**Finite Element Analysis**

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**Module No. # 01**

**Lecture No. # 04**

Last class we have seen, using a variational methods, few examples by taking various forms of approximations like quadratic approximations, cubic approximations, quartic approximations of the trial solutions, and we also looked at how to solve an eigen value problem.

In today's class, what we will do is we will solve the same examples which we have done yesterday, using modified Galerkin method.

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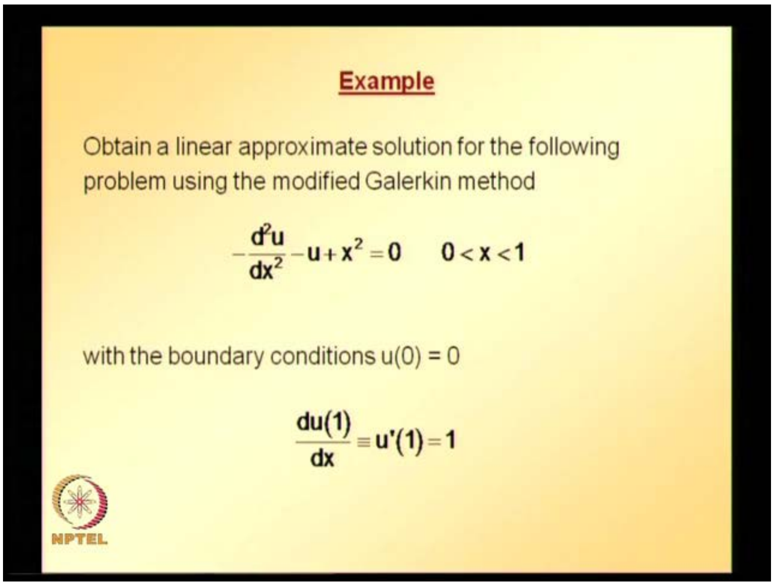


So, this is the problem statement. Obtain a linear approximate solution of the problem using modified Galerkin method. And you are given the differential equation, and also the problem domain. And this is a second order differential equation. So, you can guess you require two boundary conditions to solve this problem. The two boundary conditions that are given are here. And you can easily check using the thumb rule that I already gave you earlier, that the first boundary condition turns out to be the essential boundary condition and the second boundary condition turns out to be a natural boundary condition.

Once again I will repeat. This is a second order differential equation. So, those boundary conditions of order 0 to p minus 1; here p minus 1 is 0. So, zeroth order boundary conditions are essential boundary conditions, and those boundary conditions of order p to 2 p minus 1 are natural boundary condition. And if you check here, the order of differential equation is 2. So, 2 p is equal to 2. So, p is equal to 1.

So, those boundary conditions of order 1 are natural boundary condition. So, that way you can check the second boundary condition is natural boundary condition. So, here we are going to use modified Galerkin method.

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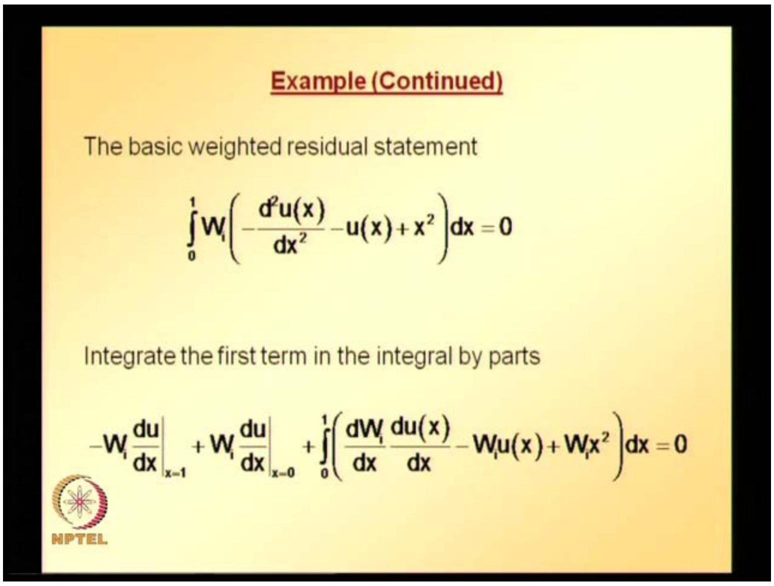


So, you know, the basic weighted residual statement for any weighted residual method is this. Multiply the given differential equation with a weight function, integrate over the problem domain, and equate it to 0. The weight function depends on the method that you choose. And if it is a least square weighted residual method, it is partial derivate of e with respect to the unknown coefficients. And if it is collocation method, it is direct delta function. And if it is a Galerkin method, weight function is going to be partial derivative of coil solution with respect to the unknown coefficients.

So, before we proceed further, what we need to do is, we look at the any higher order derivative terms, and we will use integration by parts and reduced to the lower order terms. So, if you see this equation, the first term is having second derivative of u. So, we can use integration by parts on the first term; that is, W times minus second derivative of u with respect x square, you apply integration by parts on the term and it gets simplified to what is shown there.

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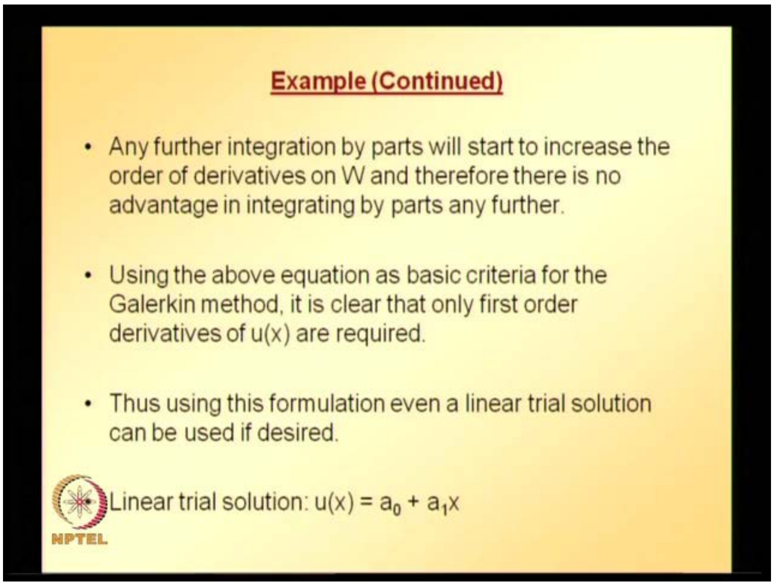


And you may think that why do not we use one more time integration by parts, but it is not going to help us anywhere. If you see why, any further integration by parts will start increase the order of derivatives on w, weight function.

So, actually the purpose is we want to use integration by parts, to balance or whatever derivatives, a higher order derivatives are there on the trial function , you want to transfer it to the weight function, but not to increase the order of derivative on the weight function. So, any further integration by parts will start to increase order of derivatives on weight function .Therefore, there is no advantage in integration by parts any further.

Using above equation as a basic criteria for Galerkin method, it is clear that only first order derivatives of u are required. So, you require only first order derivative of u. So, you can start with a linear trial solution. Thus using this formulation, even a linear trial solution can be used if desired.

And if you recall, this problem is solved using a variational method, and there we minimum trial solution, minimum order of trial solution we have taken is, second order; that is, we started out with a quadratic trial solution. And here we can even go for linear trial solution.

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So, the linear trial solution that is assumed is here. u is a naught plus a 1 x, and rest of the procedure is similar. So, what you need to do is, you need to make sure this trial solution is admissible; that is ,you need to substitute the essential boundary condition into this and reduce the number of unknown coefficients. So, what is the essential boundary condition that is given here, u evaluated at x is equal to 0.

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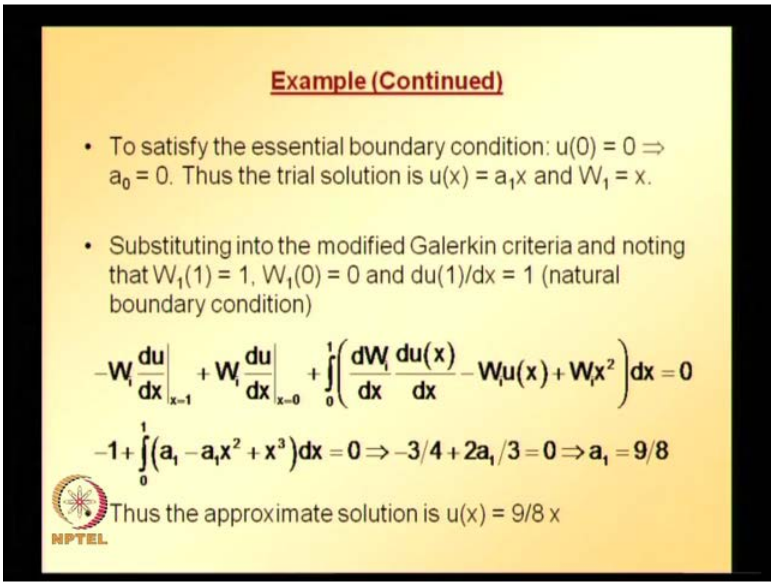


So, to satisfy the essential boundary condition u evaluated at x is equal to 0 is 0, if you substitute that condition u, it results in a naught equal to 0. So, the trial solution becomes u is equal to a1x. So, there is only one unknown coefficient to be determined. And you know, for Galerkin method, weight function is partial derivative of u with respect to the unknown coefficients. So, it turns out that weight function is equal to x.

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Substituting into the modified Galerkin criteria, whatever equation we have earlier and noting that now, you have W1 is equal to x. So, W 1 value at x is equal to 1 is going to be 1, W 1 value at x is equal to 0 is going to be 0. And also you are also given natural boundary condition for this particular problem, that is derivative of u with respect to x evaluated at x is equal to 1 is 1.

So, you plug in all this information into the previous equation. The first equation is reproduced, but our equation we have earlier, that is reproduced; the first 1, and into that equation, these all information is substituted; that is, W 1 evaluated at x 1 is equal to 1,W 1 evaluated at x is equal to 0 is 0, derivative of u evaluated at x is equal to 1 is 1. All this information is substituted at (( )) and carrying out the integration, you will get one equation, and one unknown. You can solve for the unknown coefficient a 1.

So, once you get unknown coefficient a 1, you can back substitute this a1 into the trial solution, and then you get the approximate solution for this problem.

