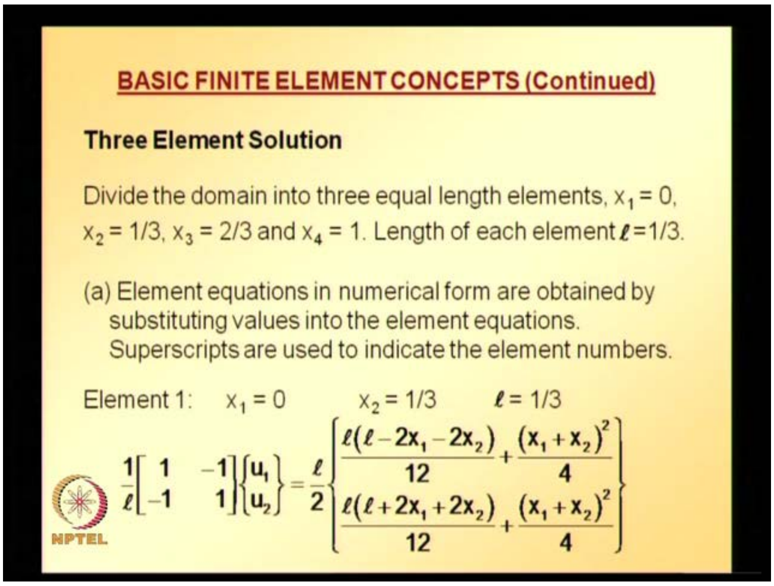

Here 3 element solution, let us look at 3 element solution for this problem means 3 element solution means we are dividing the entire solution domain into 3 elements, and the nodal coordinate of node 1 is 0, node 2 is 1 over 3, node 3 is 2 over 3, node 4 is1, and all elements are of same length equal to 1 over 3. Element equations in numerical form are obtained by substituting the nodal values into element equations that you already just now derived, that is you derived element equations with boundary conditions without boundary condition, so depending on the element locations you can select one of this equation system, and you can plug in the values of the nodal coordinates and length of the element, and then you can get the elemental equations for element 1, 2, 3. So, superscripts are use to indicate the element numbers, for element 1 first node is at x is equal to 0, second node is at x is equal to 1 1 third, third node is sorry a length of this element is 1 third.

And if you see this element 1, x is equal to 1 is not part of this element, so we need to use element equations corresponding to elements the equations that we developed without boundary conditions. We have the equation system without applying the boundary condition element equation system, in that you plug in x 1 is equal to 0, x 2 is equal to 1 third, and l is equal to 1 third you get this equation. So, this is the equation system for elements without boundary condition is reproduce here, in this you can substitute what is x 1 x 2 l.

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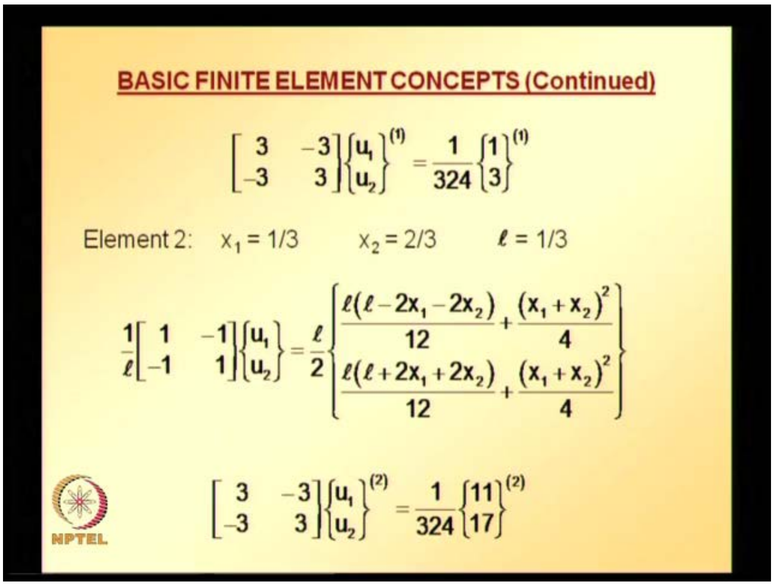

That results in element equations for element 1, superscript indicates that it is for element 1. Similar, exercise you can do for element 2, element 2 is also an element for which x is equal to 1 is not part of it, so you can use same equation system that is element equation without boundary conditions, you plug in x 1, x 2 local x node 1 coordinate, local node 2 coordinate, and length of that particular element you get element equations for element 2.

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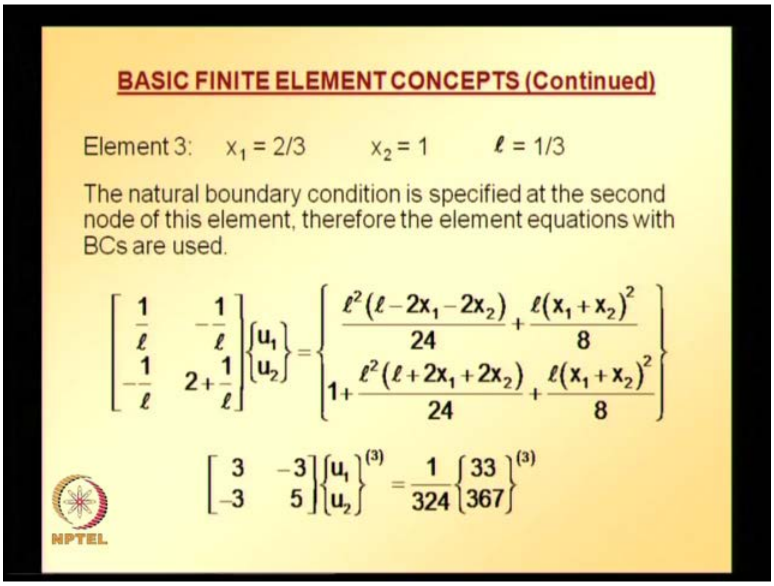

And now element 3, the local node 1 coordinate is 2 over 3, local node 2 coordinate is 1, here for element 3 x is equal to 1 is part of this element, so we need to use the element equation system or element equation that we developed with boundary conditions, here natural boundary condition is specified at the second node of this element, therefore, element equation with b c's are used here, and plugging into this equation system x 1 x 2 values and length of the element we get element equations for element 3. So, we are looking for a 3 element solution, we got element equations for element 1, element 2, and element 3, and please remember we are solving a 1 dimensional problem, and we have 4 nodes, 3 elements means 4 nodes because each element has 2 nodes.

