Finite Element Analysis
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Lecture No. # 39

Today's class, we look at three-dimensional elasticity problems. So far, we have looked at planar and axisymmetric elasticity problem, which are special case of these three-dimensional elasticity problems. If you recall the plane stress plane strain problems, that we looked at are special case of three-dimensional elasticity problems, and also axisymmetric problems that we looked at is also a special case of three-dimensional elasticity problems. Only thing is when structure loading, material properties satisfy certain criteria, we can model a three-dimensional elasticity problem as a two-dimensional one.

If it is plane stress plane strain case, we have three stress components, and three strain components that we actually computed at each element after solving for displacements whereas, if you look at axisymmetric elasticity problems, we computed actually four components of stresses and strains in each element or a each integration point. So, in plane stress plane strain problems, the constitutive matrix is of dimension 3 by 3; in axisymmetric elasticity problems, the constitutive matrix is of dimension 4 by 4. When the structure loading, and material properties do not permit us to make any of these assumptions of plane stress plane strain for axisymmetric case, then we need to model the entire three-dimensional problem using finite element method.

In that case, we are going to have six components of stresses, and six components of strain, and the constitutive matrix which relates stresses with strains is going to be 6 by 6. And also, if you recall for plane stress plane strain problems or axisymmetric elasticity problems, at each point there are two degrees of freedom. If it is plane stress plane strain problem, it is in the x direction, and y direction; if it axisymmetric elasticity problems, it is in the radial direction axial direction to displacements components, whereas for three-dimensional elasticity problems, we are going to have three displacement components; one in the x direction, another in the y direction, the third one in the z direction.

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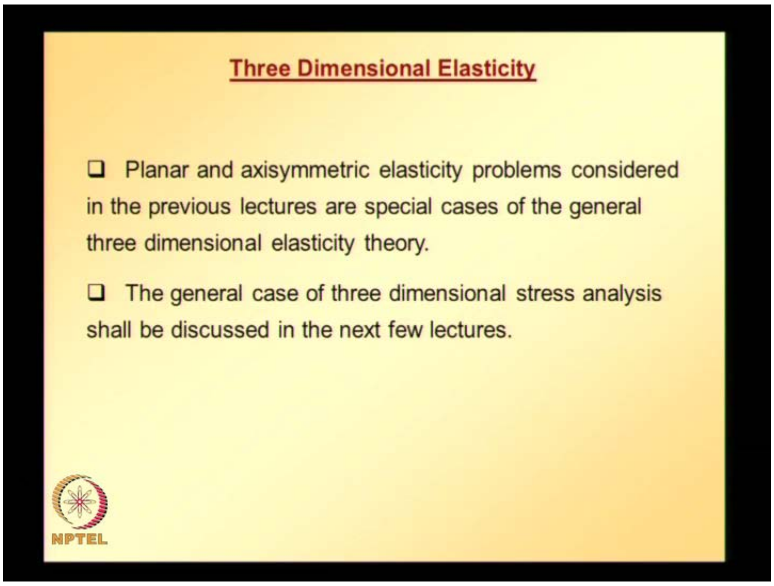

Now, let us look at three dimensional elasticity problems. Planar and axisymmetric elasticity problems considered in the previous lectures are special cases of general three dimensional elasticity theory. The general case of three dimensional stress analysis shall be discussed in the next few lectures.

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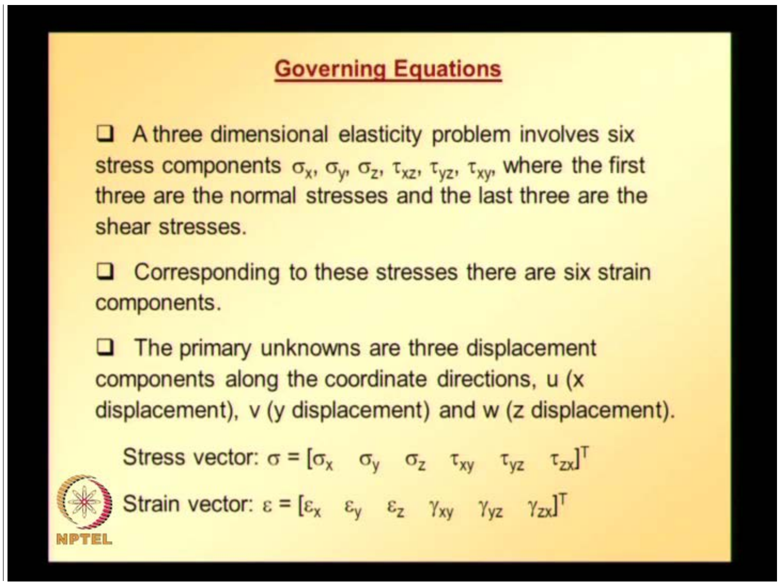

A three dimensional elasticity problem involves six stress components sigma x, sigma y, sigma z, tau x z, tau y z, tau x y, where the first three components that is, sigma x sigma y sigma z are called are normal stresses and tau x z, tau y z, tau x y or shear stresses. And corresponding to these stresses the six strain components and the primary unknowns are three displacement components along three coordinate directions, that is the along x direction, which is denoted with letter u; along y direction, which is denoted with letter v; along z direction, which is denoted with letter w. So, these are the three displacement components u v and w.

There are six stress components and all this six stress components can be put together in a vector denoted with sigma as shown here and strain vector consists of six strain components, all the six components of strain can be put together in a vector denoted with letters epsilon in this manner. Now, we need to know how the strains are related to displacements, as we already looked at for planar and axisymmetric elasticity problems under small displacements and strain assumptions.

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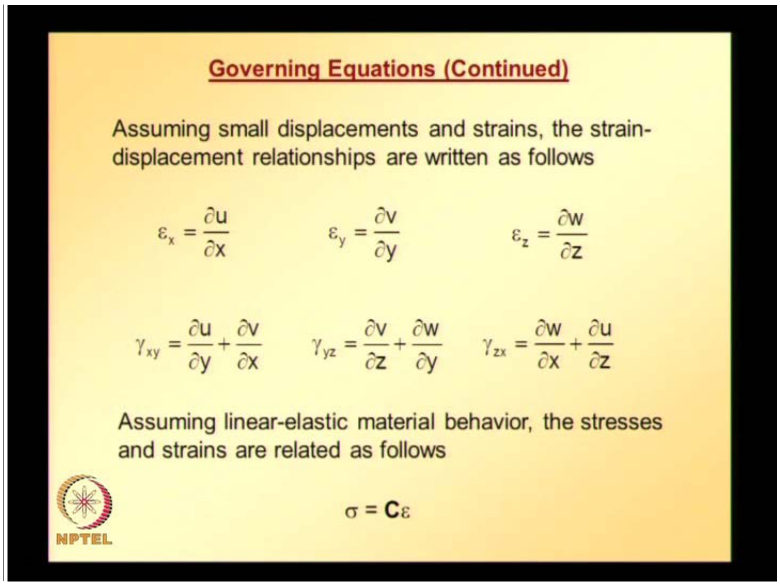

Strain displacement relations, for three dimensional elasticity problems can be written like this, that is epsilon x is partial derivative of u with respect to x, epsilon y is partial derivative of v with respect to y, epsilon z is partial derivative of w with respect to z, gamma x y is partial derivative of u with respect to y plus partial derivative of v with respect x, gamma y z is partial derivative of v with respect to z plus partial derivative w with respect to y, gamma z x is partial derivative of w with respect x plus partial derivative of u with respect to z. Assuming linear elastic behavior, the stresses and strains are related through this equation sigma is equal to C times epsilon, where c is the constitutive matrix, which is going to be of dimensions 6 by 6 since, there are six stress components and six strain components.

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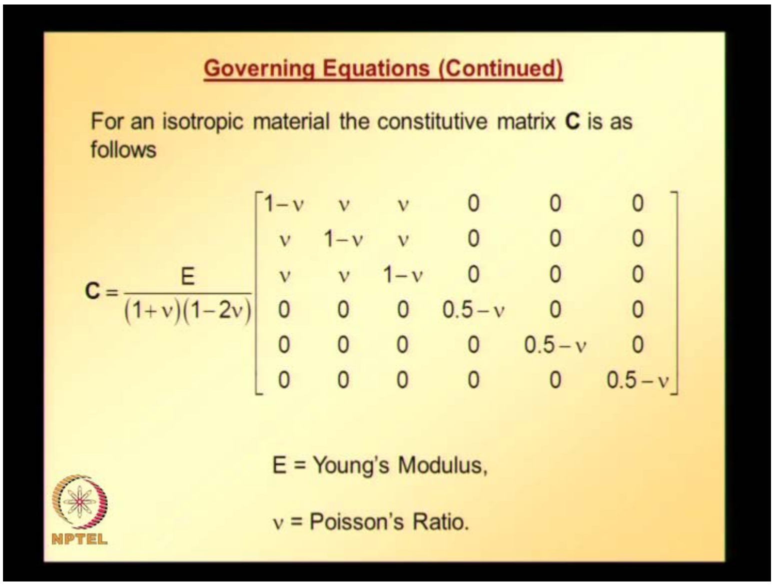

For an isotropic linear elastic material is constitutive matrix is given by this one, where E is Youngs modulus, mu is Poissons ratio. So, constitutive matrix requires two material parameters Youngs modulus and Poissons ratio.