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So, approximate solution for this problem is u x is equal to a 1 value which is 9 over 8 times x. ((No audio from 07.31 07.38))

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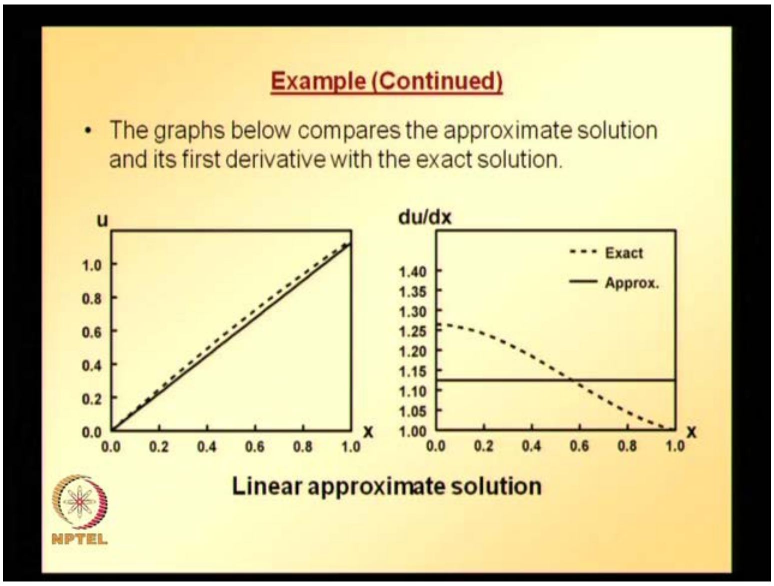


And now see, how approximate solution matches with exact solution, by plotting the exact solution verses approximate solution .We have already looked at this problem earlier using a variational method, and it is mentioned at the time, the exact solution for this problem is, u is equal to 180 over 139 x minus 21 over 139 x square. So, that exact solution and approximate solution you can over lay on each other. This is how they match. And also you can take the derivative of the approximate solution and derivative of the exact solution. You can plot them.

And since, we started out with a linear trial solution; you can see there is a large error in the derivative of the approximation; whereas, the approximation itself is fairly accurate. And if you want to further reduce the error on the derivative of the approximate solution, what you need to do is you can go and start with or you can start with a higher order trial solution; that is you can take a quadratic or cubic or higher order.

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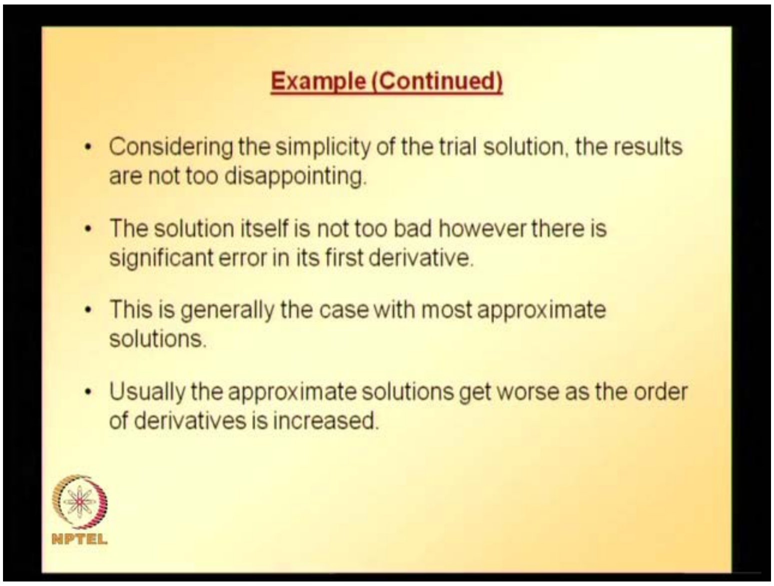


Now, let us look at what is, what are the some points related to this technique and this problem. Considering the simplicity of the trial solution, results are not too disappointing; means whatever results you have seen, the approximate solution is itself good, but the derivative of approximate solution has some error. Solution itself is not too bad; however, significant error in its first derivative, that is what you observed, and this is generally the case with most approximate solutions.

Usually approximate solutions get worse as the order of derivative is increased. Whatever we have seen there, we plotted only the first derivative, but if you again take one more derivative, error you will see much more than what you have error in the first derivative.

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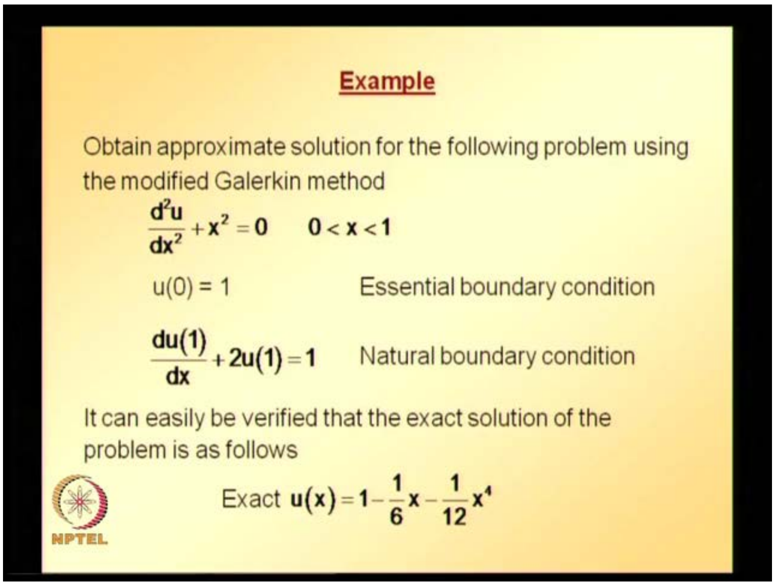


And now we look the next problem. This problem also we solved using variational method, and what we will do is we will solve the same problem using Galerkin method, modified Galerkin method. The problem statement is given here. Second derivative of u with respect to x square plus x square is equal to 0, problem domain is 0 to 1, essential boundary condition u evaluated at x is equal to 0 is 1, and natural boundary condition derivative of u evaluated at x is equal to 1 plus 2 times u evaluated at x is equal to 1 is 1. And this problem, exact solution is already given to you. And the exact solution for this problem is u is equal to 1 minus 1 over x 1 over 6 x minus 1 over 12 x power 4.

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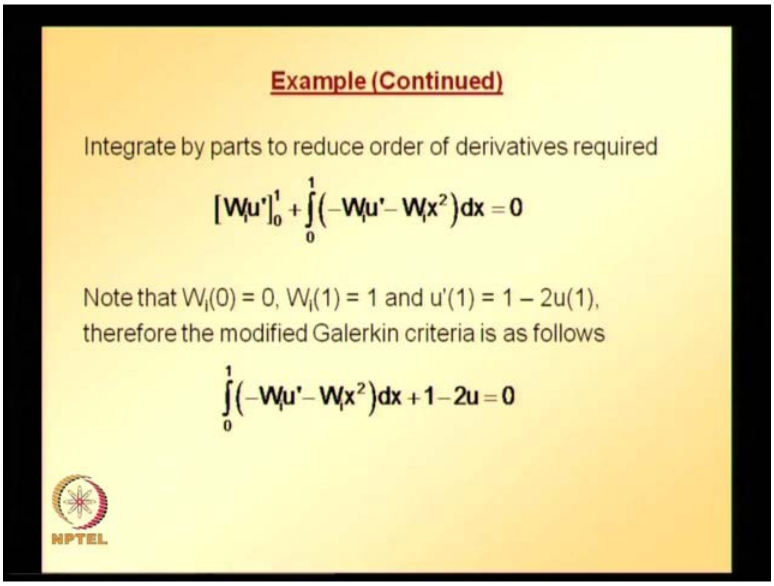


So, now let us go through the procedure. First we need to select trial solution and make it admissible. To make trial solution admissible, what we need to do is we need to substitute essential boundary condition and find one of the coefficients, unknown coefficient if it is possible. So, the admissible trial solution for this problem is 1 plus a 1 x plus a 2 x square. Once again, I want to emphasize here, admissible trial solution is a trial solution which satisfies essential boundary conditions.

So, you can check by substituting x is equal to 0 in this equation, whether it satisfies essential boundary conditions or not. And now once we got the admissible trial solution, you know, a weight functions for Galerkin method are defined like this; that is derivative of u with respect to the unknown coefficient is what is weight function. And here you have two unknown coefficients.

One is a 1, another is a 2. So, you get two weight functions. W 1 is derivative of u with respect to a 1, and W 2 is derivative of u with respect to a 2. And w 1 turns out to be x and W 2 turns out to be x square. And what is the Galerkin weighted residual statement? It is the given differential equation is multiplied with a weight function, integrated over the problem domain, equated to 0. And again you need to identify which term is having higher order derivatives. And use integration by parts and reduce the order of derivative on that term and transfer the derivative to weight function.

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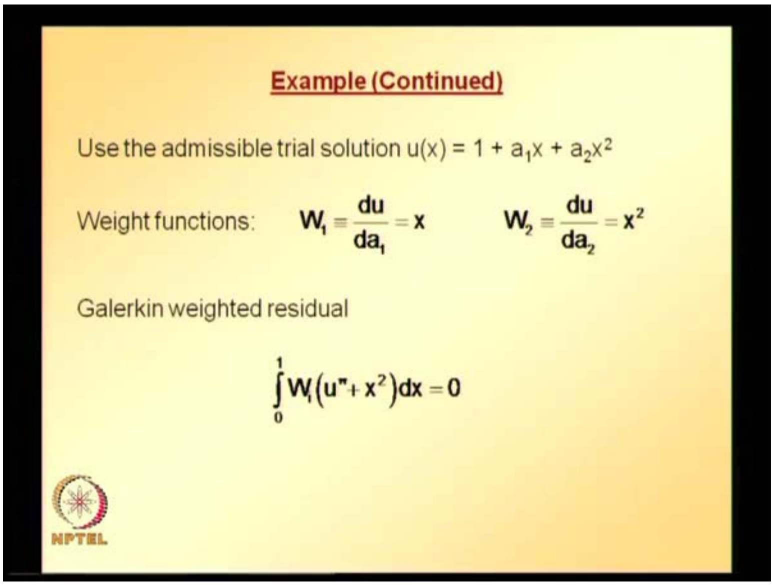


As I mentioned in the previous problem, you need to be very judicious in deciding how many times you want to use integration by parts because we do not want to differentiate weight function also too many times.

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So, now using integration by parts, it results in this one. That is integration by parts is applied only on the first term. And now, we already know what W 1 is, and what W 2 is. And W 1, if you recall, it is x. W 2 is x square. So, what I can do is, both are functions of x. So, I am here writing as w i, w i is i takes values 1 and 2. Both W 1 and W 2 evaluated at x is equal to 0 or 0 and w 1 evaluated sorry w i, where i takes values 1 and 2 evaluated at x is equal to 1 is equal to 1. And the natural boundary condition is already given there in the problem statement.