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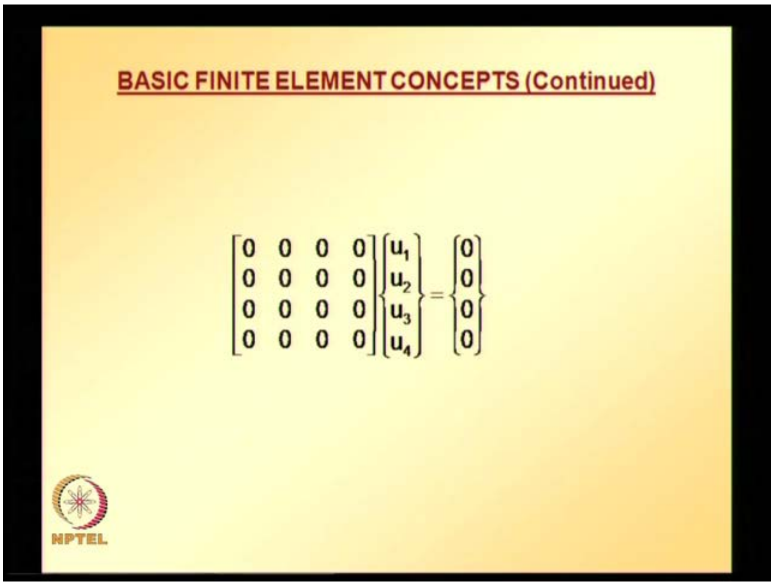

So, the global equation system will be of size same as the number of unknowns that needs to determine, that is you have 4 unknowns, so global equation system will be 4 by 4. What are the unknown quantities? Unknown quantities are u 1, u 2, u 3, u 4. What is stiffness like matrix? The matrix consisting usually it is a good practice in finite element method, especially when you are coding to initialize the matrices and vectors, before you start using them, because whenever you allocate some memory in computer for some particular areas, what happens is earlier somebody must have used so something some residue are leftovers will be there, and if you use directly without initializing it may leads to some errors.

So, better you always initialize matrices and vectors, so that is same ah technique is here adopted that is initialized matrices are shown here matrix and vector stiffness like matrix and force like vector, so now we need to decide, or we need to find where the contribution from element 1 goes in, into the global equation system where element 2 contribution goes in, and where element 3 contribution goes in. And if you see local node 1 node 2, corresponds to global node 1 node 2, so element 1 contribution goes into 1 and 2 rows in global system of equations.

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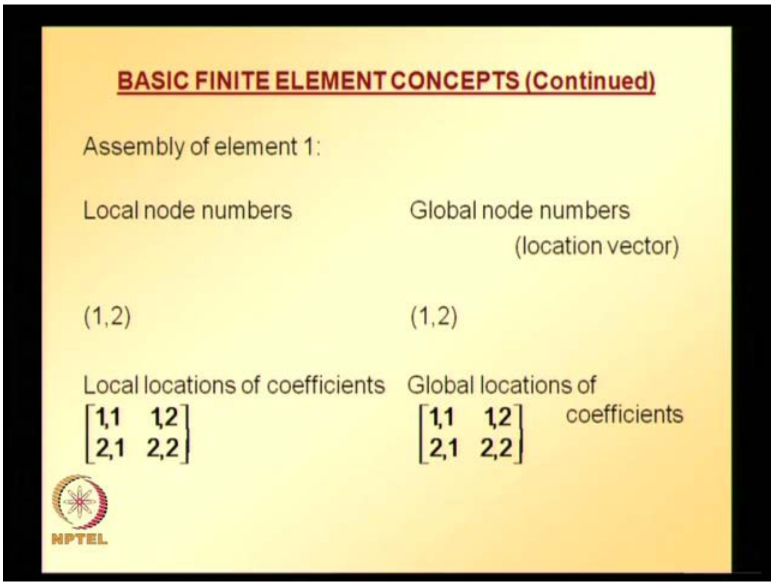

So, how to find where the contributions from element 1 goes in that you can that information you can get in from the local node numbers and global node numbers, you can prepares something like this. Local node numbers 1 and 2 corresponds to global node numbers 1 and 2, so local locations of coefficients global locations of coefficients are related in this manner, whatever you have local means for element 1 if you see, whichever is there at 1 1 location that goes into the global equation system at 1 1, whichever is that at 1 2 goes into 1 2 similarly, there as. So, now let us look at element assembly for element 2, or whatever what you can do is with this information you can substitute the contribution of element 1.

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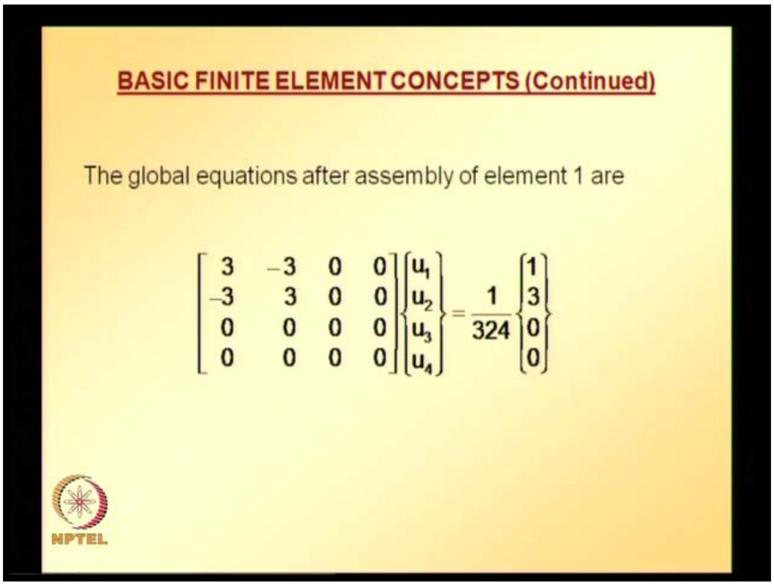

So, this is a where element 1 contribution goes into the global equation system, and global that is global stiffness matrix and global force vector, when I say global stiffness matrix it is not exactly stiffness matrix, it is stiffness like matrix and force like vector.

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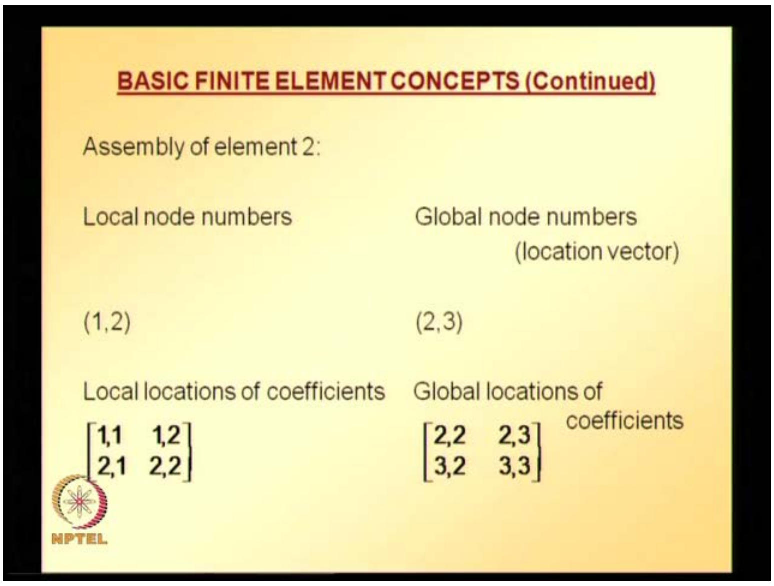

So, now element 2 contribution where it goes in it is easy to find, you just note down local node numbers 1 and 2 what they corresponds to the global node numbers, for this particular problem local node numbers 1 and 2 for element 2 corresponds to global node numbers 2 and 3, so you can prepare a matrix showing the local locations of coefficients relation to the global locations of coefficients relation. Now with this you can whatever is there at 1 1 location in the equation system of element 2 it goes into 2 2 location, and whatever is there at 1 2 it goes into 2 3 location of global equation system, similarly, 2 1 goes into 3 2, 2 2 goes into 3 3.