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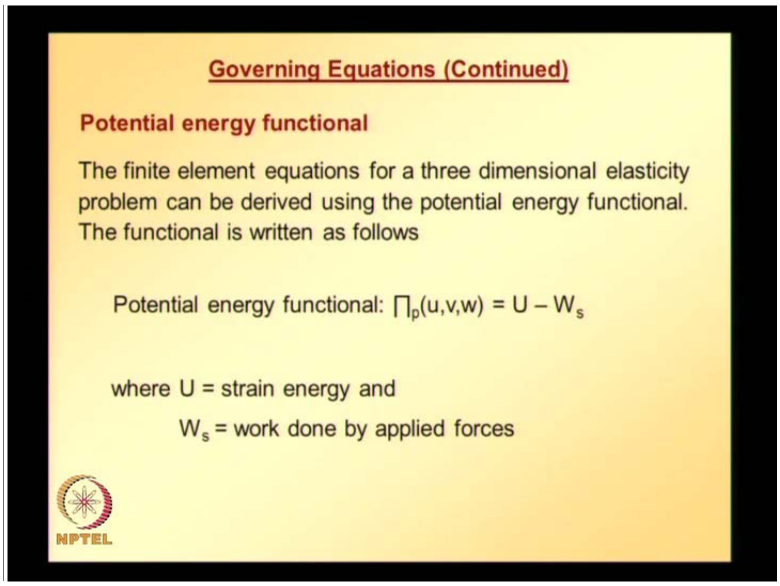

To derive finite element equations, we require potential energy functional. The finite element equations for three dimensional elasticity problems can be derived using potential energy functional. This potential energy functional can be written as follows. Potential energy functional is denoted with letter pi. Pi is equal to u minus w s, where u is strain energy and w s is work done by the applied forces. This potential energy functional is going to be function of all the three displacement components that is u v and w. This is similar to plane stress plane strain problems or axisymmetric elasticity problem that we already looked at except that potential energy functional, there is function of only displacement component along two directions. So, now let us look at how strain energy looks like.

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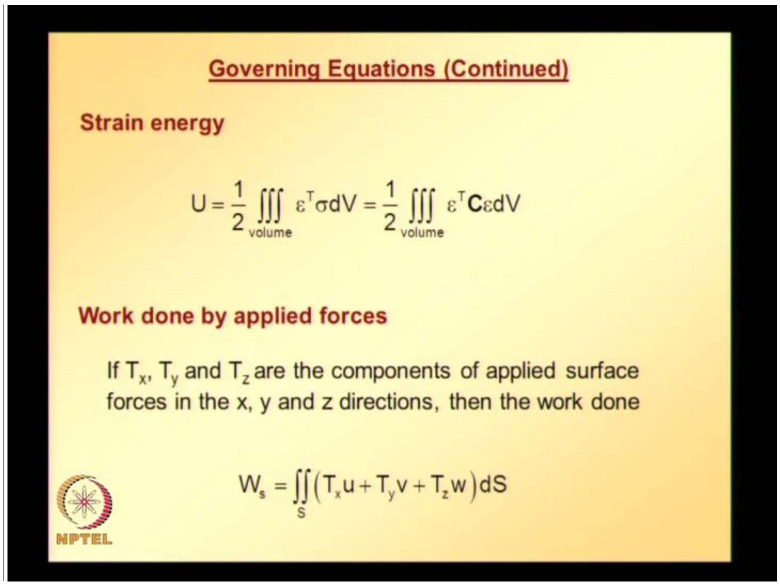

Strain energy can be calculated by evaluating this volume integral half volume integral, epsilon transpose sigma, substituting epsilon is equal to c times, sigma is equal to c times epsilon, this can be further written as half volume integral epsilon transpose c epsilon. Please note that this formula can be used only under linear elastic assumptions or small displacements and strain assumptions. Now, work done by the applied forces is given by components of traction along x y z directions multiplied by the corresponding displacements and evaluated over the surface over, which fraction components are applied. So, work done by the applied forces is given by this surface integral.

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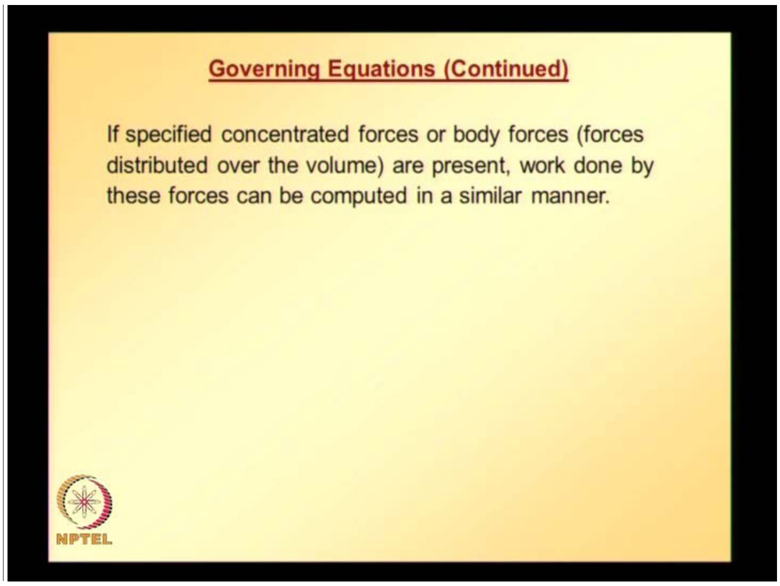

And if the specified concentrated forces or body forces are present, work done by these forces can be calculated in a similar manner. So, we looked at how to evaluate strain energy and also how to evaluate work done by the applied forces. We are ready to formulate or we are ready to derive finite element equations for three dimensional elasticity problem. Before, that we need to make a choice of the type of element that we are going to use for solving three dimensional elasticity problems. Here, basically as a part of these three dimensional elasticity problems, we are going to look at four node tetrahedral element, which is basically a linear element for solving three dimensional elasticity problems and also we look at eight node linear solid element commonly known as brick element and 20 node solid element with curved edges.

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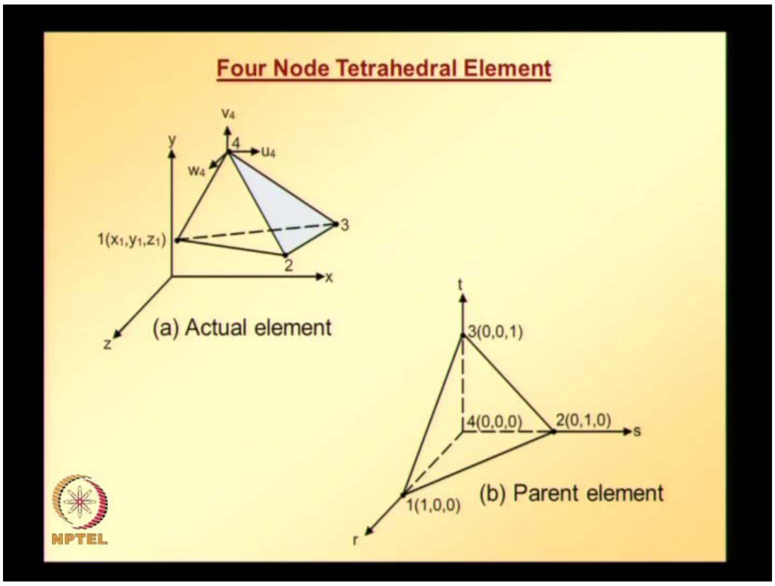

In this lecture, let us start with four node tetrahedral elements, a typical element is shown here. Four node tetrahedral elements, at each node there are three degrees of freedom displacement component along x direction y direction z direction. Total there are four nodes, this is the simplest element that we can have for solving three dimensional problems. This element is going to be linear along x direction y direction z direction, so it is counter part of three node triangular element, which is the linear element for solving two dimensional problems. Similar, to three node triangular element, which we usually called it as c s t, constants strain of triangular element.