So, that is same thing is reproduced here. So, all this information; that is w i evaluate at x is equal to 0, W i evaluated at x is equal to 1 is 1 and the natural boundary condition. You can substitute all this information into the first equation that results in the last equation. Substitute trial solution into the weighted residual to get system of equations. ((No audio from 14:37 to 14:44 and 14.48 to 14.51))

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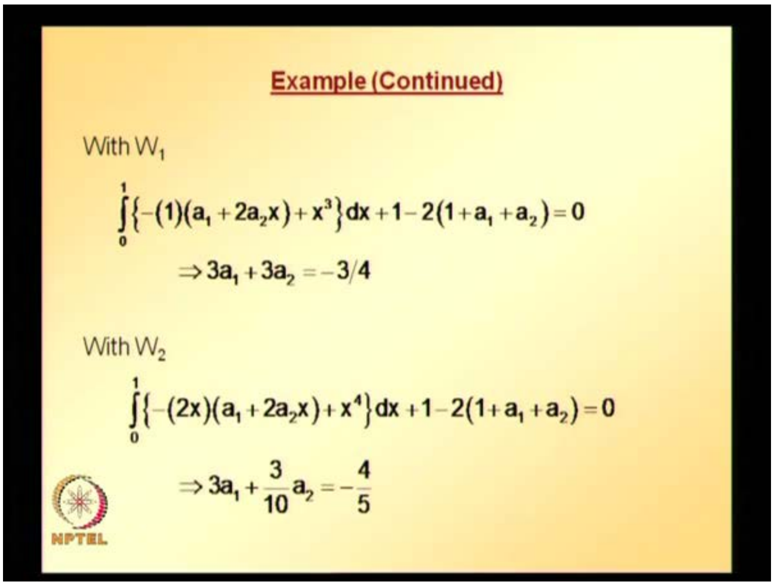


So now, i taking value 1 results in this equation. i takes values 1 and 2 here, because we need to determine two coefficients which is a 1 a 2. So, i takes values 1 and 2. So, this is the equation corresponding to i taking value 1, and this can be simplified and which results in this equation.

And when i takes value equal to 2, you get this equation .That is you need to substitute W 2 and the corresponding trial solution and derivative of trial solution and which simplifies to this equation. Here you can see there are two unknowns to be determined a 1 and a 2 and two equations. So, you can solve these 2 equations simultaneously and get the coefficients a 1 and a 2.

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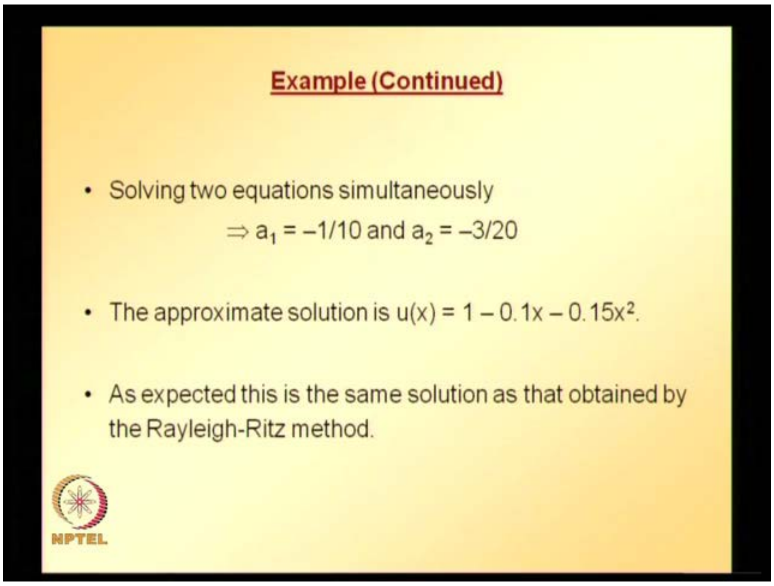


Solving two equations simultaneously results in a 1 is equal to minus 1 over 10, a 2 is equal to minus 3 over 20.

Now, you know what you need to do with these coefficients. You need to substitute back these coefficients a 1 a 2 into the admissible trial solution we started out with. What is admissible trial solution we started out with? It is u is equal to 1 plus a 1 x plus a 2 x square. So, you substitute a 1 is equal minus 1 over 10 and a two is equal to minus 3 over 20, you get the approximate solution.

And if you compare this solution whatever you obtained using modified Galerkin method, this is exactly same as what you already obtained using variational method or Rayleigh-ritz method. And this is expected because, if a particular problem can be solved using these two methods starting with the same order of trial solution, you will get exactly same solution. If you do not commit any mistakes, you should get exactly and this is one way of checking your procedure, whether you are procedure that you adopted is correct or not.

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And now let us look at terms to summarize before we look at what is the difference between or what is how can you differentiate between approximate solution techniques and finite element method, before that let me summarize what we have done so far.

We looked at various methods; weighted residual methods, we looked at least square weighted residual method, collocation method and also Galerkin method basic formulation, Galerkin method modified formulation and also variational method. And we also looked at some problems, what the advantages of these techniques, we illustrated through some examples. So, now, we can just see before we proceed to the finite element method, let us make a note of what are the differences between the approximate solution techniques and finite element method.

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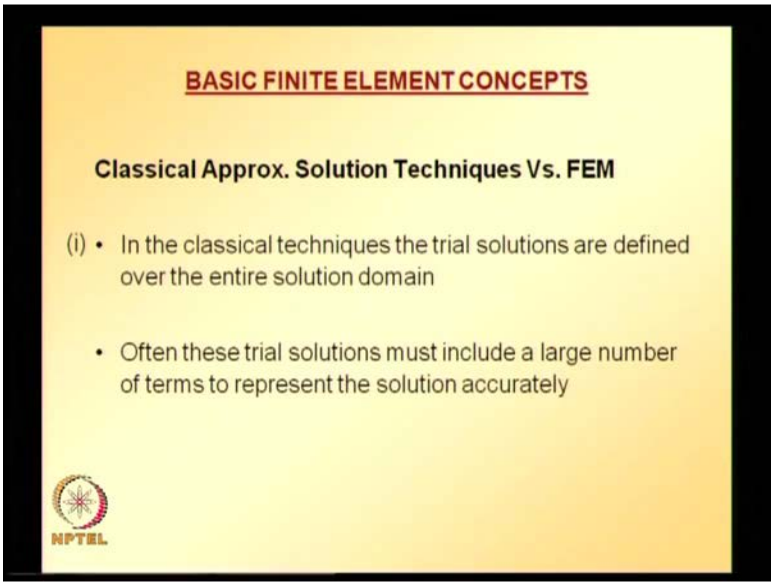


Basically, finite element method is essentially an extension of classical approximate techniques. So, the difference between the two techniques; that is, classical approximation solution techniques and finite element method are as follows here. In classical techniques, as you already experienced, when you looked at the problems, in class techniques the trial solutions are defined over the entire solution domain.

That is you started out, if you take a any problem, you started out with a trial solution; that is u is equal to a naught plus a 1 x plus a 2 x square and that whatever a trial solution you started out with is applicable for the entire solution domain or it is defined over the entire solution domain.

Often this trial solution must include large number of terms to represent the solution accurately. This also you have experienced. As you increase the order of trial solution; that is, if you go from linear trial solution to quadratic trial solution or cubic and quartic, the solution accuracy increases. So, if for a particular problem if lower order trial solution is not is a not capturing the exact solution accurately, then we need to include large number of terms to represent solution accurately.

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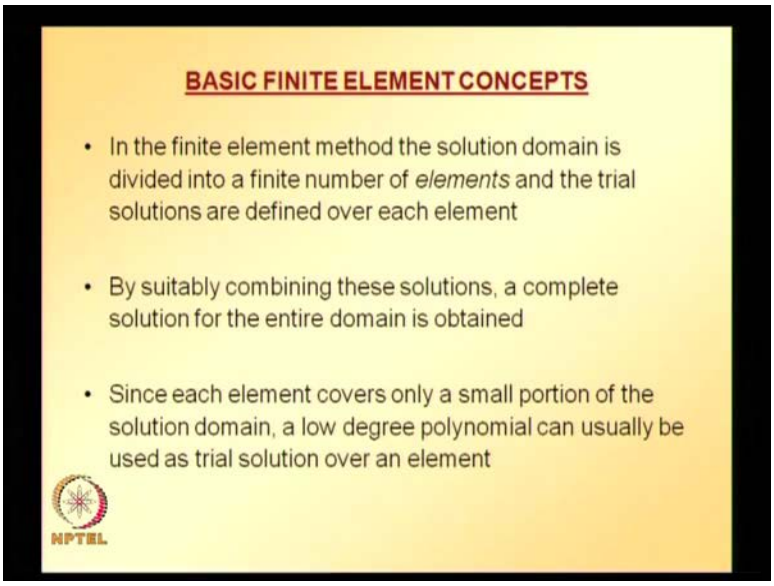


In finite element method, which we are going to see in a while, in finite element method, the solution domain is divided into finite number of elements and the trial solutions are defined over each element. By suitably combining these solutions, a complete solution for the entire domain is obtained.

So, what we will we be doing is, we will be defining trial solution for each of the element separately, and once we solved for all the unknown coefficients, you will go back to each element, and then we will push process it; that is what this means, the second point.

Since each element covers only a portion of solution domain, a lower degree polynomial can usually be used as trial solution over an element. That is we can use instead of going for quadratic polynomial you can use linear polynomial.

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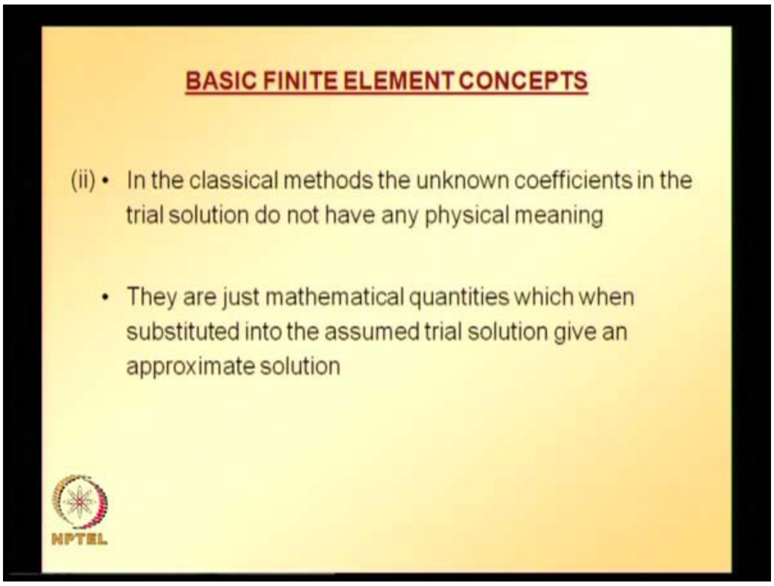


In classical methods, the unknown coefficients in the trial solution do not have any physical meaning. That is, this a naught a 1 a 2 whatever you assumed in the classical methods for trial solution; those coefficients do not have any physical meaning. They are just mathematical quantities, which when substituted into the assumed trial solution give approximate give an approximate solution. ((No audio from 21.39 to 21.46))

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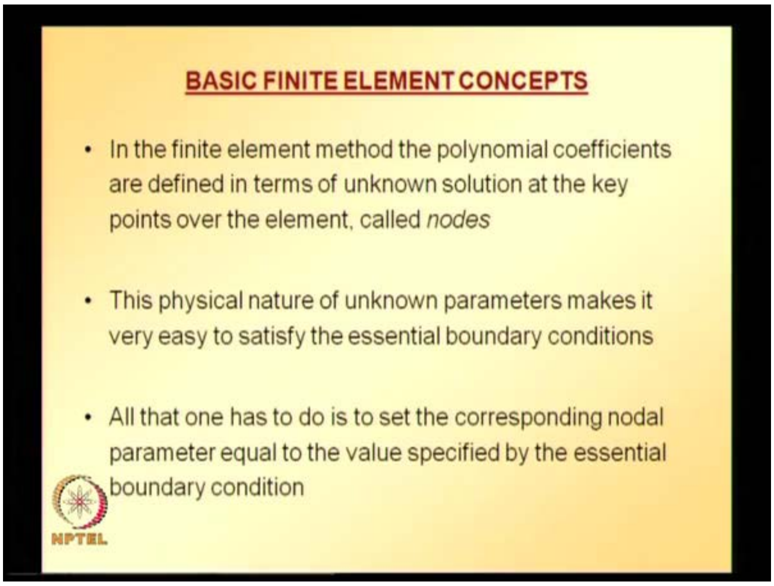


In finite element method, the polynomial coefficients are defined in terms of unknown solutions at key points over an element called nodes. We will see this more details of it in a while. So, the difference between classical approximate techniques and finite element method is the polynomial coefficients are expressed in terms of nodal values in finite element method. So, this physical nature of unknown parameters makes it very easy to satisfy essential boundary conditions. ((No audio from 22.26 to 22.34))

And why it is so simple if we do this kind of things in expressing polynomial coefficients in terms of nodal values? All that one has to do is to set the corresponding nodal parameter equal to the value specified by the essential boundary condition.