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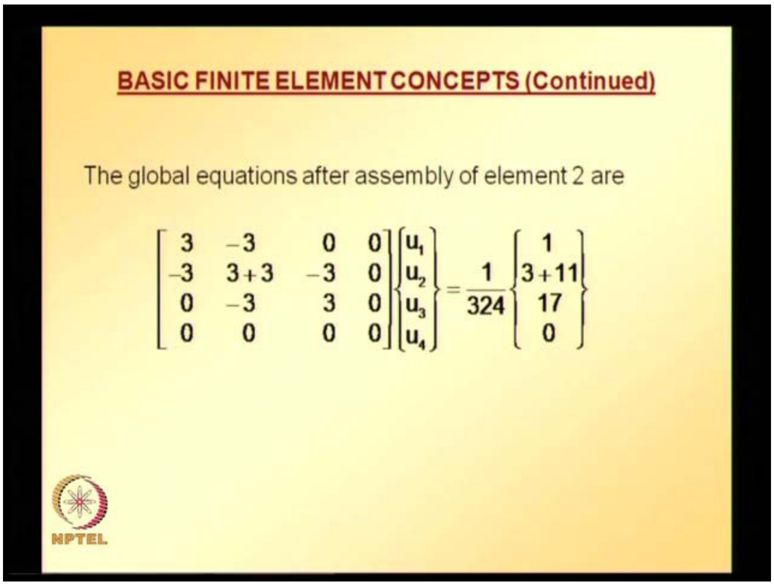

So, with this information we can you can put the a contribution of element 2 into the global equation system.

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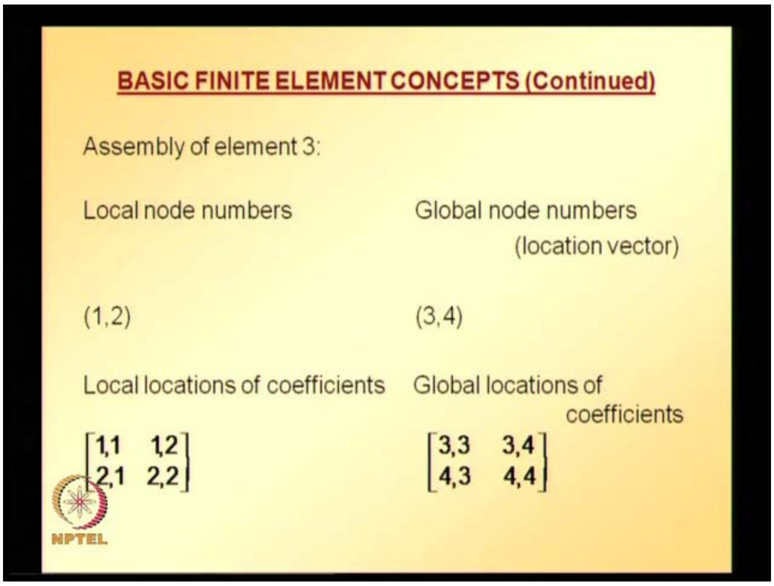

Similarly, element 3 contribution, same procedure.

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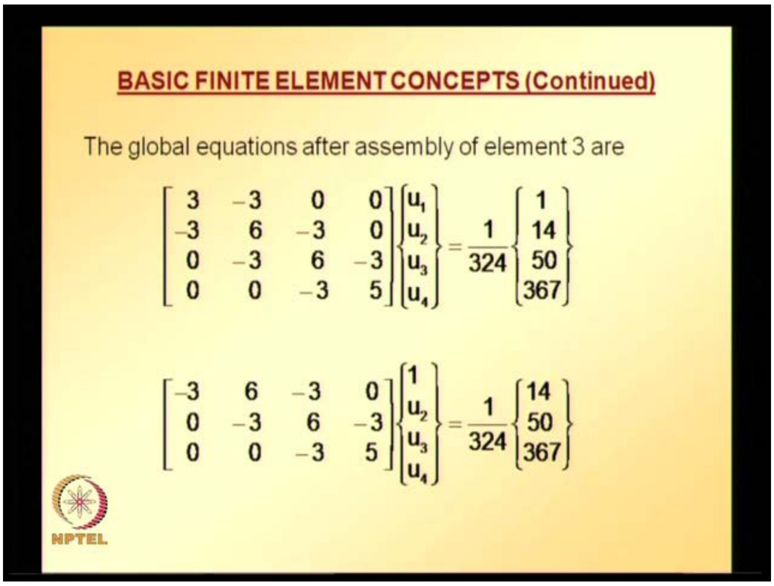

So, this is final global equation system without applying the boundary conditions, here we have apply essential boundary condition, whatever we have done is, we have by taking element equation system with boundary condition without boundary conditions we took care of natural boundary condition, but we have not still impose the essential boundary condition, essential boundary condition for this particular problem is u 1 value is given, u 1 is equal to 1 is essential boundary condition. So, what you can do is in the global equation system you substitute u 1 is equal to 1, and wherever essential boundary condition is specified wherever essential boundary condition is specified that row you can remove from the global equation system, that is from stiffness like matrix you remove that row, and also in force like vector you remove that particular component.

So, this is a 1 way of solving this equation system by imposing the essential boundary condition, I will show you the another way of solving the same equation system, and now this can be rearranged in this manner.

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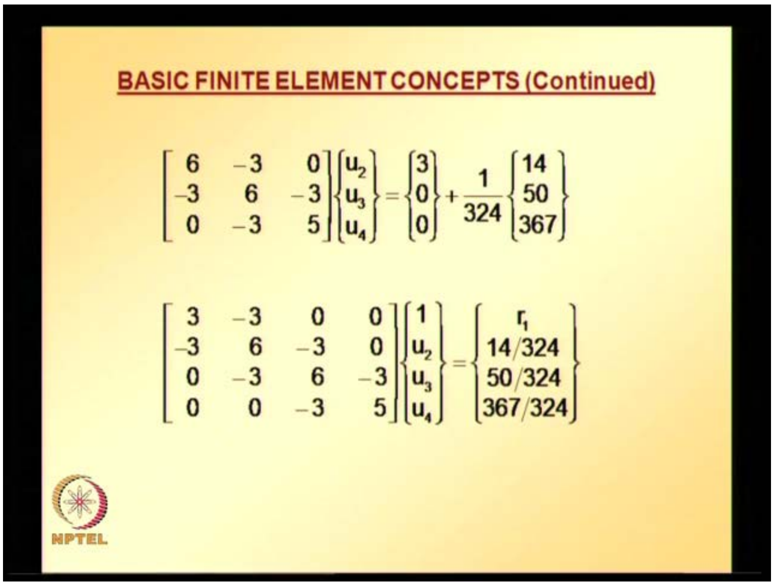

And now if you see the left hand side, you have all the numerical values you can easily find what the value of that vector, and once you get the values of those vector, you can solve for u 2, u 3, u 4 by using any of the numerical techniques like gauss elimination method. So other way of I mentioned to you there are 2 ways, 1 is you can either remove the rows and corresponding components in the force like vector, the other way is it is based on simple reasoning. So, the second approach is based on recognizing the fact that when essential boundary condition is specified, the corresponding derivative term is unknown, during derivation of functional this term was neglected with stipulation that only admissible functions will be considered as trail solution, thus the right hand term corresponding to the none nodal value is actually unknown, therefore, correct form of global equation after imposing the boundary condition is given here.