Similarly, four node tetrahedral elements is a constant strain element in three dimensions. It is convenient to use isoparametric mapping concept to develop finite element equations in a manner, similar to that we already used for triangular elements for two dimensional cases. We need to map this actual element on to a parent element like this; parent element is shown on the right hand side. The nodal coordinates of all the four nodes of parent element are indicated in the figure. Node one is located at 1, 0, 0, node two is located at 0, 1, 0, node three is located at 0, 0, 1, node four is located at the origin 0, 0, 0 in r, s, t coordinate system. So, actual element is going to be mapped on to this parent element, when we are deriving element equations.

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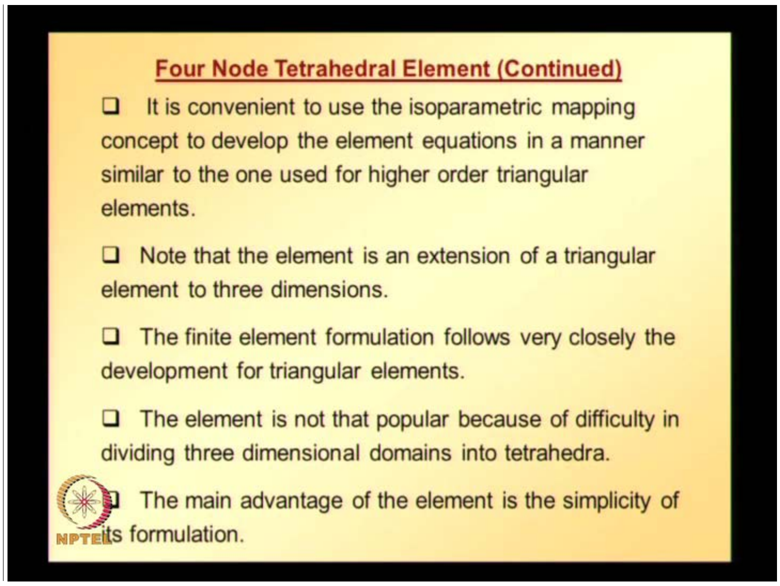

So, this four node tetrahedral element is an extension of linear triangular element, which we already looked at for two dimensional problems to three dimensions extension of triangular element to three dimensions. So, the finite element formulation follows very closely the development of triangular elements. However, this element is not that popular; because of difficulty in dividing three dimensional domains into tetrahedra and also main advantage of this element is it simplicity in its formulation. So, now let us look at how to derive shape functions for this four node tetrahedral element and the procedure that we are going to adopt the similar to that we already adopted for two dimensional problems like triangular elements in case of two dimensions.

So, we start with a polynomial having number of coefficients equal to the number of nodes of the particular element and by substituting the trial solution values at the nodes we solve for the coefficients and substitute back this coefficients into the trial solution and group terms having similar coefficients, which corresponds to the nodal values and then we are going to arrive at finite element shape functions.

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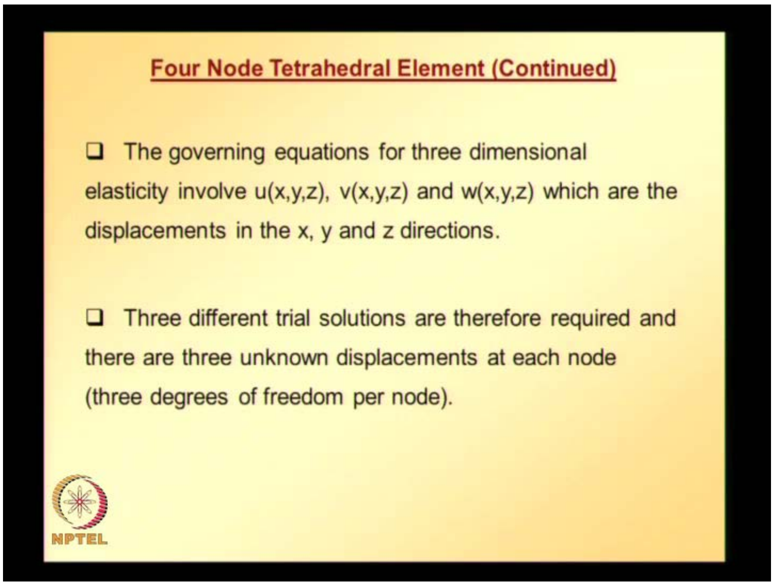

The governing equations for three dimensional elasticity problems involve u, v, w, which are displacements along x, y, z directions. Three different trial solutions are therefore required and there are three unknown displacements at each node three degrees of freedom per node.

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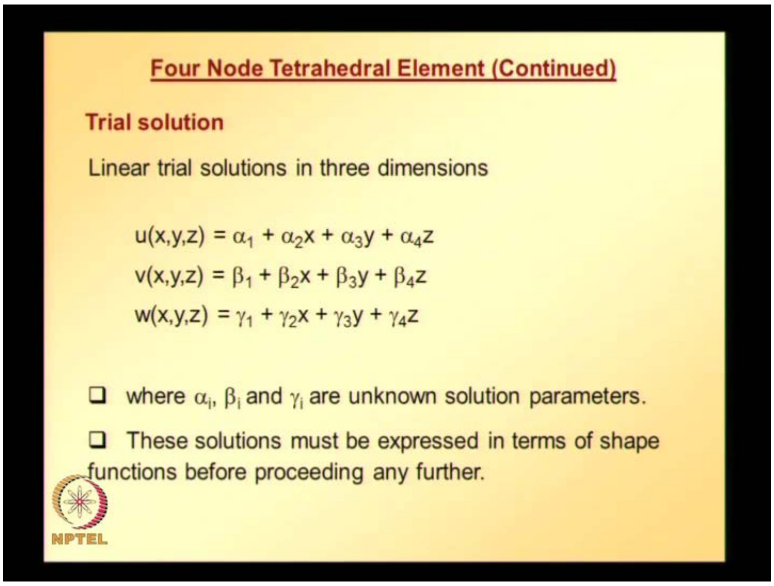

So, trial solution there are four nodes, linear trial solution in three dimensions u is equal to alpha 1 plus alpha 2 x plus alpha 3 y plus alpha 4 z. Similarly, v is equal to beta 1 plus beta 2 x plus beta 3 y plus beta 4 z. W is equal to gamma 1 plus gamma 2 x plus gamma 3 y plus gamma 4 z, where this alphas betas and gammas are unknown solution parameters and these solution parameters must be expressed in terms of finite element shape functions before proceeding any further.

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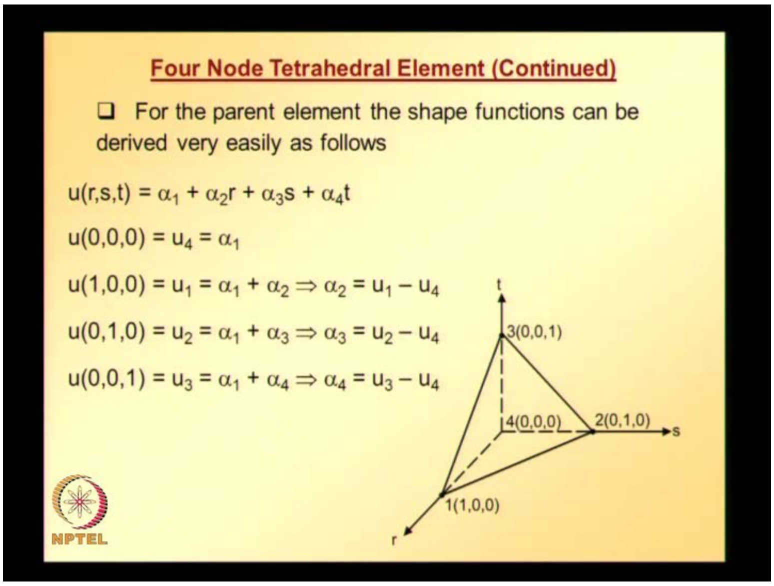

What we will do is, we will adopt procedure that we already familiar with, for the parent element the shape functions can be derived very easily as follows to start with the trial solution, substitute the nodal value of the trial solution corresponding to node four substitute r is equal to zero, s is equal to zero, t is equal to zero and equate the trial solution value to the value of the displacement or the value of the nodal value at node four. Similarly, trial solution evaluated at node 1 is equal to u 1, applying that condition, we are going to get the second equation in terms of alpha 2 and similarly applying the condition the trial solution evaluated at node 2 is going to be u 2, we are going to get the third equation and similarly, we get the fourth equation by applying the condition the trial solution value evaluated at node 3 is equal to u 3. Finally, we got four equations in terms of alpha 1 alpha 2 alpha 3 alpha 4.