So, this is a why it is very fairly easy to apply essential boundary conditions in finite element method.

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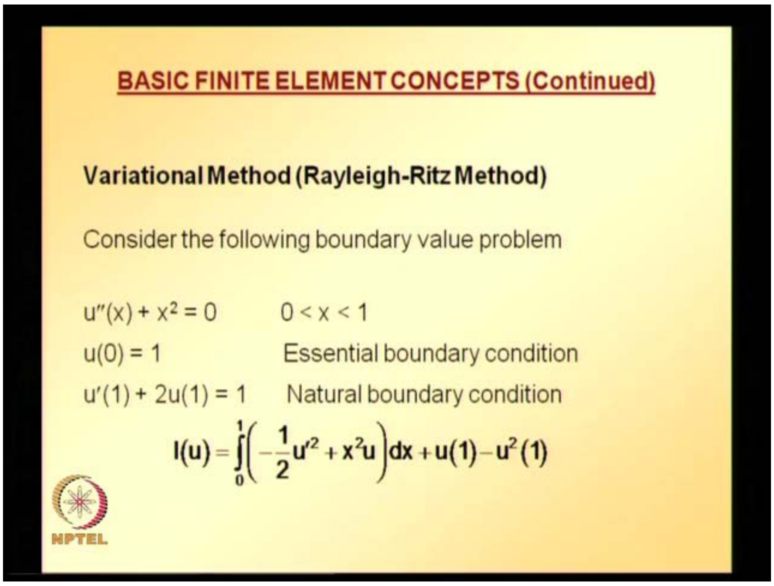


Now, what we will do is we will take one example, we will solve this or this concepts whatever concepts that is; the difference between classical approximate techniques and finite element method to make this concepts clearer, we will be solving one dimensional second order differential equation with mixed boundary conditions. To make the concepts as clear as possible, the example that we are going to look, we are going to solve in slightly two different ways.

The first approach is what may be considered as long hand approach and this approach gives you more insight into the solution process and clearly demonstrates that there is very little difference between classical approximate techniques and finite element method. So, we are going to follow two different approaches for solving same problem and then you will appreciate what is the transition between classical approximate techniques and finite element method.

The second approach follows a more traditional way of organizing finite element equations. So, first let us look at the first approach. Again this boundary value problem you already looked at and the second order boundary value problem, domain is 0 to 1, and the boundary conditions are u evaluated at x is equal to 0 is 1.

First derivative of u evaluated at x is equal to 1 plus two times u evaluated at x is equal 1 is 1. It is a natural, it is turns out the first boundary condition is essential boundary condition and the second boundary condition is natural boundary condition. You can verify this easily, and by this time you know how to get equivalent functional using variational method.

So, if you follow that procedure, equivalent functional turns out to be this one and before we proceed further, in finite element method, the first step is to select a trial solution. One of the key concepts in finite element method is that solution domain is divided into small parts called elements, and trial solutions are defined over these individual elements, assuming fairly lower order polynomials. For this problem, domain is divided into m number of elements and the length of each element is it can be different or it can be same.

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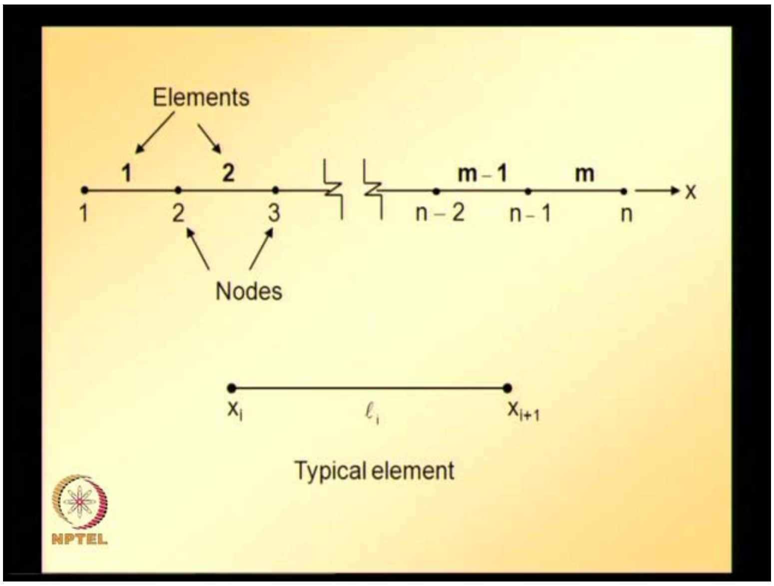

So here, the problem domain 0 to 1 is divided into m number of elements, and each element is assumed to have two key points at the extreme ends of that element, the particular element. So, each element has two nodes, and there are n number of nodes, m number of elements. We will decide what a value of m and n later in a while and this is how the solution domain 0 to 1 is divided or discretized.

So, now if you take a typical element and please note that each of these elements can have different lengths. So, there is no restriction that all the elements should be of same length. And now if you see a typical element, typical element looks like what is shown there in the second figure.

Typical elements; typical element is connecting node i with i plus 1. The x coordinate of i th node is x i, x coordinate of i plus 1 th node is x i plus 1, length of this element is l i.

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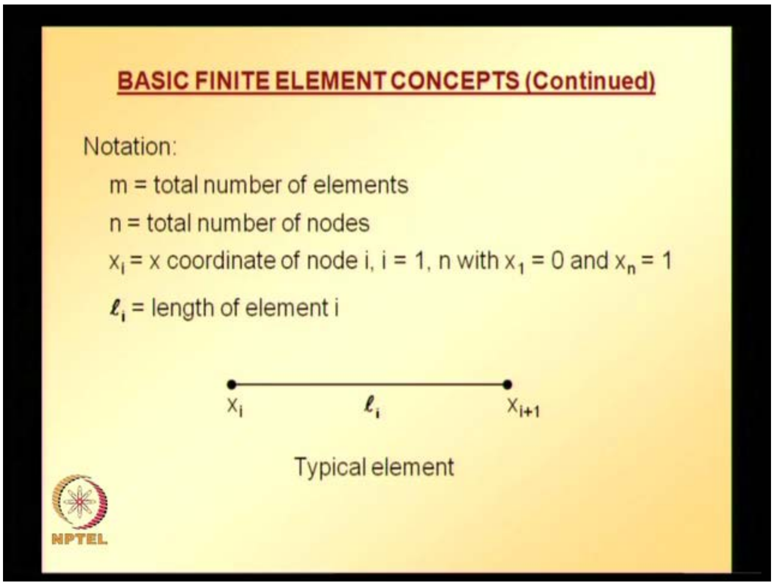


And let us see what the various quantities in this figure are. m is total number of elements, n is total number of nodes, x i is x coordinate of node i, i takes values from 1 to n. x 1 coincides with x is equal to 0, x n coincide with x is equal to 1, and l i is length of element I, and this is typical element reproduced again, and what we will do is we will assume a linear trial solution over this element.

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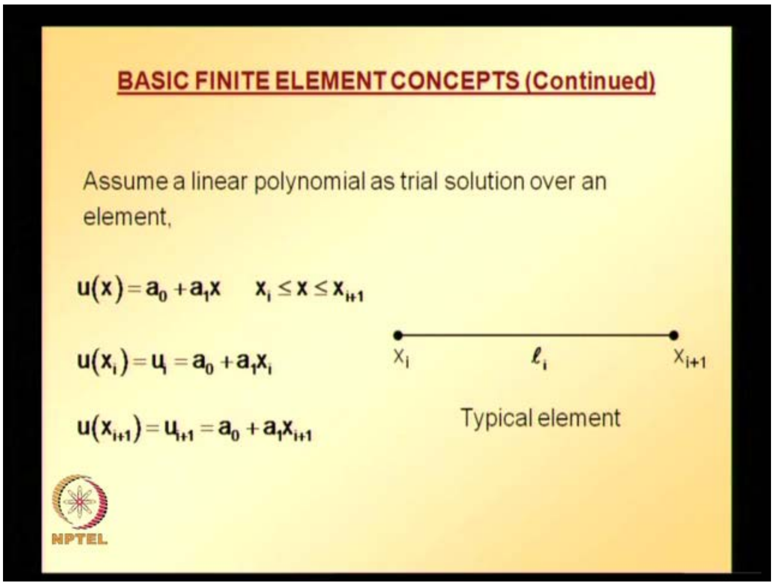


Assume linear polynomial as a trial solution over this typical element. Trial solution can be written as u is equal to a naught plus a 1 x, and the domain of this element is going from x i to x i plus 1. And one of the basic concepts in finite element formulation is that trial solution is expressed in terms of unknown solution at the nodes.

So, what we are going to do is we are going to replace this a naught a 1 with unknown solution at the nodes. These unknown nodal values act as parameters to be determined by various techniques that you already know; that is, variational or weighted residual methods. If unknown solution at node i is denoted using u i and the unknown solution at node i plus 1 is denoted using u i plus 1, what we can do is we can substitute, you can get two equations from the assumed linear polynomial trial solution; that is, u value at x is equal to i is u i, u value at x is equal to i plus 1 is u i plus 1. Substitute those two things into this equation, you get the first equation; that is, u evaluated at x is equal to x i is u i that is equal to a naught plus a 1 x i. Second equation u evaluated x is equal to x i plus 1 that is u i plus 1 is equal to a naught plus a 1 x i plus 1.