So, you just keep in mind wherever you have a no nodal value the corresponding location in the force like vector is going to be unknown, here u 1 is known that is essential boundary condition is specified, so the corresponding location in the force like vector is going to be unknown. So, that is why I do not know what is that value unknown that is why I am denoting that with r 1, now you have this equation system what you can do is you can use partitioning of matrices, and solve for u 2, u 3, u 4 and also back I once you get u 2, u 3, u 4 you can back calculate what is r 1.

And this approach I means I have shown you 2 kinds of approaches, 1 is in which you can delete the corresponding row in stiffness like matrix, and also corresponding component in still force like vector, the second approach is wherever you have known value known nodal value, the corresponding location in there right hand side you take it as unknown. So, you can follow either of this, but the second approach is conceptually better than the first approach, because what happens is using this idea global equations obtained using variational method are identical to those obtained using Galerkin method which you will see in a while.

So, whatever procedure we are following here variational procedure, but you have seen from you are the examples, that you looked at for classical approximate techniques, the exactly same solution you get if you adopt variational method or Galerkin method, so similarly, here you get exactly same equation system if you use the second approach that is wherever essential boundary condition is known at that location in the force like vector you make that component as unknown. So, you can use any either of these techniques to solve for u 2, u 3, u 4 and if you solve in either way you get these values.

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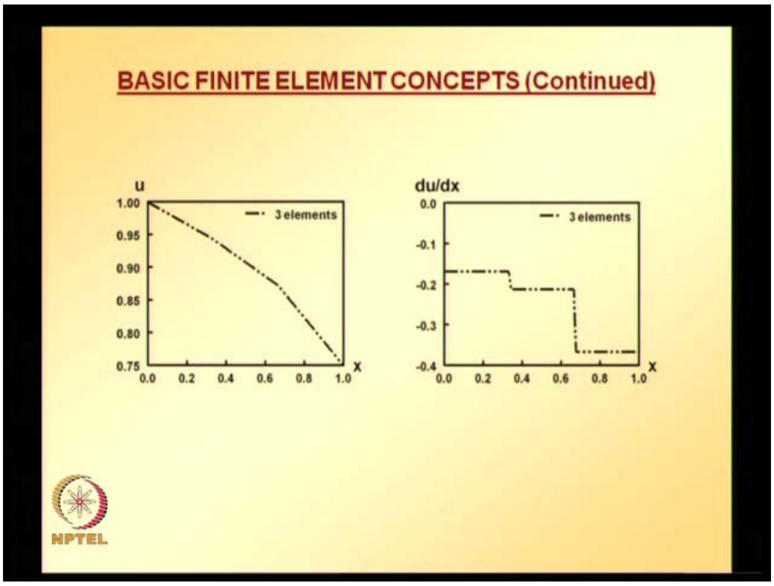

And then what you can do is, once you get the nodal values you go to the each element and do post processing find the approximate solution derivative of approximate solution the plot is shown for you for this particular problem, actually we solved this problem using the procedure similar to classical approximate techniques a just a earlier, and there

I have shown you comparative plot of 1 element, 2 element, 3 elements whereas, here only 3 elements will shown. So, what we did basically is key concepts are introduced, here our development of element equations and assembly of equations to form global equation system. So this is the new thing that you learned in this lecture, and it is necessary to develop element equations only once for a class of problem to get numerical solution all that is needed is to substitute appropriate numerical constants. The concept of assembly of element equations to form global equations gives finite element method a great deal of versatility.

The assembly process is very mechanical and means once you get the element equation substituting the nodal coordinates and length of the element all these things are mechanical and this is ideally suited for computers. These concepts will prove even more valuable when dealing with more complex 2 and 3 dimensional problems. So what will do is we will quickly solve the same problem

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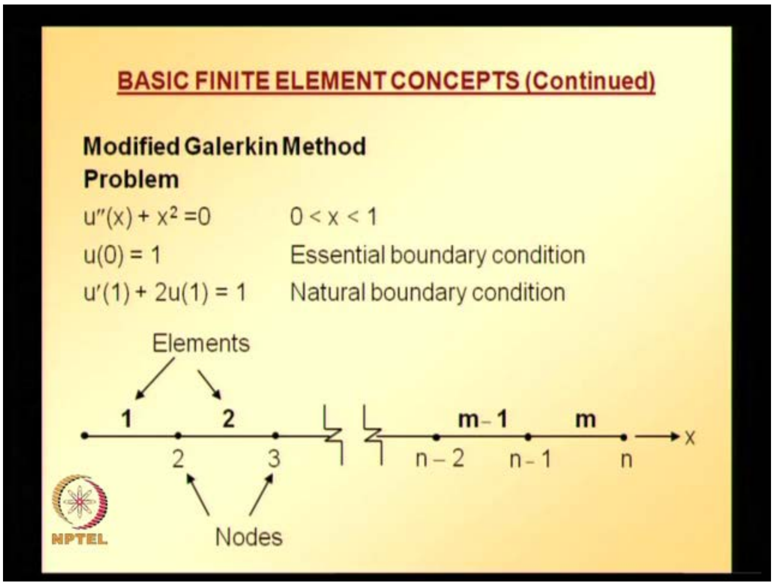

using modified Galerkin method because just now I made a statement, that exactly same equation system global equation system you will get even if you used Galerkin method, let us see whether it is true or not, here a boundary value problem it is against same boundary problem, and again it is going to be discretize using m number of elements, n number of nodes m and n we can decide later.

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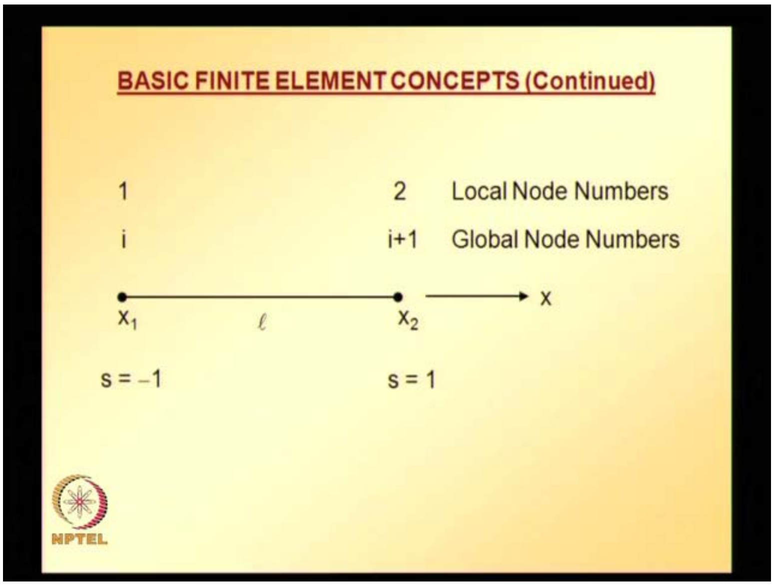

And this is typical element by this time you know how x is related to s all those equations are common.