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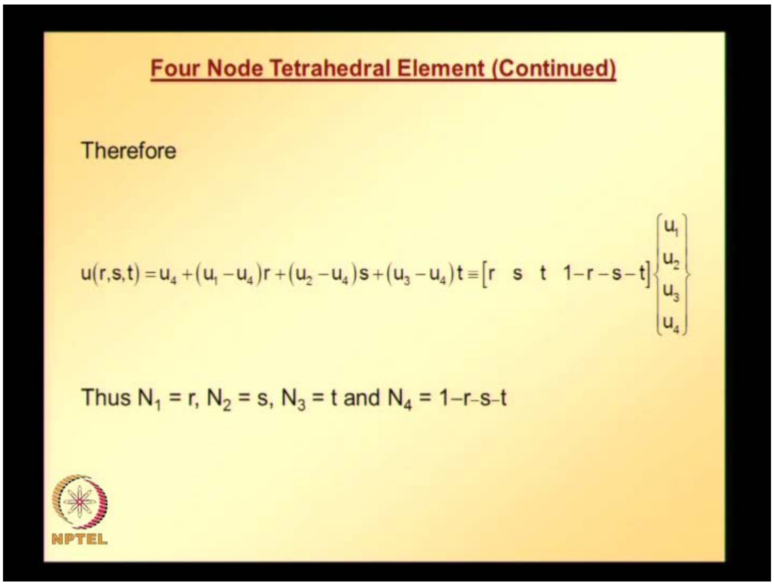

Substituting back this alpha 1 alpha 2 alpha 3 alpha 4 values into the trial solution, we get this, which can be written in a matrix and vector form as shown in the right hand side of the given equation. So, shape function corresponding to node 1 is r, shape function corresponding to node 2 is s, shape function corresponding to node 3 is t and shape function corresponding to node 4 is 1 minus r minus s minus t. So, this is how we can arrive at the shape functions for four node tetrahedral element. Basically, the procedure is similar to that we already adopted for deriving shape functions for three node triangular element, except now the dimension is increased.

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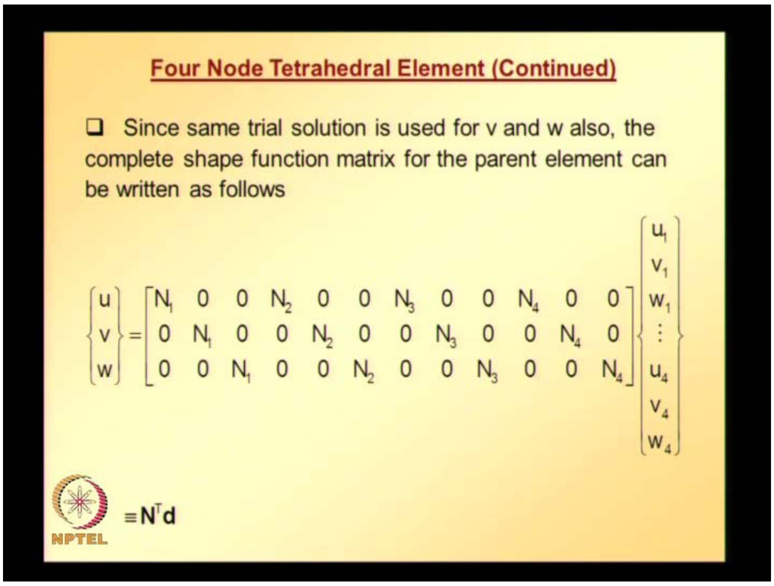

We can also solve starting with u and w, since same trial solution is used for v and w, also complete shape function matrix for the parent element can be written as follows, u expressed in terms of finite element shape functions, v in terms of finite element shape functions, w expressed in terms of finite element shape functions putting together all these things in a matrix and vector form, where we can write like this, where n transpose is matrix comprising of finite element shape functions of all the four nodes of this tetrahedral element and d is the vector consisting of nodal displacements, all the three components of displacements at each node for all the four nodes.

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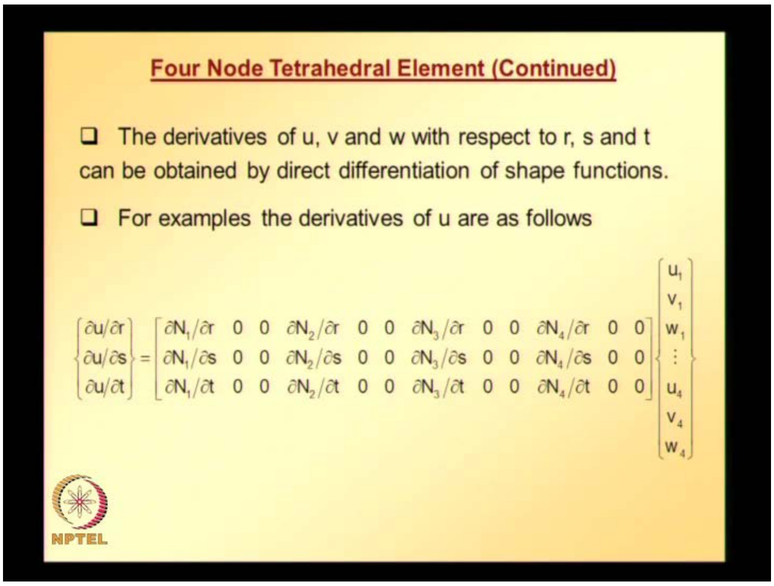

So, the derivatives of u, v, w with respect to r, s, t can be obtained by direct differentiation of shape functions. Please note that N 1 is equal to r, N 2 is equal to s, N 3 is equal to t, N 4 is equal to 1 minus r minus s minus t. So, we can easily calculate what is the derivative of this N 1, N 2, N 3, N 4 respect to r, s, t. So, derivatives of u, v, w with respect to r, s, t can be easily obtained once we know the derivatives of shape function with respect to r, s, t, which we can easily get since we already know the expressions for shape functions. For examples, derivative of u can be evaluated as follows, partial derivative of u with respect to r, partial derivative of u with respect s, partial derivative of u with respect to t can be written in a matrix and vector form like this, if you put together all the partial derivatives of u with respect to r, s, t in a vector as shown there.

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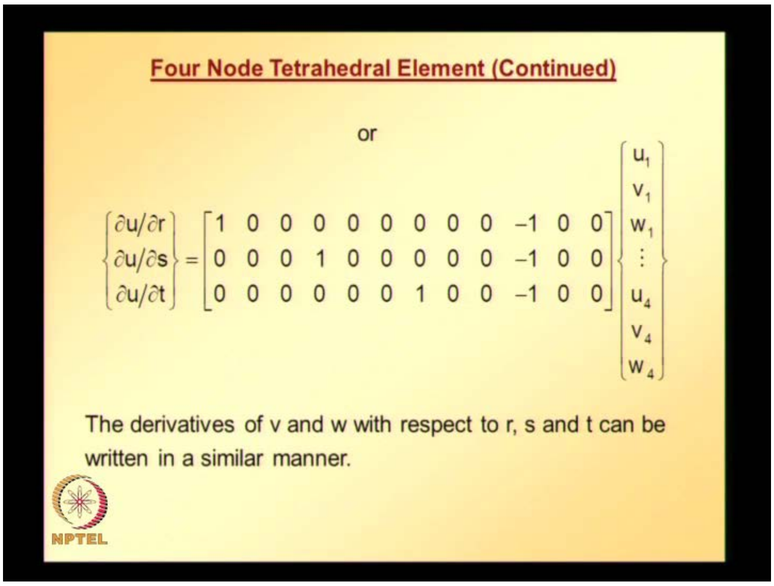

Taking partial derivatives of shape function with respect to r, s, t. It can be easily checked that, in fact partial derivatives of u with respect to r, s, t is given by this one. Similarly, derivatives of v and w with respect to r, s, t can be derived. So, this is how derivatives of displacement components with respect to r, s, t can be obtained, but if you recall strain is related to derivatives of displacement components with respect to x, y, z. We need to know what is the relationship between x, y, z coordinates and r, s, t before we start derive or before we start expressing strains in terms of finite element shape functions and displacements.