So, now you got two equations and you can solve these two equations for a naught and a 1. Subtracting the second equation from the first gives you what is a 1, and substituting the a 1 that you just got by subtracting equation 2 from equation 1, it the first equation you can obtain what is a naught. So, that is what is shown here.

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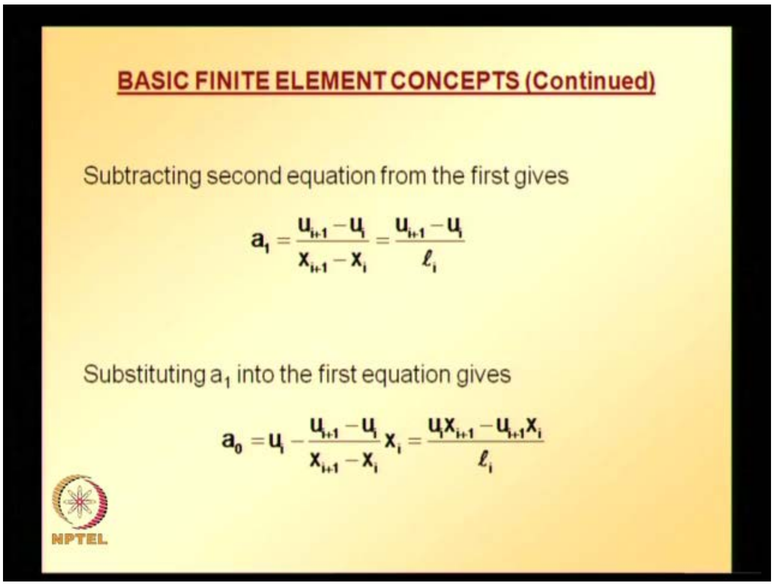


Subtracting second equation from first results and what is shown there, and please note that x i plus 1 minus x i is equal to l I; length of the element is given by the special coordinate of i plus 1 th node minus special coordinate of i th node and substituting a 1 into the first equation, gives us a naught.

And now you determine what is a naught and a 1. What you do is you can substitute back this coefficients a naught a 1 into the trial solution; linear trial solution that we started out with, and then you need to some mathematical manipulations such a way that you bring the coefficient; the terms having coefficients u i and u i plus 1 separately.

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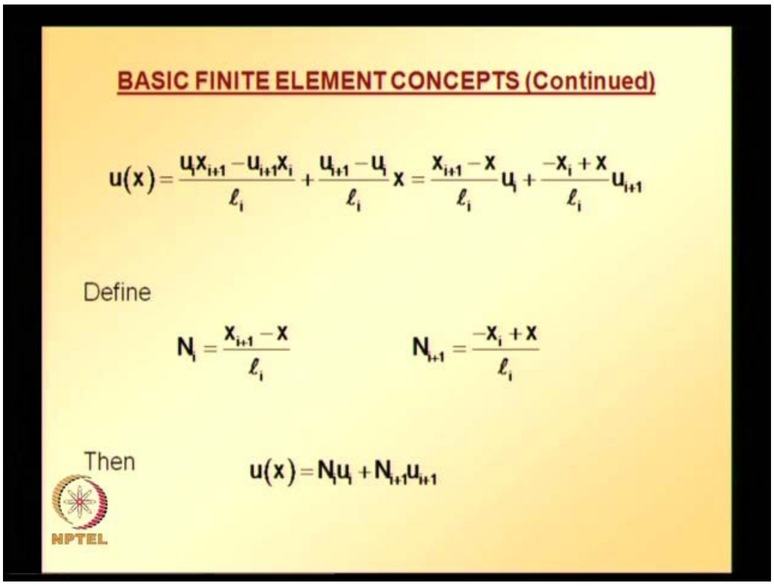


So, first step is you substitute a naught a 1 into the linear trial solution that you started out with, and the second step is you group terms having u i as coefficient, u i plus 1 as coefficient separately. Whatever term which is coefficient of u i; that is called n i or that is defined as n I, and whatever coefficient that you have for u i plus 1 that is defined as n i plus and this n i and n i plus 1 are called shape functions or interpolation function and this is one way of deriving shape function expressions, but the other simple way of deriving this shape function f expressions is known lagrange interpolation technique.

So, the equation u can be written in a compact manner like this. Once we define what is n i and n i plus 1, u is equal to n i u i plus n i plus 1 u i plus 1. If you look at this equation carefully, earlier whatever a naught is there, in that position you have u i, earlier whatever is there at are where u 1 is a 1 is there, the position at which a 1 is there, there at that position you have in this equation u i plus 1.

And if you recall, linear trial solution is u is a naught plus a 1 x, and if a naught plus a 1 x can be put in a matrix in a vector form like 1 times x in a vector and times a naught a 1 in a another vector, and if you see this 1 x are linearly independent. Similarly this n i and n i plus 1 are linearly independent.

Shape function should be linearly independent and also if you sum up this n i and n i plus 1, they will be equal to 1. So, sum of shape function should be equal to 1, and also if you take derivative of n i with respect to x and derivative of n i plus 1 with respect to x, and if you add these two derivatives, that will be equal to 0. Sum of derivatives of shape function shape function is equal to 0. These are what are called consistency conditions which will be using at a later stage.

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So, now, coming to the equation u is equal to n i u i plus n i plus 1 u i plus 1, once again i want to bring your attention to this equation. If you recall for Galerkin weighted residual method, the weight function is defined as partial derivative of u with respect to the unknown coefficient. So, here earlier you have unknown coefficients as a naught a 1; whereas, now you have unknown coefficients here as u i u i plus 1.

So, now you take partial derivative of this u with respect to u i, you get n i. Partial derivative of u with respect to u i plus 1, you will get n i plus 1. So, in weighted Galerkin based weighted residual method, weight function is going to or shape function takes the position of weight functions.

So, when you are apply the finite element technique in Galerkin based weighted residual method, weight functions are same as shape function. These three points you just keep in mind we will be using later stage.

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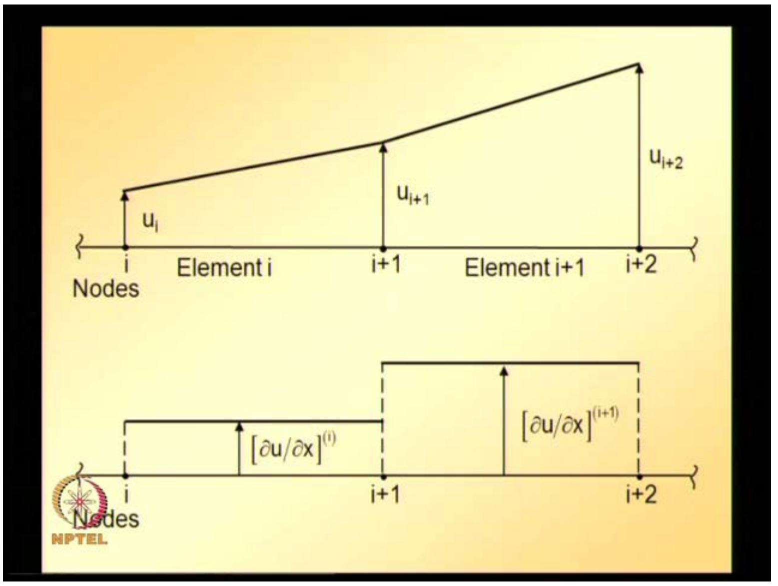


And now, these trial solutions, the first derivative, the derivatives of this trial solution and this first derivatives are shown here. If you see, the trial solution is linear and it is continuous, linear in each element and continuous across the element boundaries. So, this is what is piecewise linear, piece wise continuous; whereas, the first derivative of trial solution is discontinuous along the element edges and is constant in each of the element. And now we defined what the trial solution is for this problem, in terms of finite element shape functions and the nodal values.

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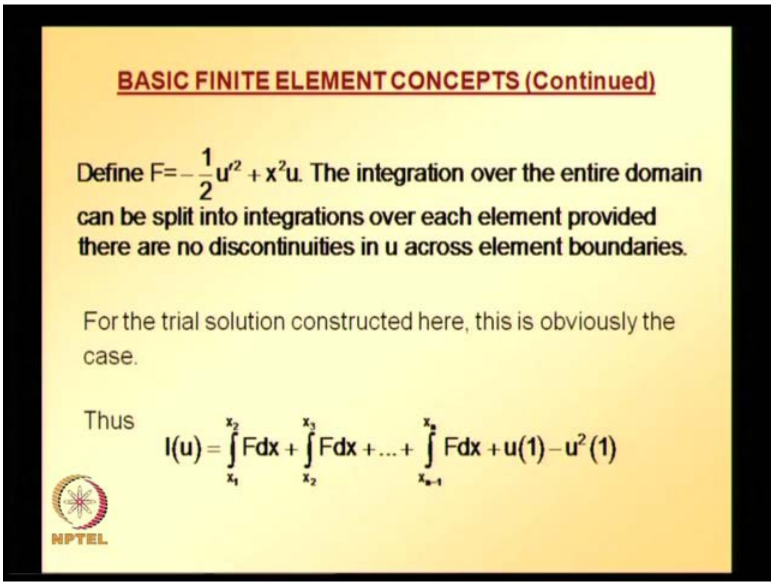


Now, we are ready to solve the problem that we are looking at. The equivalent functional for the problem that we are looking at using Galerkin method sorry variational method is here, and now into this equivalent functional, you can substitute all the trial solutions.

Before doing that, first we need to decide how many number of elements and how many number of nodes that we want to use for this particular problem. To simplify the to simplify in writing, the previous equivalent functional, here f is defined like this integration over the entire domain can be split into integration over each element provided there are no discontinuities in u across solution or element boundaries. Here if you see, u is continuous over the entire solution domain that is 0 to 1.

So, for the trial solution constructed here, there are no discontinuities across element boundaries. That you have seen in the figure that I showed you. Trial solution is piecewise continuous. So, this equivalent functional the integral 0 to 1, we can split it in this manner, where x 1 corresponds to x is equal to 0, and x n corresponds to x is equal to 1 and depending on the number of elements, you choose for this particular problem, you will get so many number of integrals there and the last two terms are boundary terms that is u evaluated at x is equal to 1, u square evaluated at x is equal to 1.

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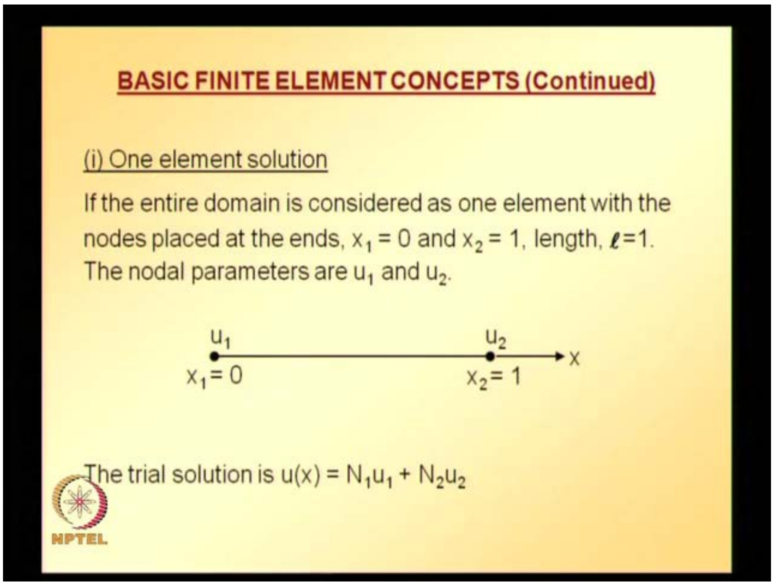


So, before we proceed further, we need to decide how many elements you want for this solution domain. So, let us try using one element; that is, we will take only single element over the entire domain. The first node coincides with x is equal to 0 and the second node coincides with x is equal to 1.