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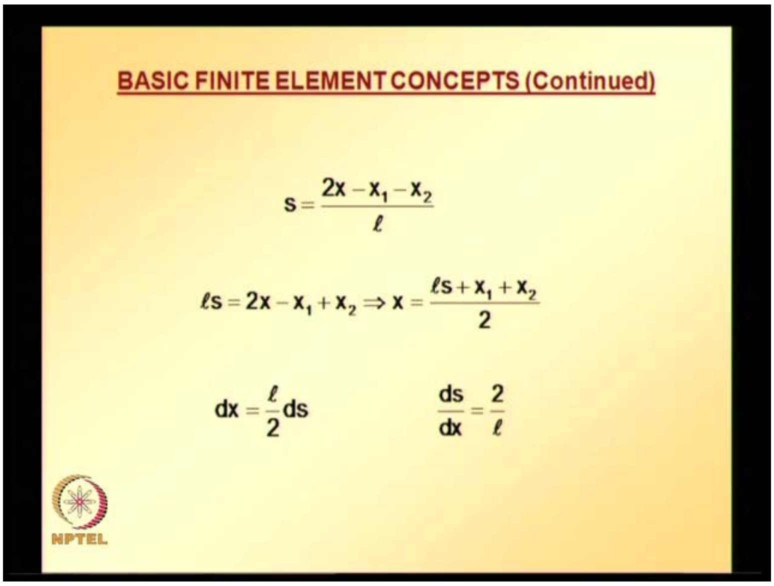

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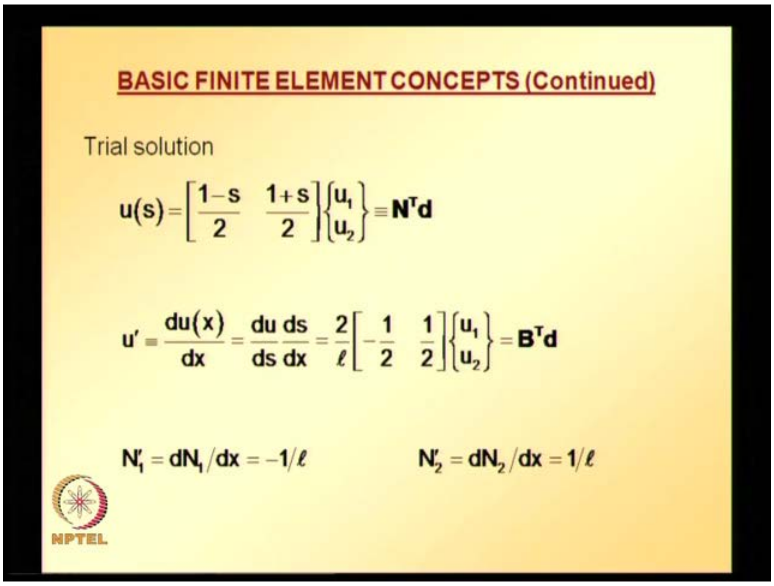

Trial solution n transpose d, derivative of trial solution b transpose d have seen all these earlier, and n transpose derivative of N 1, n 2

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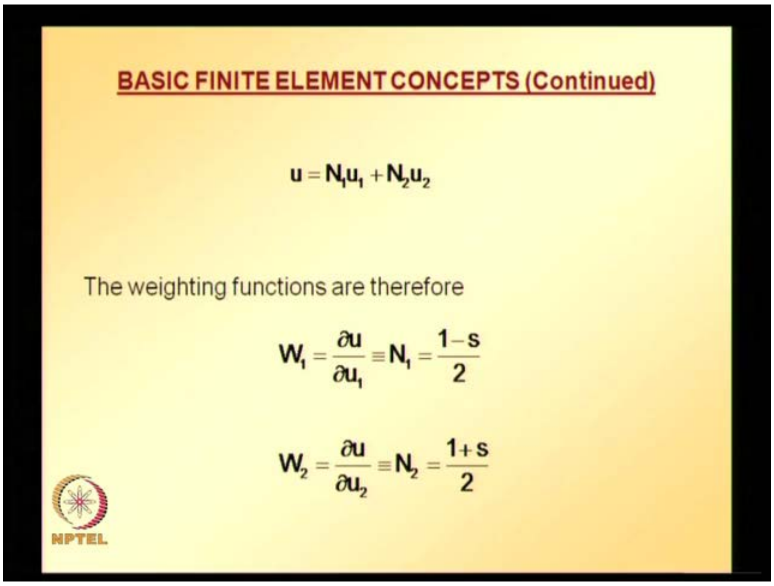

and the trial solution can be written as u is equal to N 1 u 1 plus n 2 u 2, and weighting functions are here in the beginning of the last lecture I mention that weight function in finite element method for Galerkin based finite element method is same as shape function, because how wait function is defined for Galerkin based method is partial derivative of u with respect to the unknown or nodal the coefficient, so it is partial derivative of u with respect to a 1 earlier, now a 1 value is taken by or a the coefficients a naught a 1 are replaced with u 1 u 2, so wait functions now becomes partial derivative of u with r respect to the coefficient values nodal coefficients, that is u 1, u 2, so weight function becomes same as shape function. So, these are the 2 waiting functions for a 2 node element.

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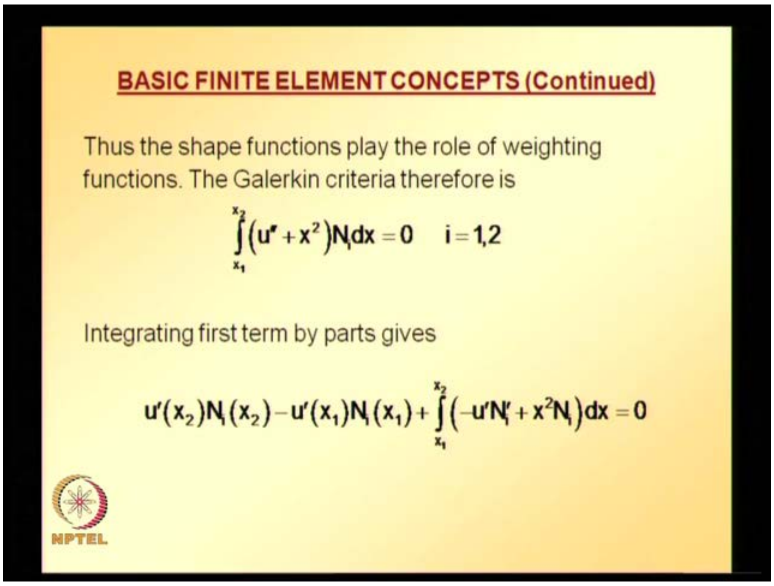

And you know the Galerkin criteria is like this, the integrate it given differential equation by multiplying with a weight function, integrate of the problem domain, here there are 2 weight function N 1, n 2, so you get 2 equations I taking values 1 and 2, and this is weight function here, substitute before we proceed what we can do is we can reduce the order of differentiation by using integration by parts.