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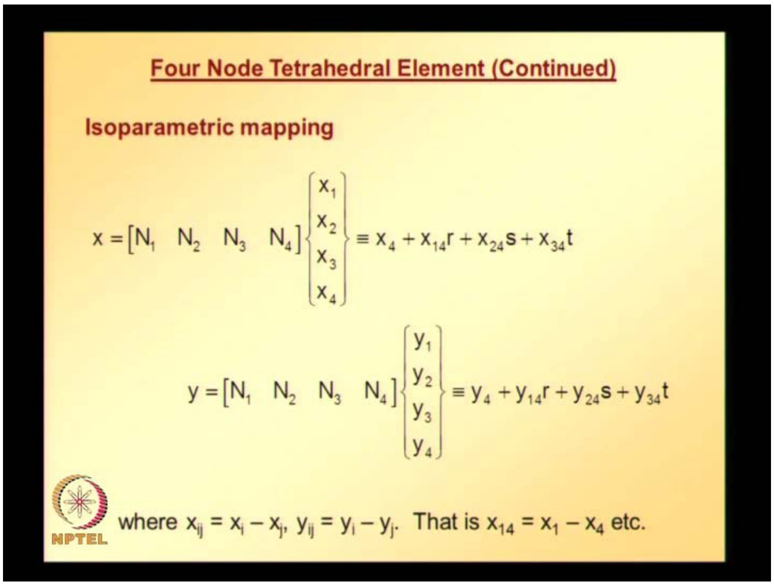

To know the relations between r, s, t and x, y, z, we can use isoparametric mapping concept. Isoparametric mapping concept based on that x is equal to N 1, x 1 plus N 2, x 2 plus N 3, x 3 plus N 4, x 4, substituting N 1, N 2, N 3, N 4, that is N 1 is equal to r, N 2 is equal to s, N 3 is equal to t, N 4 is equal to 1 minus r minus s minus t, substituting that information. This expression can further be simplified as shown on the right hand side, where x 1 4 means, x 1 minus x 4, x 2 4 denotes x 2 minus x 4, x 3 4 denotes x 3 minus x 4. Similarly, we can write a y in terms of r, s, t, where x i, x j has the meaning x i minus x j, y i, j has meaning y i minus y j. Similarly, we can also express z in terms of r, s, t using finite element shape functions of this four node tetrahedral element and when we do that we get this.

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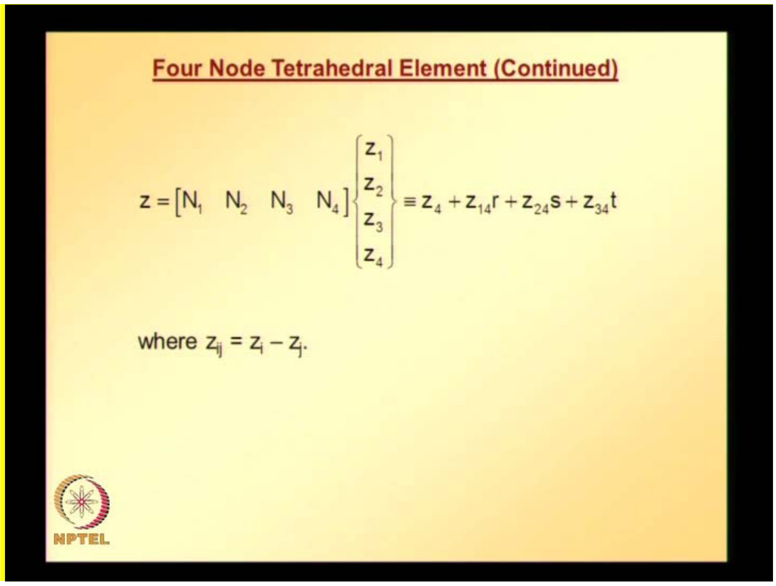

So, now we know how to express x, y, z in terms of r, s, t. So we can easily take what is partial derivative of x with respect to r, s, t; y with respect to r, s, t; z with respect to r, s, t and we already know how to calculate derivatives of displacement components with respect to r, s, t. So, we using both these information and using chain rule of differentiation, we can easily arrive a derivatives of displacement components with respect x, y, z. Once we know derivatives of displacements with respect to r, s, t and finally, we can write the strain displacement relations in a relationship in terms of finite element shape functions and the nodal displacement components at the nodes.

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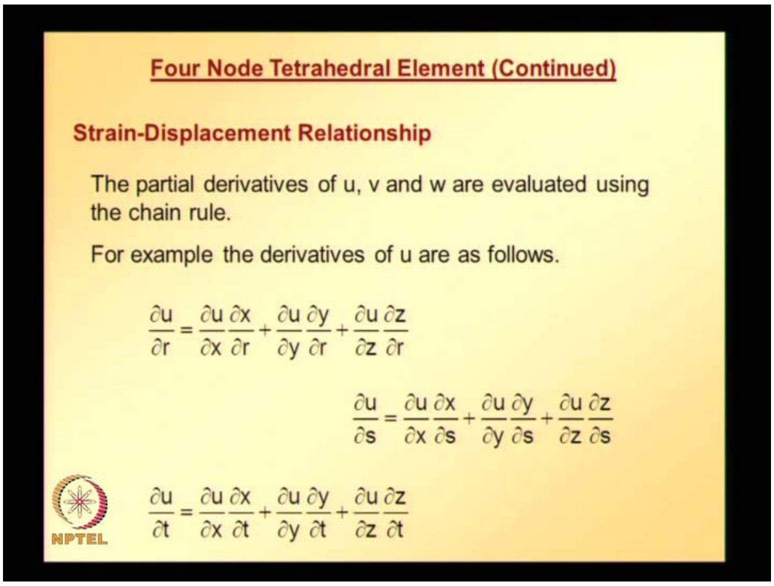

So, strain displacement relationship partial derivatives of u, v, w are evaluated using chain rule. Here derivatives of u alone are shown, partial derivative of u with respect to r can be written as partial derivative of u with respect x times partial derivative of x with respect to r plus partial derivative of u with respect to y times partial derivative of y with respect to r plus partial derivative of u with respect to z times partial derivative of z with respect to r. Similarly, we can write partial derivative of u with respect to s, partial derivative of u with respect t.

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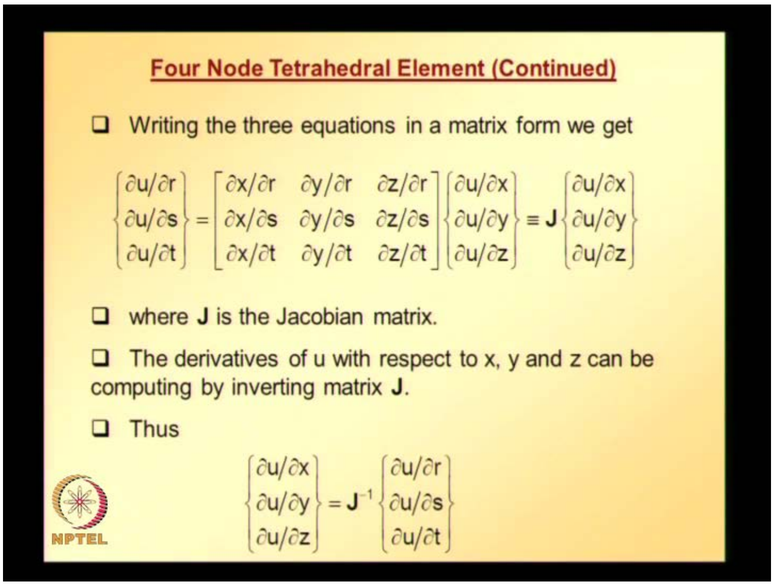

These three equations can be put together in a matrix and vector form. Writing three equations in a matrix form we get this relation, where j is used to denote partial derivatives of x with respect to r, s, t, where j is used to denote components of partial derivative of x with respect to r, s, t, y with respect to r, s, t, z with respect to r, s, t or a matrix consisting of partial derivatives of x with respect to r, s, t, y with respect to r, s, t, z with respect to r, s, t is denoted with letter j, which is called Jacobian matrix. This equation basically helps us to calculate partial derivatives of displacement component along x direction with respect to r, s, t. Once we know it is derivative with respect to x, y, z, if you want to know how to calculate partial derivatives of displacement component u along x, y, z directions.

Once, we know it is partial derivatives with respect to r, s, t, we need to know what is j inverse derivatives of u with respect to x, y, z can be computed by inverting j matrix. So, the inverse relation is like this. Since, we already know x in terms of r, s, t; y in terms of r, s, t; z in terms of r, s, t we can easily take partial derivatives and express Jacobian matrix in terms of x coordinate x, y, z coordinates. Similarly, once we express Jacobian matrix in terms of x, y, z coordinates of all the four nodes, we can easily find what is j inverse.