If the entire domain is considered as one element with nodes placed at ends of the domain; that is, x at x is equal 0, that is x 1 is equal to 0, x 2 is equal to 1, length of the element is 1. The nodal parameters are the unknowns that that are to be determined or denoted with u 1 and u 2. u 1 corresponds to node 1, u 2 corresponds to node 2, and now for this element, you know what is node 1 and what is node 2.

So, you can easily write a trial solution as n 1 u 1 plus n 2 u 2, and from the nodal coordinate information that is given to you here, you can easily find what is n 1 and what is n 2 and also essential boundary condition is prescribed for this problem at u evaluated at x is equal to 0 that is 1.

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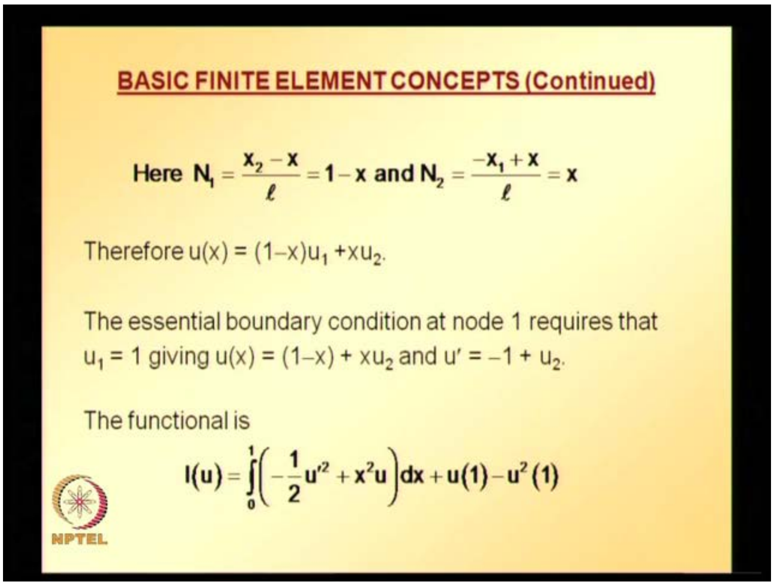


So, n 1 n 2 substituting the nodal coordinates, the trial solution becomes this, essential boundary condition at node 1 requires u 1 is equal to 1. When you substitute u 1 is 1 into the previous equation or at the trial solution, it becomes u is equal to 1 minus x plus x times u 2 and the derivative of it; you can easily find, it turns out to be minus 1 plus u 2. So, all this all this quantities that is trial solution and derivative of trial solution are required for us to plug in to the equivalent functional. So, we need to plug into this functional. ((No audio from 42:08 to 42:16))

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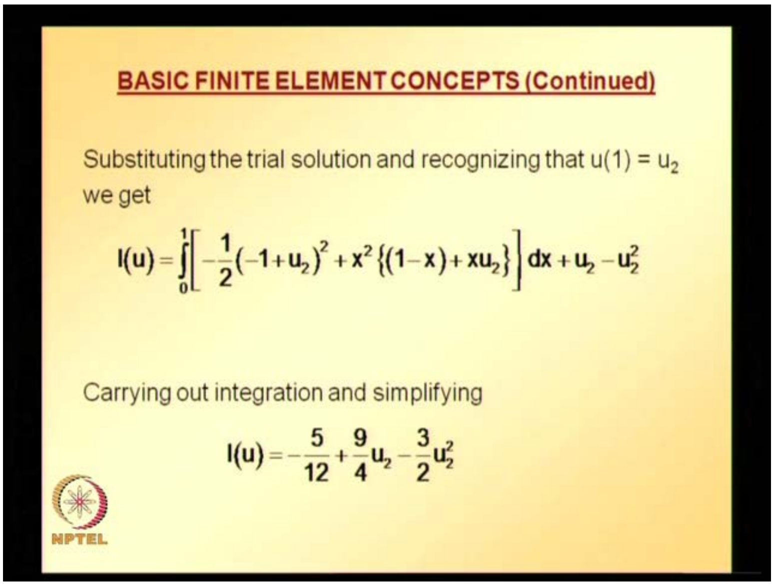

Substituting trial solution and recognizing that u evaluated at x 1 is equal to 1 is nothing but u 2; the equivalent functional becomes this and you need to… here u i i as a function of u and u in turn is a function of u 2, so, i becomes function of only u 2 and if you simplify this equation by integrating, it is going to be function of u2 a 1 carrying out integration and simplifying results in this equation, and now we need to invoke the condition that variation of i should be equal to 0 which is possible only when partial derivative of i with respect to u 2 or here since i is function of only u 2, it is derivative of i with respect to u 2 should be equal to 0, and that is called stationarity condition.

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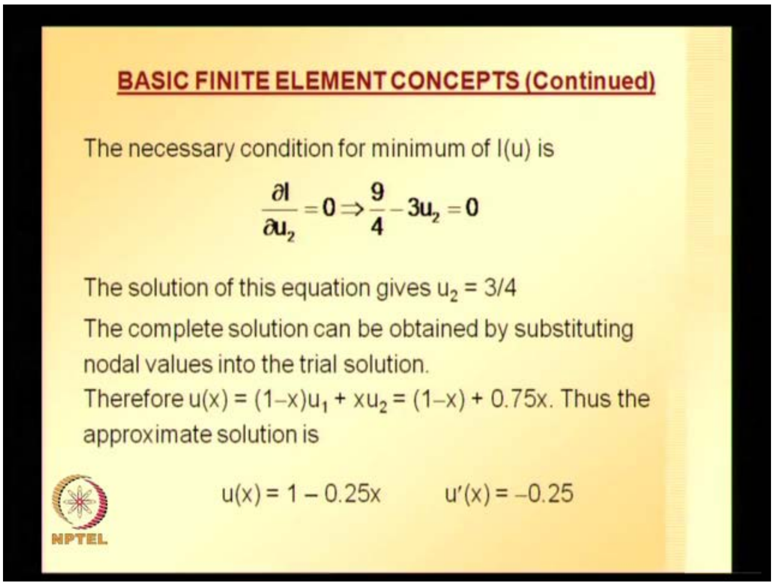


So, necessary condition for minimum of i is partial derivative of i with respect to u 2 equal to 0, which leads to one equation, one unknown, and you can solve for this unknown u 2 and that substitute this u 2 into the trial solution, you get complete solution. Therefore, the approximate solution is u is equal to, if you simplify after substituting u 1 u 21 values into the trial solution that we started out with it and simplifying that equation the approximate solution turns out to be u is equal to 1 minus 0.25 x and also you can take derivative of this which turns out to be a minus 0.25.

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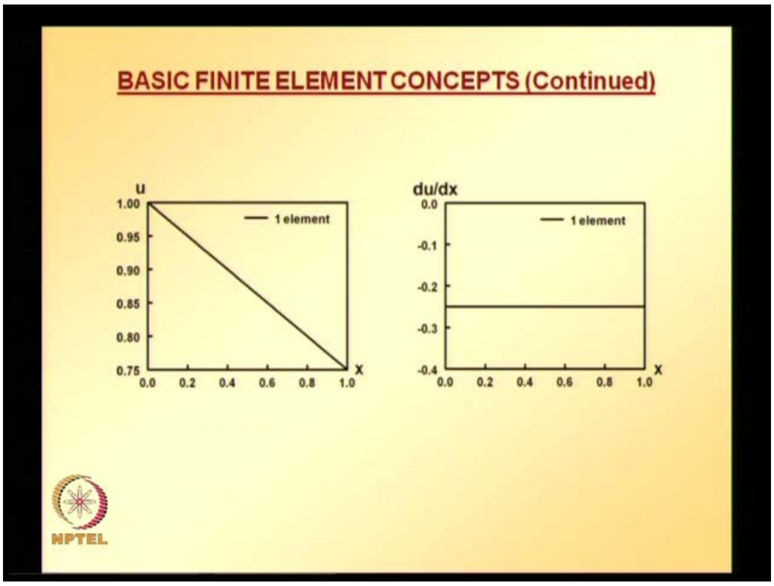


And this plot shows one element solution, and derivative, the approximation of the derivative obtained using one element.

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We do not know whether this one element is good enough or not. So, what we can do is we can increase the number of elements and see whether solution is converged or not.So, now let us try using two elements.

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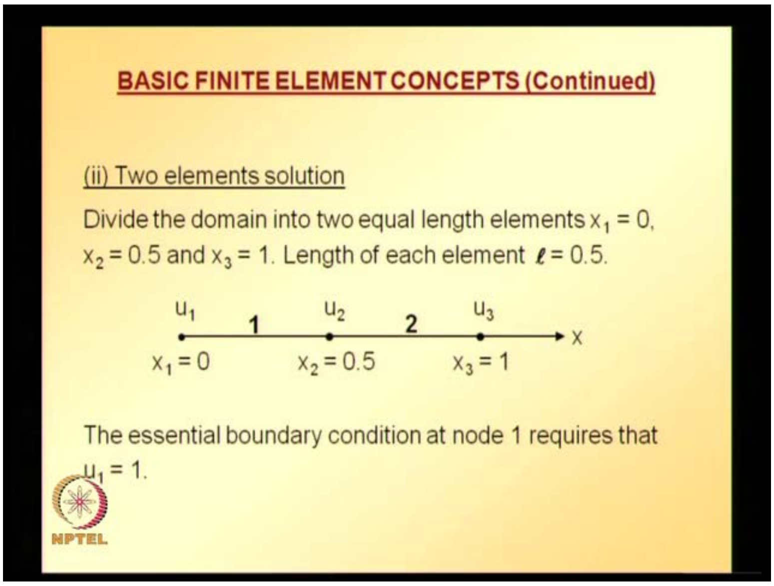

Two element solution; the solution procedure is same as for one element solution. First you need to divide or discretize the domain into two elements here because we are looking for two element solution; discretize domain into two elements, and take for simplicity, here the two elements are assumed to be of same length, it need not be. So, x the first node coincides with x is equal to 0, second node coincides with x is equal to 0.5, third node coincides with x is equal to 1, and length of each element is 0.5.

Since nodal values of the special coordinate sorry special coordinate values and the length of each element is known, we can easily write the approximate trial solution for each of these elements, and also noting that essential boundary condition is prescribed at node 1 which is u evaluated at x is equal to 0 is 1, it turns out that u 1 is equal to 1.