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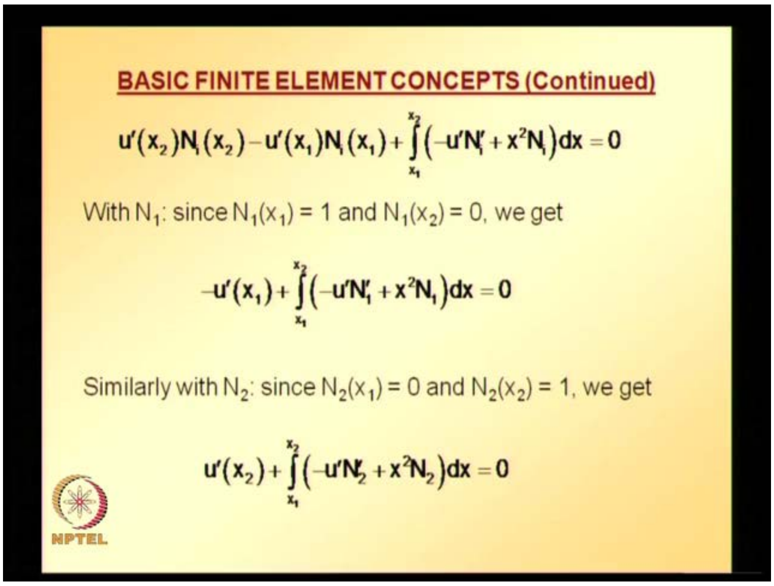

And note that and this can be further simplified, and we need before we proceed further we need to note that N 1 value or shape function satisfies kronecker delta property. So, N 1 value at x is equal to x 1 is 1, N 1 value at x is equal to x 2 is equal to 0, you make a note of it and substitute the same information into the equation, when I is equal 1 you get this equation, and similarly, n 2 is equal to 0, at x 1 n 2 is equal to 0, n 2 is equal to 1 at x 2, so this information is substituted by taking I is equal to 2, so you get 2 equations, here u prime is nothing but derivative of u with respect to x.

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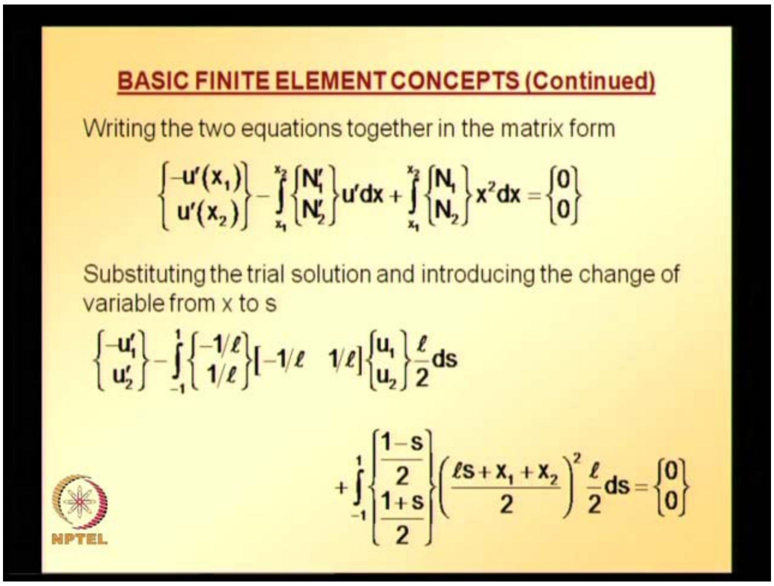

And these 2 equations can be written in a matrix form in this manner, and plugging in values of N 1 n 2, and changing the limits of integration x 1 x 2 2 minus 1 2 1 in s coordinate system d x is replaced with l over to d s, and x is replaced with corresponding values of s, and doing all these we get this equation system which we need to perform or simplify this further, final elemental equation system looks like this.

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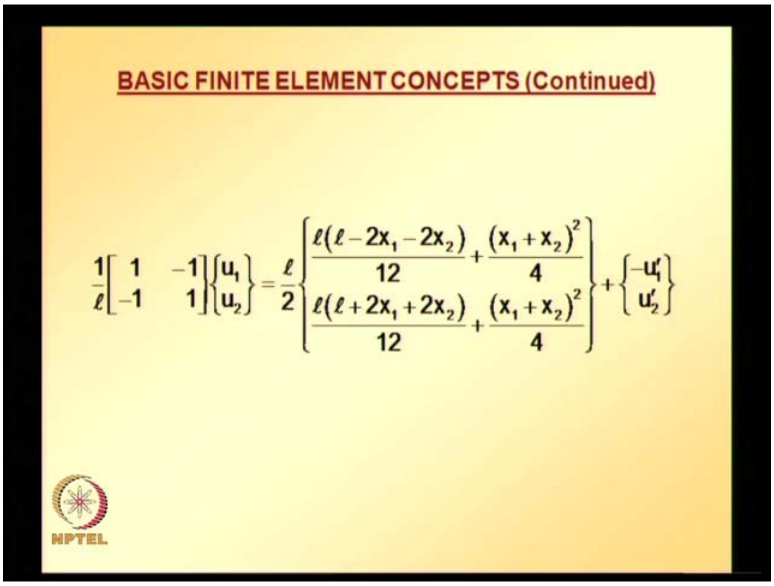

Typical elemental equation system, now what we can do is we can use this please remember there is a there is nothing in these Galerkin method the equation system with applying boundary with applying natural boundary conditions, without applying natural boundary condition, such kind of situation is not there, you will get only 1 equation system for a typical element.

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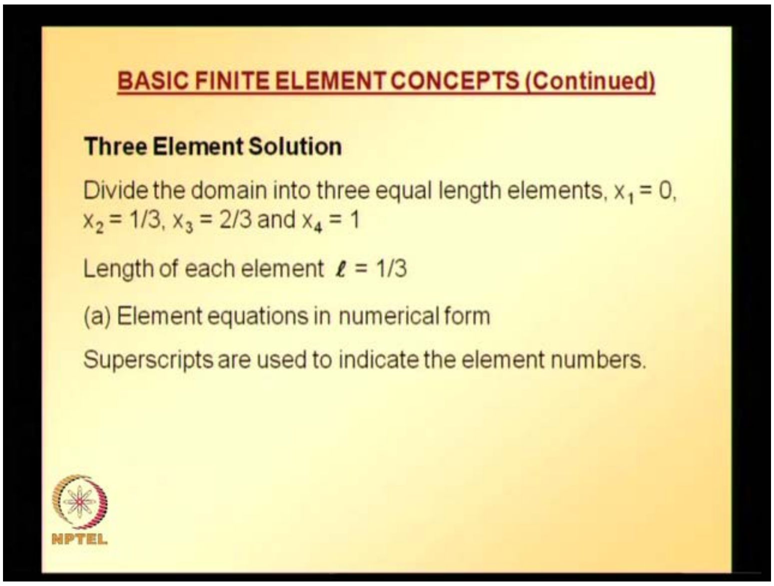

And using this equation system you can start assembling the elemental equations, you decide the discretization, 3 element solution, solution domain 0 to 1 is discretize using 3 elements, node 1 at x is equal to 0, node 2 at x is equal to 1 over 3, node 3 at x is equal 2 over 3, node 4 at x is equal to 1, and length of each element is l over 3, and again same thing superscripts are used to indicate the element numbers.