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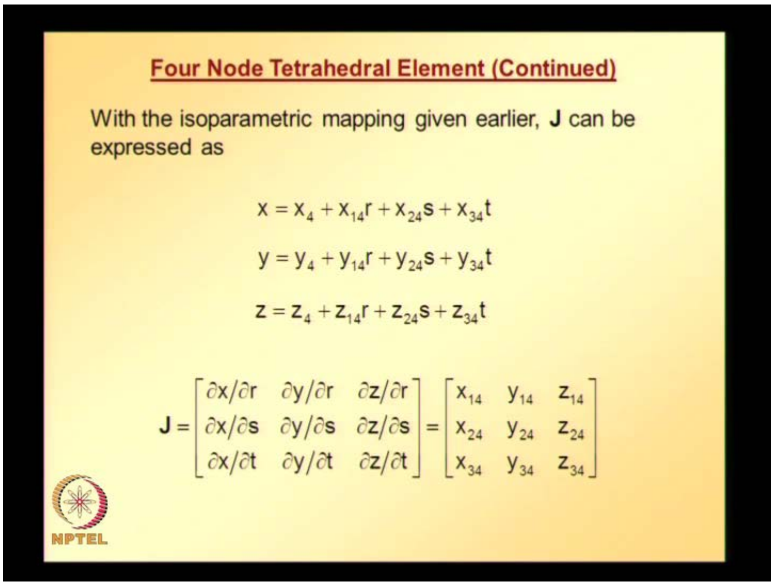

So, taking derivatives of the relations that we derived based on isoparametric mapping. These are the three equations that we derived based on isoparametric mapping taking partial derivatives of these with respect to r, s, t. We can easily check that Jacobian matrix is indeed given by this where x 1 4, 2 4, 3 4, y 1 4, 2 4, 3 4, z 1 4, 2 4, 3 4 has meaning that we already discussed earlier that is x i j is nothing but, x i minus x j, y i j is y i minus y j, z i j is z i minus z j. With that understanding, we can easily once we know all the coordinates of all the four nodes of this tetrahedral element we can easily calculate, what is j numerical value? Once we have numerical value of j, we can easily find what is j inverse?

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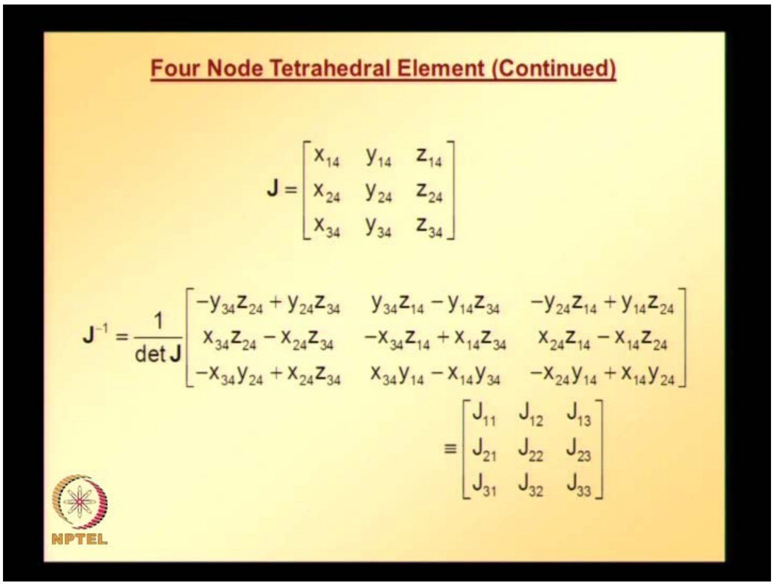

So, j is given by this which is going is a 3 by 3 matrix, so we can easily find what is j inverse analytically in this manner. So, once we know the nodal coordinates for all the four nodes of tetrahedral element, we can easily plug in that into this equation and get what is j inverse and here determinant of j is given by this equation.

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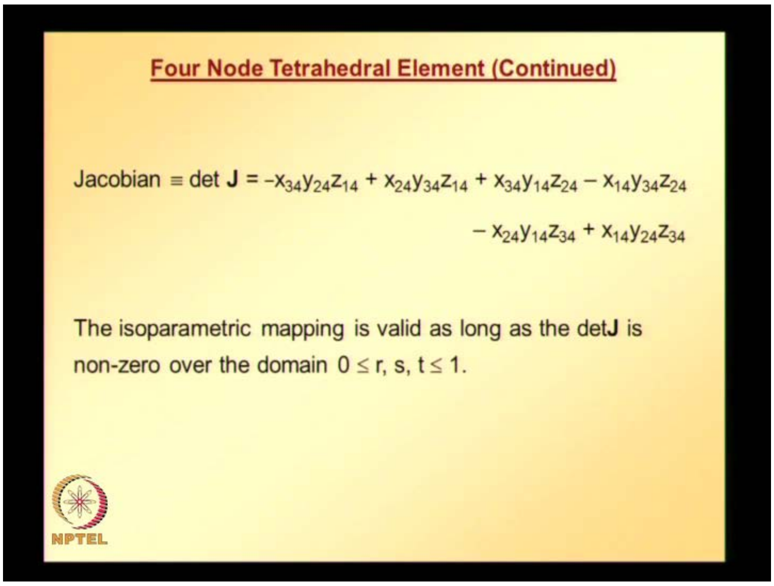

If you recall determinant of j has some importance for a valid isoparametric mapping determinant of j should be non zero over the entire element domain. Here, we are dealing with four node tetrahedral elements, so element domain goes from r, s, t going from 0 to 1. So, the isoparametric mapping is valid as long as the determinant of j is non zero over the domain r, s, t between 0 and 1, entire domain of parent element. So, we are ready to express derivatives of displacement components with respect to x, y, z in terms of nodal values.

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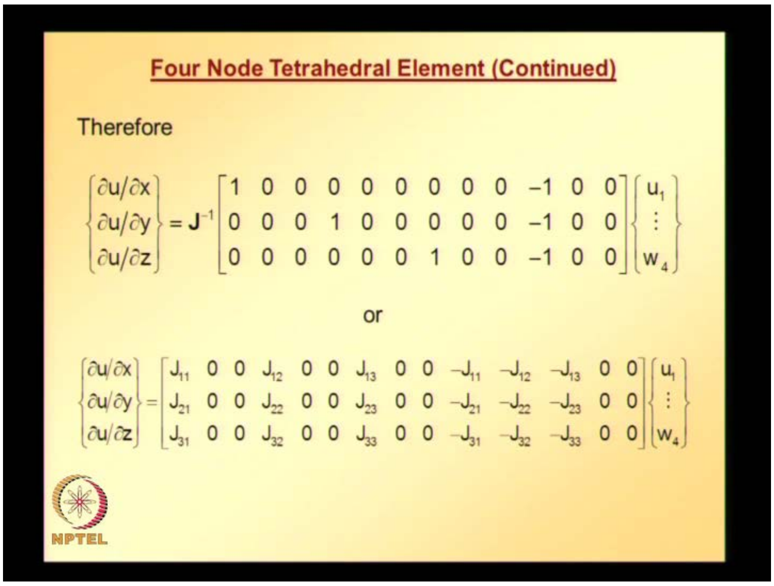

This equation gives the relationship between partial derivatives of u with respect x, y, z and substituting all the information that we just derived. This equation, which relates partial derivative of u with respect x, y, z with the nodal displacement components for all the four nodes of this tetrahedral element and plug in the components of j inverse and simplifying we get this equation. Similarly, we can derive equations for partial derivatives of p with respect x, y, z; partial derivatives of w with respect to x, y, z and they look like this.

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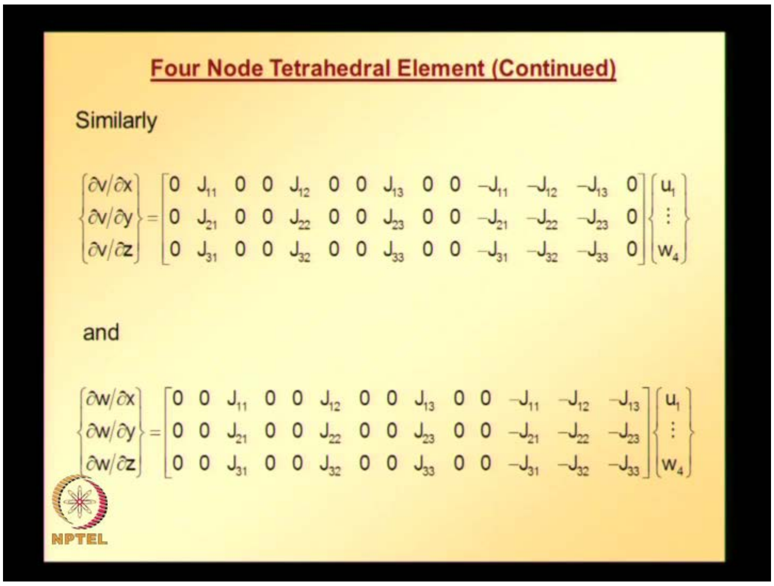

So, now we are ready to express strains in terms of nodal displacements.