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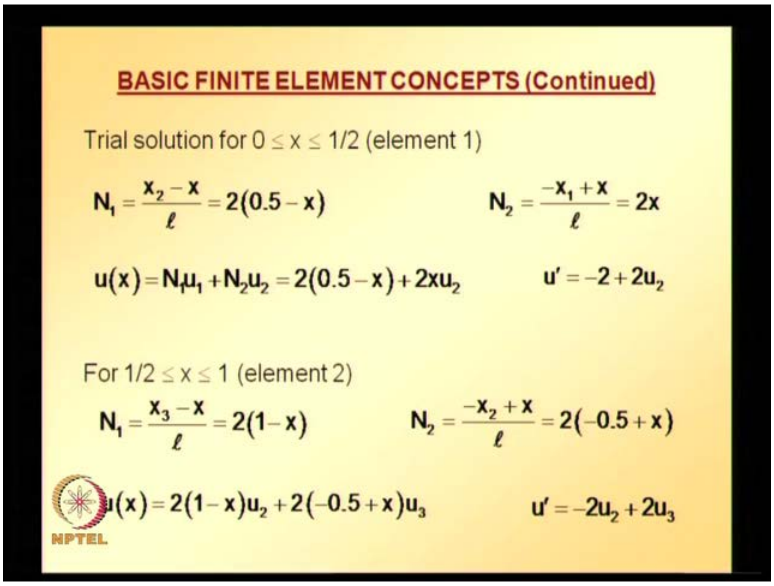


So, trial solution for element one; substitute the special coordinates corresponding to node 1, node2; here you have two elements. So, for each element, you have a locally node 1 node 2, and global node number is different from local node number. For element 1 local nodes 1 and 2 coincides with global nodes 1 and 2. So, you can plug in the corresponding special coordinates of nodes and get n 1 n 2, and once you get a n 1 n2, you can write approximate trial solution. Here u 1 value is also substituted, u 1 is equal to 1; that is already given, and derivative of u is you can easily check it turns out to be minus 2 plus 2 u 2.

And now, trial solution for element 2. For element two local node 1 coincides with global node 2, local node 2 coincides with global node 3. So, shape functions for element 1 sorry element 2, no at local node 1 local node 2 are given here, and here both u 2 u 3 the nodal values at node 2 and node 3 are unknown.

So, the trial solution turns out to be u times 1 minus x. u is equal to 2 minus 1 minus x u 2 plus 2 minus 2 times minus point 5 plus x u 3, and derivative of it is minus 2 u 2 plus two u 3. Substituting all this quantities; that is, trial solution and derivative of trial solution into the equivalent functional, here there are two elements. So, you will have two integrals; integral over the first element plus integral over the second element.

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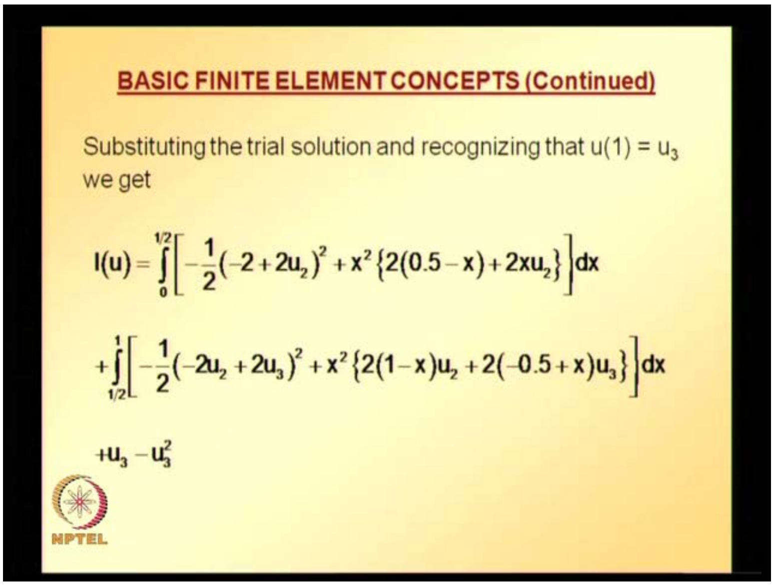


And the first element domain is from 0 to half and second element domain is from half to 1, and the last term is boundary term which corresponds to, in this particular discritization, x is equal to 1 corresponds to u 3. So, u evaluated at x is equal to 1 is u 3.

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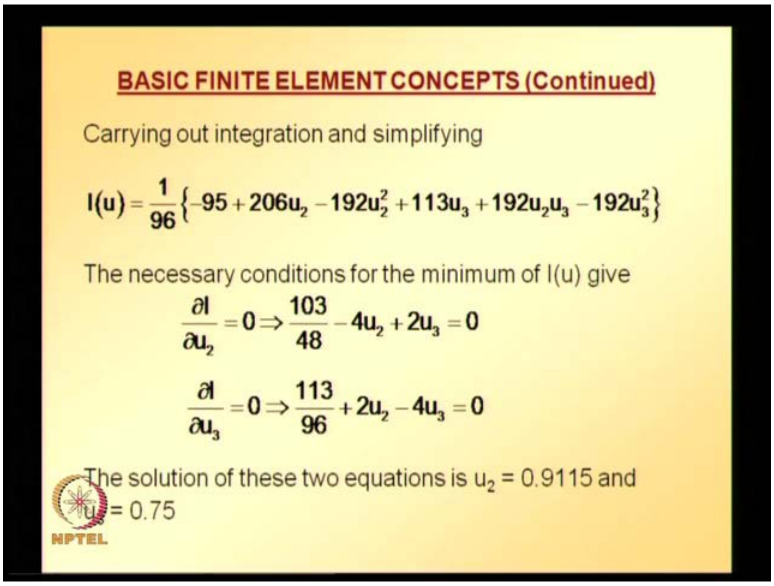


Substituting that information, u i as a function of u turns out to be this and simplifying this integral, you get i as a function of u 2 and u 3 and then applying this stationarity conditions, you get two equations, two unknowns.

First equation is partial derivative of i with respect to u two is equal to 0, and second equation is partial derivative of i with respect to u 3 is equal to 0, two equations two unknowns. Solve for these two unknowns; u 2 u 3, and solution of these equations gives u 2 is equal to 0.9115 and u 3 is equal to 0.75.

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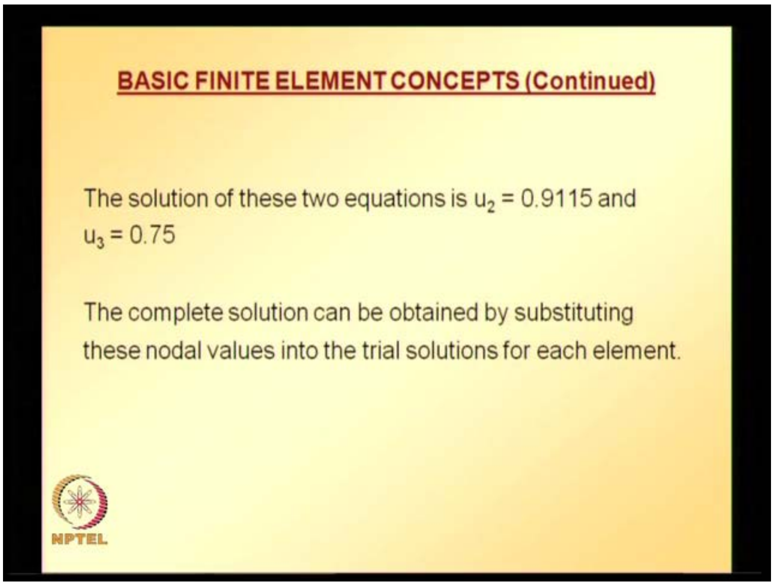

And once we get these two nodal values, we can go back to each element, complete solution can be obtained by substituting this nodal values into trial solution for each element.

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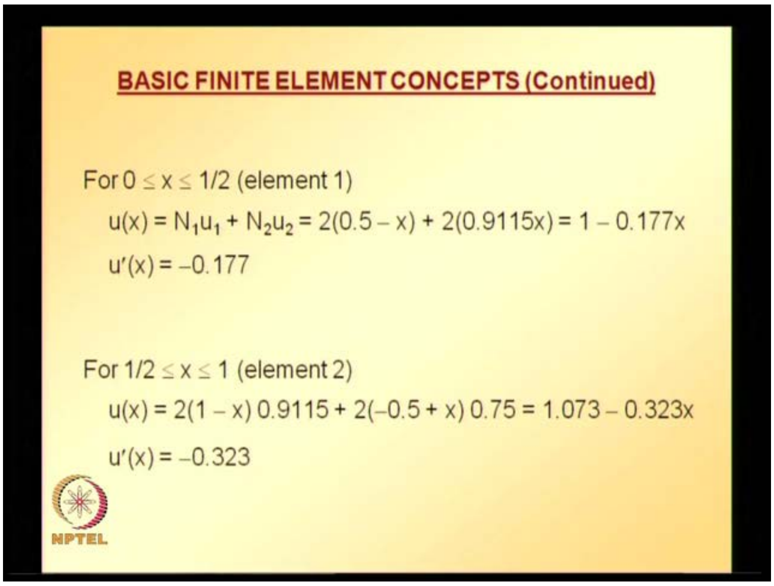


So, go to the first element. You know what is trial solution in the first element and you substitute u 1 is already substituted. Now, you got u 2 value. Substitute and simplify that gives to you approximate solution in element one, and take derivative of it you get this one, and similarly go back to element 2, substitute the nodal values of u 2 u 3, you get the approximate solution, and take derivative of it, you get (( )).

And now we got approximate solution element 1 and derivative of it in element 1, approximate solution in element 2 and derivative of it in element 2. So now, we are ready to plot.

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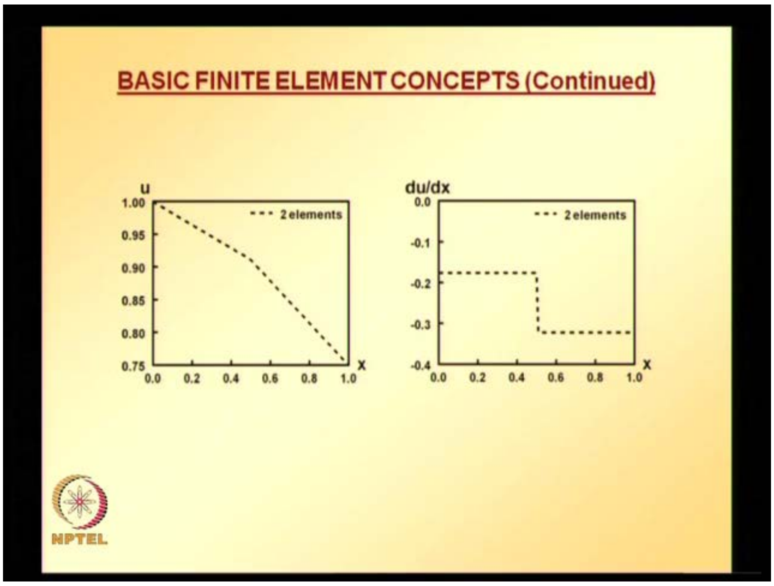


This is how solution looks using two elements and derivative of solution looks using two elements. Still we are not sure whether our solution is converged or not.