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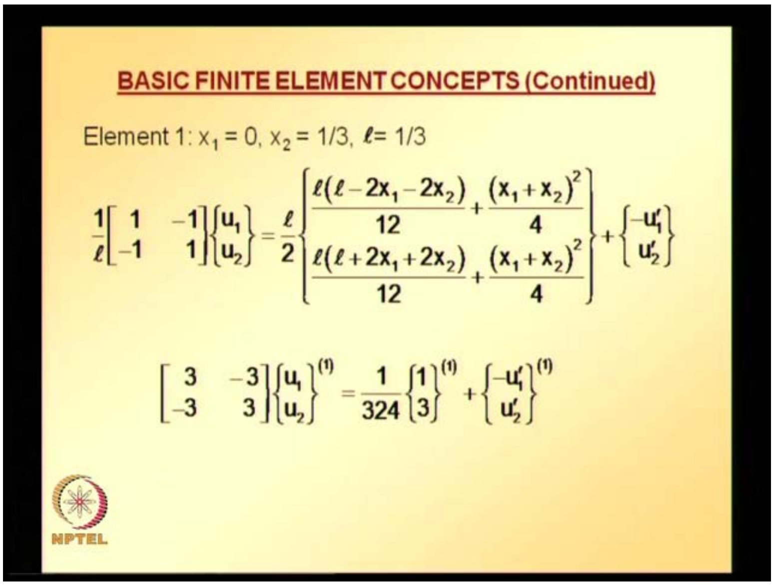

So, for element 1 these of the special coordinates and the length of the element, substitute all this information into the typical equation system you get this one, and substituting you get the superscript indicates element number, similarly, you do it for element 2, and element 3.

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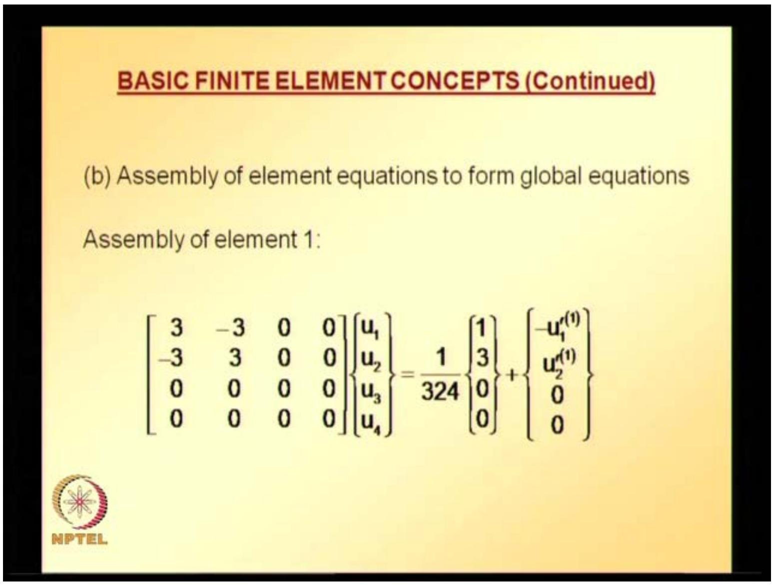

Assembly of element equations to form global equations this is similar to what we have seen for variational method or rayleigh-ritz method, the first element contribution goes into 1 and 2 rows and columns, so the global equation system.

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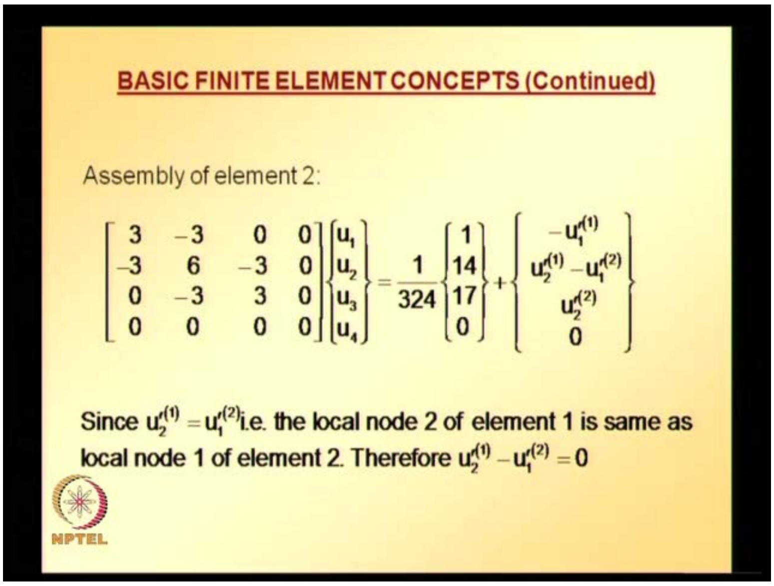

Second element contribution goes into 2 and 3 rows and columns, and if you see here on the right hand side you have sorry on the left hand side you have u u 2 prime for element

1, u 1 prime for element 2, u 2 prime for element 1 minus u 1 prime for element 2, actually please note that second node of element 1 is same as first node of element 2. So, those 2 quantities are same local node 2 of element 1 is same as local node 1 of element 2. So, those are same so these that quantity is going to be 0, so now let us look at a contribution of element 3.

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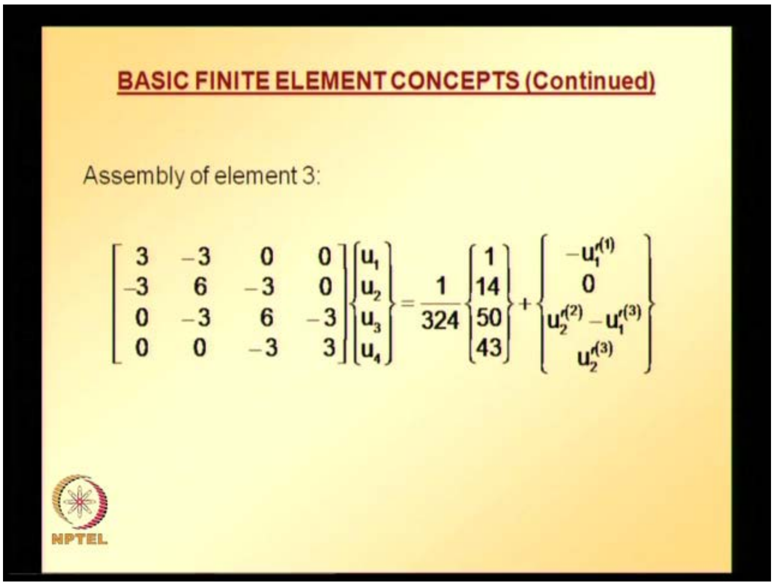

Similarly, you note that local node 2 of element 2 is same as local node 1 of element 3.