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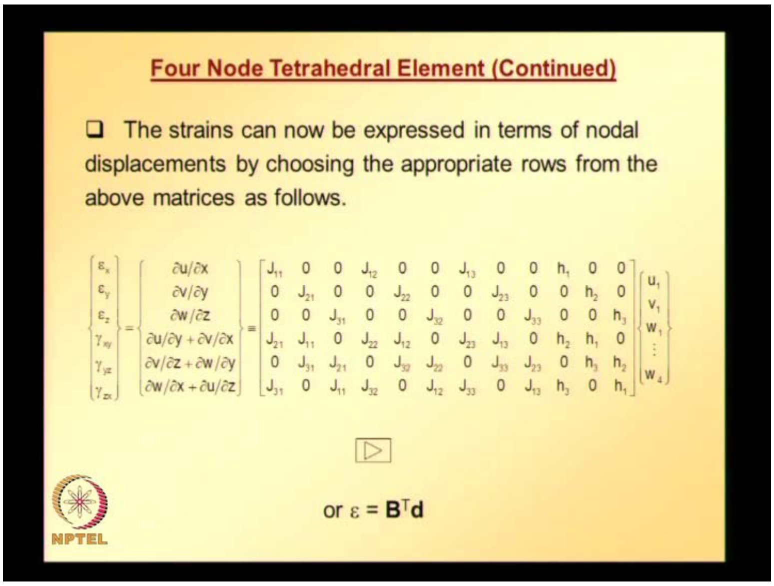

Strains now can be expressed in terms of nodal displacements by choosing appropriate rows from the above matrices. Since, strain components are defined like this, so putting together all the strain components the corresponding definitions that is epsilon x is equal to partial derivative of u with respect x, epsilon y is equal to partial derivative of v with respect to y, epsilon z is equal to partial derivative of w with respect to z, gamma x y is partial derivative of u with respect to y plus, partial derivative of v with respect to x and so on putting together all that information and substituting the partial derivatives of displacements with respect x, y, z. We get the right hand side, which relates strains with nodal displacements and which can be compactly written as epsilon is equal to B transpose d.

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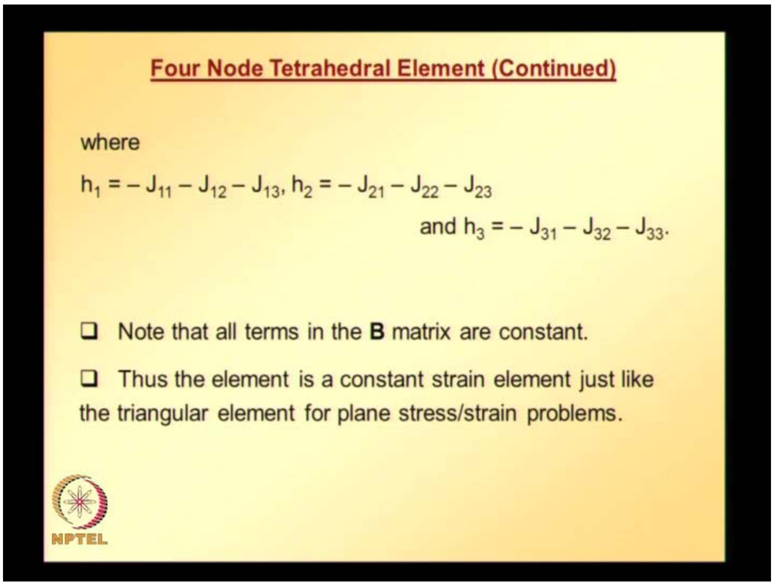

And here h 1, h 2, h 3 are defined and this h 1, h 2, h 3 are basically they are related to the components of inverse of Jacobian through these equations. If you see this strain displacement relation, it can be easily verified that all terms in the B matrix are constant. So, this element is constant strain triangular element just like three node triangular element for plane stress plane strain problems.

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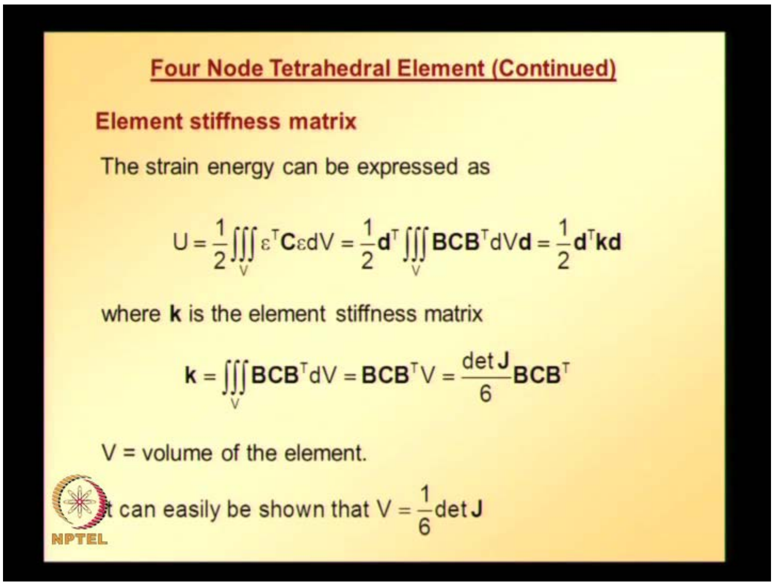

So, we express strains in terms of displacements nodal displacements, so now we are ready to derive element stiffness matrix, for that we need to go back to the strain energy definition. Strain energy can be written or can be expressed through this equation; substituting epsilon is equal to B transpose d, that information into this equation. We can evaluate strain energy through finite element approximation as half d transpose k d, where k is element stiffness matrix and it is given by this volume integral B C B transpose. Since, B is constant and since constitutive matrix is also constant, we can pull them out of the integral, finally k is given by B C B transpose times v, v is nothing but volume of tetrahedral element.

If you recall, when we are deriving finite element equations for three node triangular element, area of triangle are determinant of j is twice the area of triangle. Similarly, determinant of j here for four node tetrahedral element is six times volume of the tetrahedral element or volume is given by 1 over 6 determinant of j, substituting that we can evaluate element stiffness matrix using this equation. Please note that here the assumption or this integral is simplified in this manner based on the condition that B is not constant. We need to adopt some kind of numerical integration like Gaussian Quadrature, v is the volume of element and it can be easily shown that volume of element is one sixth of determinant of j.

So, this is how one can evaluate element stiffness matrix for four node tetrahedral elements. So we are ready, except that we do not know what is force vector?

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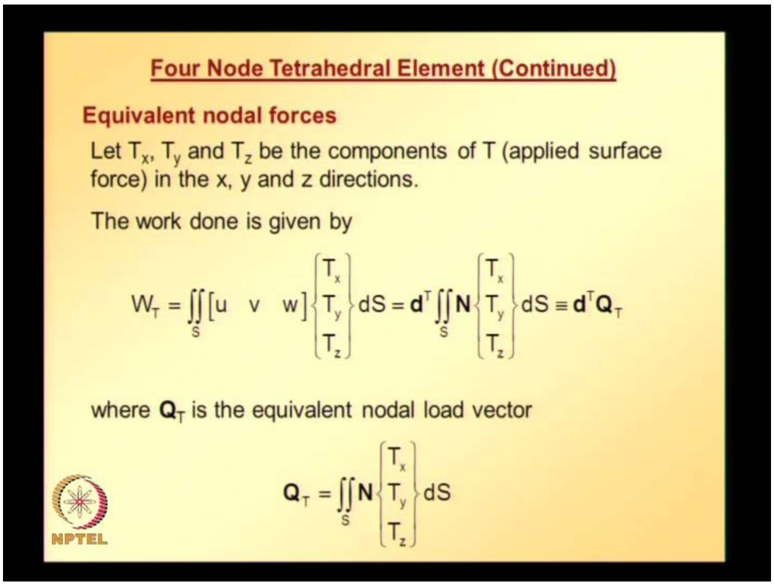

So, now let us look at equivalent force vector. Equivalent nodal forces, let T x, T y, T z with the components of fraction applied in x, y, z directions. So, work done is given by this displacement components times traction components integrated over the surface over which tractions are specified and substituting displacement, in terms of finite element shape functions displacements u, v, w, in terms of finite element shape functions and nodal values. We get what is given on the right hand side of this equation, which can be further simplified as d transpose Q T, where q is the equivalent nodal load vector defined as surface integral matrix consisting of finite element shape functions times the components of tractions.