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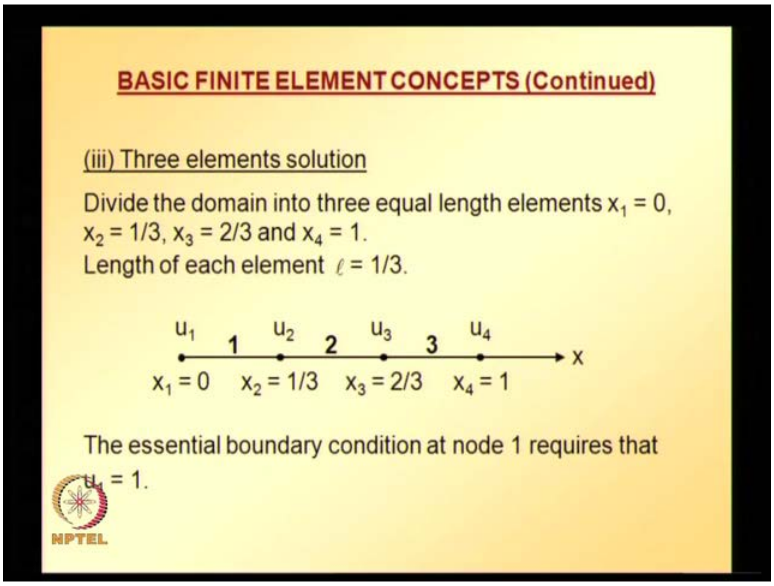


So, we can go further and use three elements. So, divide the solution the problem domain into three elements, each element having length 1 over 3 and all elements have same length and the nodal the coordinates of nodes are shown in the figure there.

Node 1 corresponds to x is equal to 0, node 2 to corresponds to x is equal to 1 over 3, node 3 corresponds to 2 over 3, node 4 corresponds to 1, and essential boundary condition is prescribed at node 1, and also natural boundary condition are u evaluated at x is equal to 1corresponds to u 4 now. Noting all these things, what you need to do is you need to go to element 1, develop the trial solution, derivative of trial solution.

Similar procedure you need to repeat for element 2 and element 3, and then substitute all this information into the equivalent functional, and apply the stationarity conditions and solve for u 2, u 3, and u 4.

Again, once you get this u 2, u 3, u 4 you go back to each element and get the trial solution and derivative of it. And whatever the task that you are performing after getting the nodal values u 2 u 3 u 4, that is what is called post processing. So, you do post processing to get to go to the each element, and then find the approximate solution in each element.

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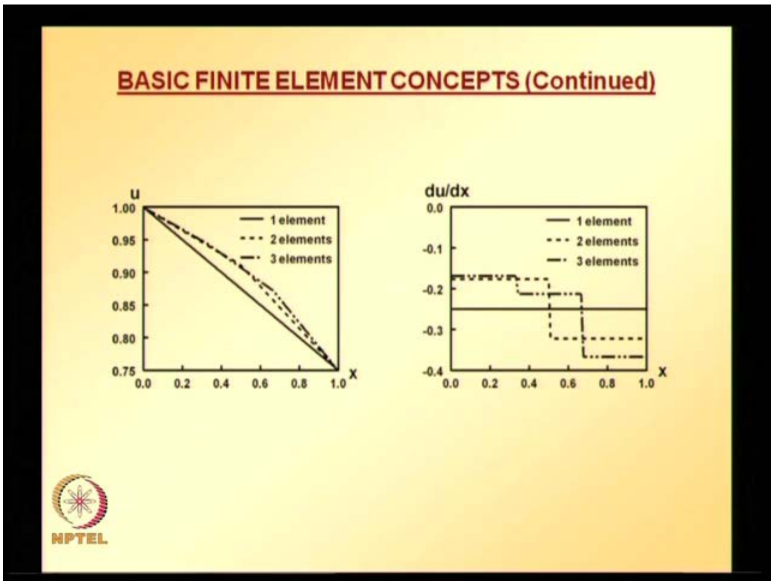


All that information, all that procedure details are not given and the plot is shown here, comparative plot of approximate solution that is obtained using one element, two element and three elements. As you can see here, when you use one element, the difference between one element and two element solution is quite large when compared to the difference between the two element solution, three element solution.

And you can also plot a comparative plot of each of this discritization for first derivative of u and that looks like that. And as you can see from these two figures, the solution is fairly converging from two elements to three elements; whereas, derivative of this approximate solution is still not converging when we go from two elements to three elements. And if you want to further reduce error, you can go for four element solution and as you can see here, as you increase number of elements, the effort that you have to put in solving the solving for the unknowns is becoming more.

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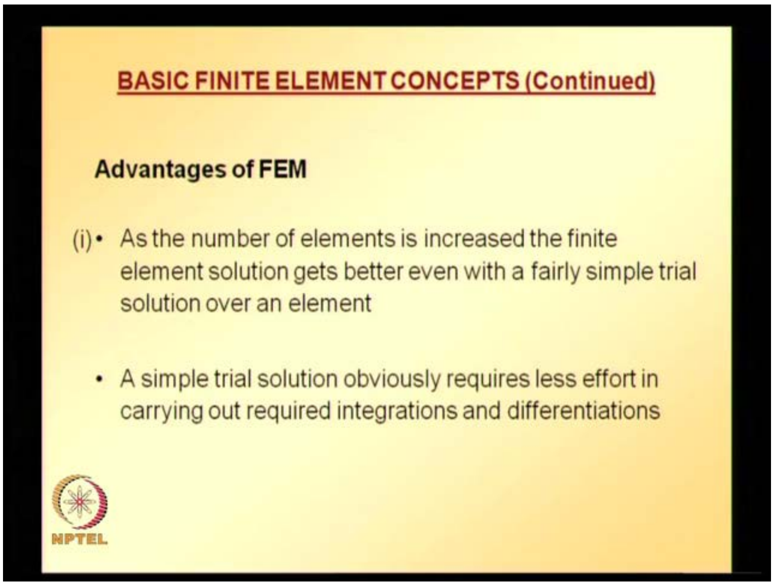


So, let us see what the advantages of finite element method are. So, what we did is we solved a problem using Rayleigh-ritz method or variational method by substituting finite element approximations of trial solutions and derivative of trial solutions, and we did convergent study, and based on the observations that we obtained by solving this or by with this experience.

The advantages of finite element are: as the number of elements is increased finite, finite element solution gets better even with fairly simple trial solution over an element.

You please remember that we have use the simplest trial solution that we can use; that is, linear trial solution, and you also observed that as the number of elements is increased, finite element solution is getting better. A simple trial solution; obviously, require less effort in carrying out the required integrations and differentiations.

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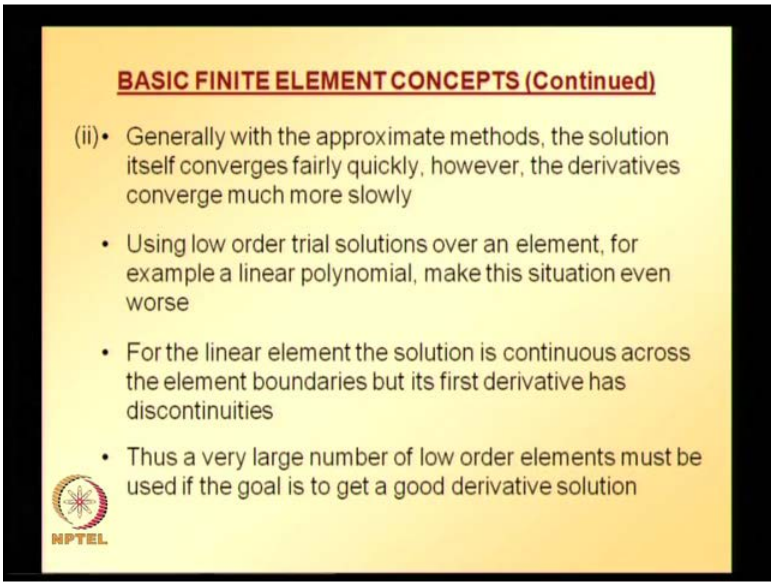


Generally with approximate methods, solution converges, solution itself converges fairly quickly; however, derivatives converge more slowly. This is what we have observed. Using lower order trial solutions over an element for a simple for example, linear polynomial makes this situation even worse.

For linear element, solution is continuous across element boundaries, but its first derivative has discontinuities, and what you need to do if you want to use good derivative approximation? You need to use large number of lower order elements.

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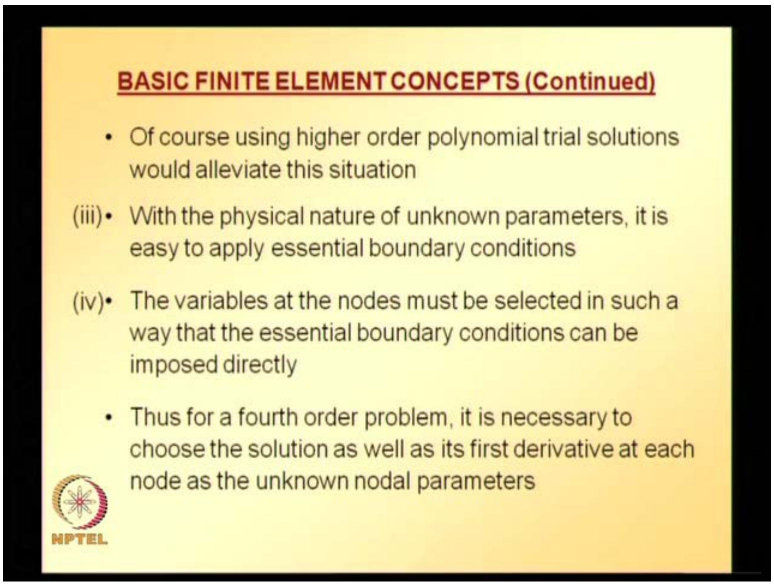

Of course, you can also use higher order polynomial trial solution to alleviate this problem. And another advantage of finite element method is you know the physical nature of unknown parameter. So, it is easy to apply the essential boundary conditions.

Variables at nodes must be selected such a way that essential boundary condition can be imposed directly, and this is one of the important points that you have to keep in mind. Suppose you have a boundary condition prescribed at a particular location, and please make sure that you have a node at that location, because unless you have a node, we cannot impose that boundary condition.

So, variables at nodes must be selected such a way that essential boundary condition can be imposed directly, and we will see this last point later; that is, for a fourth order problem it is necessary to choose solution as well as its first derivative at each node as the unknown nodal parameters.

We will continue in the next class rest of the things.