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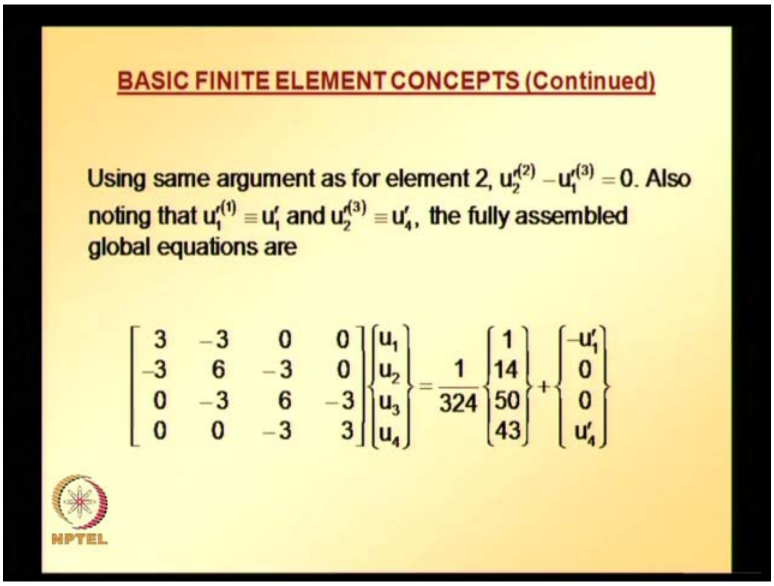

So, with that argument this u 2 prime for element 2 minus u 1 prime for element 3 is equal to 0, and now finally, the global equation system looks like this, and now we need to apply natural boundary condition u prime evaluated at x is equal to 1 is given for this problem, x is equal to 1 corresponds to node 4, so u prime evaluated at x is equal to 1 is given as 1 minus 2 times u evaluated at x is equal to 1, and that needs to be substituted in place of u 4 prime.

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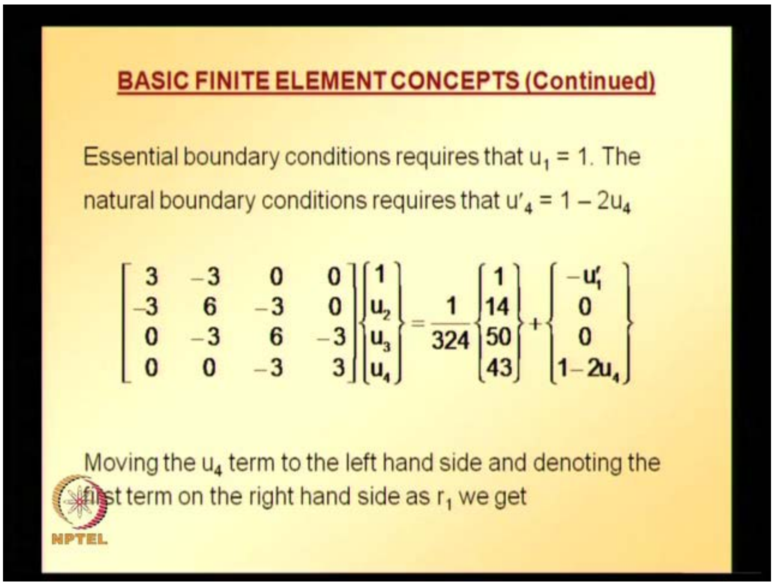

So, that is what is done here, essential boundary condition requires that u 1 is equal to 1, natural boundary conditions requires that u 4 prime is equal to 1 minus u evaluated at x is equal to 1, this is an 3 element solution, so node 4 corresponds to x is equal to 1, so it becomes u 4, so global equation system looks like this, and this can be rearranged to bring this u 4 to the right hand side moving u 4 term to the left hand side an denoting the first term on the right hand side as r 1, we get this equation system.

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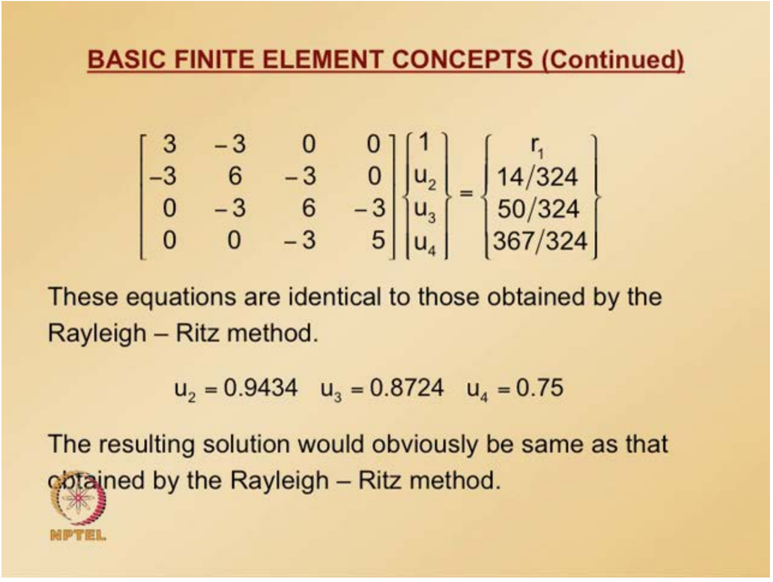

And if you compare this with what we got using variational method, this is exactly same equation system, these equations are identical to those obtain using variational method or rayleigh-ritz method, so solving this equation system by partitioning of matrices you get u 3, u 2, u 3 and u 4.

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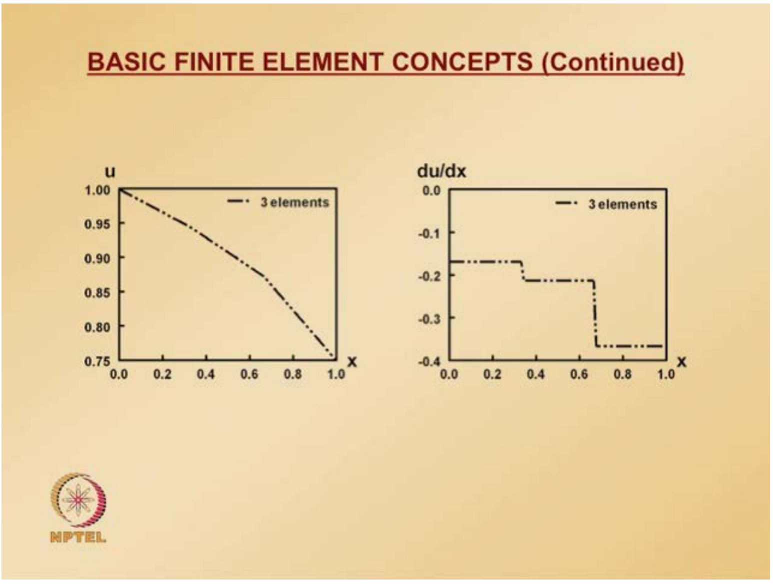

And rest of the procedure is same you can go to each element and find the element solutions, and then plot the approximate solution and derivative of approximate solution. We will continue in the next class.