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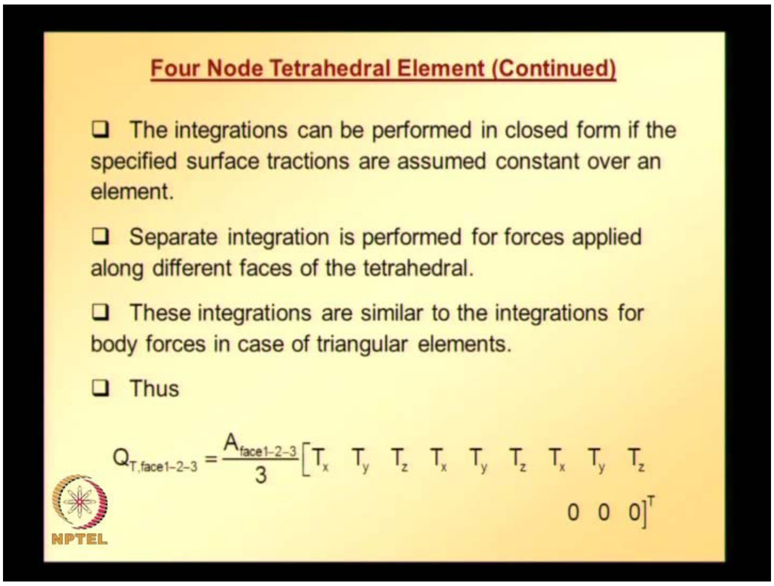

To evaluate this Q T integration can be performed in a closed form, if specified surface tractions are assumed to be constant over the element and this is for one face of the tetrahedral element and similar integrations needs to be perform separately for forces applied along different faces of tetrahedral elements. These integrations are similar to the integrations for body forces in case of triangular elements. So, substituting N matrix comprising of finite element shape functions if all the four nodes of this tetrahedral element and simplifying, we get this Q T on one of the face joining nodes 1-2-3 that is the reason why Q and subscript T face 1-2-3 is shown there.

It is given by area of face 1-2-3 divided by 3 times the components or a vector consisting of components of tractions that are applied. It can easily be noticed since we are dealing with face joining nodes 1-2-3, if you see this vector it is having non zero components corresponding to nodes 1-2-3 and zero components corresponding to node 4 and also it can be noticed that the total load on the face 1-2-3 is divided equally among all the three nodes defining that face. Again, this equivalent nodal load vector is based on the condition that T x, T y, T z are constant, if they are not constant then we need to do numerical integration of previous equation, assuming T x, T y, T z as functions of spatial coordinates.

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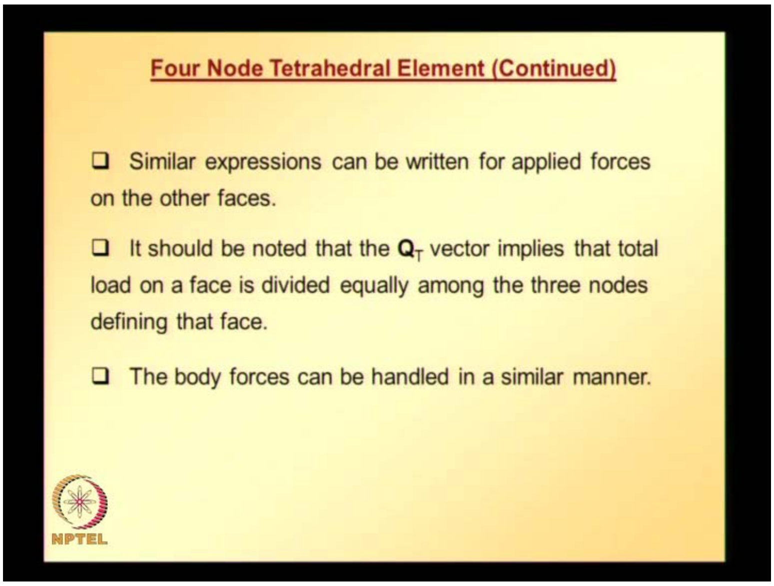

And similar expressions can be written for forces applied on other faces. It should be noted that Q T vector implies that total load on face is divided equally among the three nodes defining that face. Body forces can be handled in a similar manner. So, we looked at how to evaluate equivalent nodal force vector for the applied traction components along any of the faces and we know how to evaluate element stiffness matrix. We can easily assemble element stiffness matrix and equivalent load vector given a discretization and once we evaluate this quantity that is element stiffness matrix and equivalent nodal load vector or each of the elements. We need to assemble global stiffness matrix and global force vector base following similar procedure that we followed for two dimensional problems based on the element connectivity.

We can easily put the contribution from each of the element in the corresponding locations in the global stiffness matrix and get the global stiffness matrix and also global nodal force vector, then applying appropriate essential boundary conditions and if the essential boundary condition turns out to be zero, we can delete that particular row and column of the global stiffness matrix and arrive at the reduced equation system, which we can solve for the nodal displacements and all this procedure is similar to that we already look that for two dimensional problems.

Once we get the nodal displacements, we can calculate strains through strain displacement relations and once we arrive at strains, we can calculate stresses through stress strain relation. The stresses that we are going to obtain or going to be in terms of x, y, z coordinates that is sigma x, sigma y, sigma z. To know whether a particular structural component or mechanical component satisfies certain failure criteria or not we need to know what are called principal stresses, which are different from sigma x, sigma y, sigma z and all the shear components that is tau x y, tau y z, tau z x.

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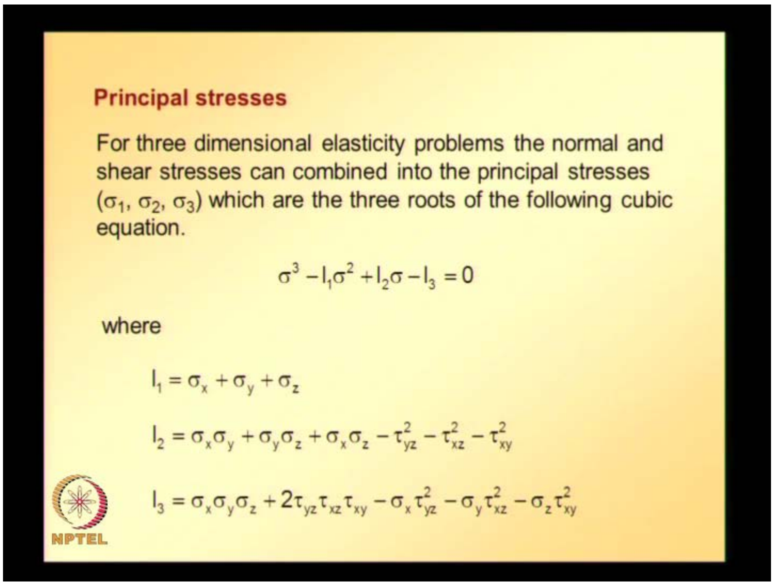

Now, let us look at how to calculate this principal stresses. For three dimensional elasticity problems the normal and shear stresses can be combined into principal stresses, which can be used for further applications in any of the failure criteria. These principal stresses are going to be the roots of the following cubic equation, which is actually in terms of stress invariants i 1, i 2, i 3 are called stress invariants, where i 1 is defined like this i 2, i 3. So, once we know the all the components of the stresses in a particular point, we can easily evaluate what is i 1, i 2, i 3 using the normal stress components and shear stress components. Once we get the numerical values of i 1, i 2, i 3, we can solve this cubic equations for the three roots that gives us sigma 1, sigma 2, sigma 3, which corresponds to the three roots of the cubic equations that is shown here.

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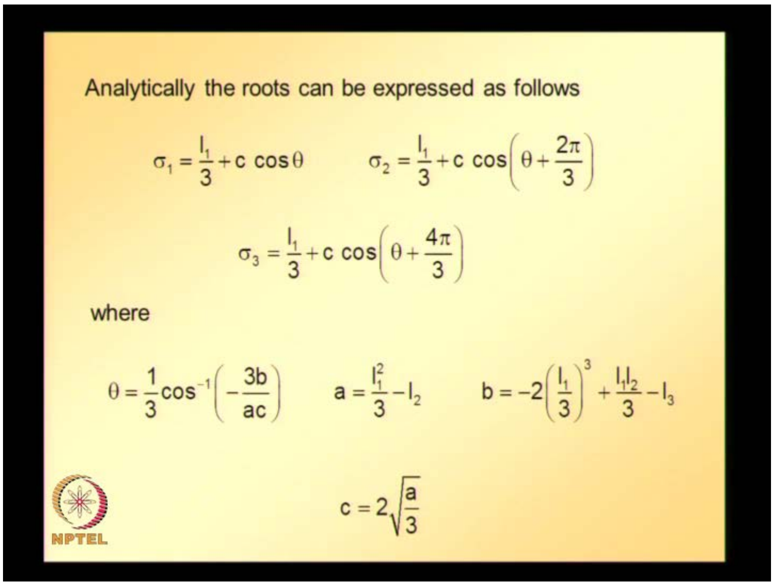

Or analytically the roots of that cubic equation can be expressed as follows. Sigma 1 is given by i 1 over 3 plus c times cos theta, sigma 2 is given by i 1 over 3 plus c times cos theta plus 2 pi over 3, i 3 is given by i 1 over 3 plus c times cos theta plus 4 pi over 3, where theta is defined like this, one-third cos inverse minus of 3 b over a c, where a b c are defined like this, in terms of stress invariants. So, either we can evaluate sigma 1, sigma 2, sigma 3 by solving the cubic equations cubic equation, which is in terms of stress invariants numerically or we can use this analytical solutions to get sigma 1, sigma 2, sigma 3 for further application in any of the failure criteria.